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Uniqueness of Uniform Decomposition Relative to a Torsion Theory

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Abstract

As a consequence of classical Krull-Remak-Schmidt Theorem, a uniqueness theorem for finite direct sum decomposition relative to uniform modules with local endomorphism rings in torsion theories is reviewed.

Keywords: τ-uniform module, τ-injective hulls, τ-essentially equivalent, Krull-Remak-Schmidt Theorem, torsion theory

1. Introduction

In this note all rings are associative with identity and all modules are unitary left modules. For a ring R, let $\tau := (\mathcal{T}, \mathcal{F})$ be a torsion theory on R-Mod. Modules in $\mathcal{T}$ will be called *τ-torsion* and modules in $\mathcal{F}$ are said to be *τ-torsion free*. Given an R-module, $\tau(M)$ will denote the τ-torsion submodule of M. Then $\tau(M)$ is necessarily the unique largest τ-torsion submodule of M and $\tau(M/\tau(M)) = 0$. For the torsion theory $\tau := (\mathcal{T}, \mathcal{F})$, $\mathcal{T} \cap \mathcal{F} = 0$ and the torsion class $\mathcal{T}$ is closed under homomorphic images, direct sums and extensions; and the torsion-free class $\mathcal{F}$ is closed under submodules, direct products and extensions (by means of short exact sequence). If the torsion class $\mathcal{T}$ closed under submodules, a torsion theory τ is called hereditary. (For more torsion theoretic terminology see also (1-3).

Let R be any ring and let τ be a hereditary torsion theory on R-Mod. For an R-module M, a submodule N of M is called *τ-dense* (respectively, *τ-pure* (or *τ-closed*)) in M if M/N is τ-torsion (respectively, τ-torsion-free). Cleary $\tau(M)$ and M both are τ-pure submodules of M. The unique minimal τ-pure submodule K of M containing N is called a *τ-closure* (or *τ-purification* in the sense of (3)) of N in M .

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An R-module M is *τ-injective* if and only if $Ext_R^1(T,M)=0$ for all τ-torsion R-module T. Equivalently, M is τ-injective if and only if M is τ-pure submodule of $E(M)$. The τ-closure of a module M in an injective hull $E(M)$ of M is called a *τ-injective hull* of M and is denoted by $E_T(M)$. (See (4)).

Let N be a submodule of a module M. Then N is called *τ-essential* in M if it is τ-dense and essential in M. Clearly, M is τ-dense essential submodule of $E_T(M)$ and $E_T(M)/M = \tau(E(M)/M)$. Every module has a τ-injective hull, unique up to an isomorphism (See [4, Theorem 2.2.3]). Thus $E_T(M)$ is unique up to an isomorphism. Here $E_T(M)$ is an essential τ-injective submodule of $E(M)$ and it is the minimal such submodule of $E(M)$ ([4, Lemma 2.2.2 (i)]). In other words, $E_T(M)$ is a τ-injective τ-essential extension of M.

A nonzero module U is called *τ-uniform* if every nonzero submodule of U is τ-essential in U (See (3, 5, 6)).

In this article, as a consequence of classical Krull-Remak-Schmidt Theorem, we show that if $A_1,A_2,\cdots,A_m$ and $B_1,B_2,\cdots,B_n$ be τ-uniform R-modules, and $A=A_1\oplus A_2\oplus\cdots\oplus A_m$ and $B=B_1\oplus B_2\oplus\cdots\oplus B_n$ are τ-essentially equivalent, that is, there are τ-essential submodules $A'\subseteq A$ and $B'\subseteq B$ such that $A'\cong B'$, then $m=n$ and there exists a permutation σ of $\{1,2,\cdots,n\}$ such that A_i and $B_{\sigma(i)}$ are τ-essentially equivalent for every i. Our interest in this result comes from the works (7-9) and especially the work of Krause (10) in abelian categories. This result can be deduced by Krause's theorem (10), but in this article we adopt the proof in torsion-theoretical concept.

Diracca and Facchini (9) proved a similar result for uniform objects in abelian categories using a different equivalence relation defined on objects, namely they say that two objects A and B belong to the same monogeny class if there exist two monomorphisms $A\to B$ and $B\to A$. Krause proved the same result as in (9) using another equivalence relation defined on objects, namely they say that two objects A and B are essentially equivalent if there exist essential subobjects $A'\subseteq A$ and $B'\subseteq B$ such that $A'\cong B'$. However, two definitions are related in the sense that finite sums of uniform objects are essentially equivalent if they belong to the same monogeny class.

2. The Proof

We say that two R-modules A and B are *τ-essentially equivalent* if there exist τ-essential submodules $A'\subseteq A$ and $B'\subseteq B$ such that $A'\cong B'$. Observe that this defines an equivalence relation on R-Mod.

Lemma 1. *Let M be a uniform (τ-uniform) R-module. Then $E_T(M)$ is uniform (τ-uniform). In particular, if M is uniform (τ-uniform) then $E_T(M)$ is indecom-posable.*

Proof. Straightforward.

Recall e.g. from (11) that a ring is a *local ring* in case it has a unique maximal ideal.

