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Some Results on the Ideals of Real-Valued Lower Triangular Toeplitz Matrices

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Abstract. In this article, we discuss some results on the ideals of real-valued lower triangular Toeplitz matrices (LTTM). Specifically, we provide the basic ring structure, and look at the ideals of LTTM. We provide new findings concerning the ideals of LTTM.

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1. Introduction

Toeplitz matrices are used in many areas such as signal processing theories and solution to difference equations [1–4, 6, 7]. Moreover, a large class of matrices are shown to be similar to Toeplitz matrices [5]. Recently, it is also shown that every matrix is a product of Toeplitz matrices [8].

In this paper, we provide a few interesting results on the ideal structure of the set of real-valued Lower Triangular Toeplitz matrices (LTTM), where a Toeplitz matrix is defined as $T = [t_{i,j}]$ with $t_{i,j} = t_{i-j}$.

Example 1.1. Consider

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & a & 0 \\ d & c & b & a \end{bmatrix}$$

for $a, b, c, d \in \mathbb{R}$. Then, $A \in LTTM(4, \mathbb{R})$, the set of $4x4$ real-valued Lower Triangular Toeplitz Matrices (LTTM). That is, for $0 \le i, j \le n$, $(A)_{(i-j)=0} = a, (A)_{(i-j)=1} = b, (A)_{(i-j)=2} = c, (A)_{(i-j)=3} = d$, and 0 otherwise.

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2. Preliminaries

It is obvious that LTTM is an abelian group under matrix addition. In this paper, we will focus on the matrix multiplication operation, x, in order to understand the ring structure of LTTM, in particular its ideals. Before we provide our results, let's define a few algebraic tools used in our arguments.

Definition 2.1. Let I_k be a matrix in $LTTM(n,\mathbb{R})$ such that $(I_k)_i = 1$ for $i = k$ and 0 otherwise.

Example 2.2.

$$I_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad I_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. Ring Structure

Under the matrix multiplication operation, it is also obvious that $LTTM(n,\mathbb{R})$ is associative, and has a multiplicative inverse, namely the nxn identity matrix. Also, the product of two lower triangular matrices is a lower triangular matrix. Thus, for the most part LTTM satisfy the ring properties, and Identity matrix is naturally a Toeplitz matrix. What is left is to prove the closeness before labeling $LTTM(n,\mathbb{R})$ a ring.

Theorem 3.1. *LTTM(n,$\mathbb{R}$) is closed under matrix multiplication.*

Proof. Let's first note that any nxn matrix A is a member of $LTTM(n,\mathbb{R})$ if and only if $(A)_{i-j} = (A)_{(i+1)-(j+1)}$ for $0 \le i, j \le n$. We will be using this fact in our proof.
Let A and B be matrices from $LTTM(n,\mathbb{R})$. Then,

$$(AB)_{i-j} = \sum_{r=j}^{i} a_{i-r}b_{r-j} = a_{i-j}b_{j-j} + a_{i-j-1}b_{j+1-j} + \cdots + a_{i-i}b_{i-j}$$

and

$$(AB)_{i+1-(j+1)} = \sum_{r=j+1}^{i+1} a_{i+1-r}b_{r-j-1} = a_{i+1-j-1}b_{j+1-j-1} + a_{i+1-j-2}b_{j+2-j-1} +$$
$$\cdots + a_{i+1-i-1}b_{i+1-j-1} = a_{i-j}b_{j-j} + a_{i-j-1}b_{j+1-j} + \cdots + a_{i-i}b_{i-j}.$$
$$\therefore (AB)_{i-j} = (AB)_{i+1-(j+1)} \quad \text{for} \quad 0 \le i, j \le n.$$

Thus,

$$AB \in LTTM(n,\mathbb{R}).$$ □

Now that we know $LTTM(n,\mathbb{R})$ is a ring, we check to see if $LTTM(n,\mathbb{R})$ is a commutative ring in the next theorem.

Theorem 3.2. *LTTM(n,$\mathbb{R}$) is commutative under matrix multiplication.*

Proof. Let A and B be matrices from $LTTM(n,\mathbb{R})$.

$$(AB)_{i-j} = \sum_{r=j}^{i} a_{i-r}b_{r-j} = a_{i-j}b_{j-j} + a_{i-j-1}b_{j+1-j} + \cdots + a_{i-