Lemma 2. *Let M be a τ-uniform R-module. Then the endomorphism ring of $E_T(M)$ is local.*

Proof. Let M be a τ-uniform R-module. Then M is uniform. By Lemma 1, $E_T(M)$ is uniform. Let us denote $A=E_T(M)$. On the other hand for any $f\in End(A)$, $Kerf\cap Ker(1_A-f)=0$. If $Kerf=0$ then $f(A)$ is τ-injective, thus $f(A)$ is a direct summand of A. By [4, Theorem 2.2.3] this implies $f(A) = A$ and so f is an isomorphism. For $Kerf \neq 0$, then $Ker(1_A - f) = 0$ and $1_A - f$ is an isomorphism.

Following technical Lemma plays the key role.

Lemma 3. *Let A and B be R-modules. Then A and B are τ-essentially equivalent if and only if $E_T(A)$ and $E_T(B)$ are isomorphic.*

Proof. Suppose A and B are τ-essentially equivalent, i.e., let $A' \subseteq A$ and $B' \subseteq B$ be τ-essential submodules such that $A' \cong B'$. Since A' is a τ-essential submodule of A, it is essential and τ-dense in A. By [4, Lemma 2.2.5], we have $E_T(A') \cong E_T(A)$ (in fact, they are equal). Similarly one shows that $E_T(B') \cong E_T(B)$.

On the other hand, assume $\varphi: B' \to A'$ is an isomorphism. Denote by $i: A' \to E_T(A')$ and $j: B' \to E_T(B')$ the inclusion homomorphisms. It follows that the composite $B' \to A' \to E_T(A')$ is a monomorphism. By the τ-injectivity of $E_T(B')$, there exists a homomorphism $f\colon E_T(A') \to E_T(B')$ such that $fi\varphi = j$. Since $i\varphi$ is an essential monomorphism, we have f is a monomorphic (See [1, Corollary 5.13]). By the τ-injectivity of $f\big(E_T(A')\big)$, the sequence

$$0 \to f\big(E_T(A')\big) \to E_T(B') \to X = E_T(B')/f\big(E_T(A')\big) \to 0$$

Splits, write $E_T(B') = f\big(E_T(A')\big) \oplus X$. Since $fi\varphi = j$, $j(N) \cap X = 0$ for any submodule N of B'. But we know $j(N)$ is an essential submodule of $E_T(B')$, so we have $X = 0$. Then it follows that f is an epimorphism. Thus $E_T(A') \cong E_T(B')$. Hence,

$$E_T(A) \cong E_T(A') \cong E_T(B') \cong E_T(B).$$

Conversely, assume that $\gamma : E_T(A) \to E_T(B)$ and $\gamma' : E_T(B) \to E_T(A)$ are isomorphisms. We put $A' = A \cap \gamma'(B)$ and $B' = B \cap \gamma(A)$. Then we have $\gamma(A') = \gamma(A) \cap \gamma\gamma'(B) = \gamma(A) \cap B = B'$. Since γ and γ' are isomorphism, we have $A' \cong B'$, which we expect.

Now we show A' is τ-essential in A and B' is τ-essential in B. First we show the essential condition. Since intersection of essential submodules is again an essential submodule, we have $A' = A \cap \gamma'(B)$ is essential in A and $B' = B \cap \gamma(A)$ is essential in B. On the other hand, $(A/A') \subseteq E_T(A)/A'$. By the definition of τ-injective hull, A is τ-dense in $E_T(A)$. Since $\big(E_T(A)/\gamma'(B)\big) \cong (E_T(B)/B)$ we have $\gamma'(B)$ is τ-dense in $E_T(A)$. Hence the intersection $A' = A \cap \gamma'(B)$ is τ-dense in $E_T(A)$. Thus, its submodule A/A' is τ-torsion. Similarly one shows that B/B' is τ-torsion.

Theorem 4. *Let $A_1, A_2, \cdots, A_m$ and $B_1, B_2, \cdots, B_n$ be τ-uniform R-modules. Suppose $A = A_1 \oplus A_2 \oplus \cdots \oplus A_m$ and $B = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ are τ- essentially equivalent. Then $m = n$ and there exists a permutation σ of $\{1,2,\cdots,n\}$ such that A_i and $B_{\sigma(i)}$ are τ-essentially equivalent for every i.*

Proof. Suppose $A = A_1 \oplus A_2 \oplus \cdots \oplus A_m$ and $B = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ are τ-essentially equivalent. Then by Lemma 3 and by [a, Proposition 2.2.6], we have

$$E_T(A_1) \oplus \cdots \oplus E_T(A_m) \cong E_T(A) \cong E_T(B) \cong E_T(B_1) \oplus \cdots \oplus E_T(B_n).$$

By Lemma 1, τ-injective hull of a τ-uniform module is indecomposable and by Lemma 2, has a local endomorphism ring. Then applying classical Krull-Remak-Schmidt Theorem we obtain $m = n$ and there exists a permutation σ of $\{1,2,\cdots,n\}$ such that $E_T(A_i) \cong E_T(B_{\sigma(i)})$ for every i (see [1, Theorem 12.9]). By Lemma 3, A_i and $B_{\sigma(i)}$ are τ-essentially equivalent for every i.

As we state in introduction, Theorem 4 can be deduced by Krause's arguments as follows. In the hypotheses of Theorem 4, Krause's hypotheses also had and so we have that $n = m$, and there is a permutation σ such that A_i is essentially equivalent to $B_{\sigma(i)}$. Since these modules are now τ-uniform by hypotheses, each essential submodule, being non-null is also τ-dense and hence τ-essential. Therefore A_i is τ-essentially equivalent to $B_{\sigma(i)}$.

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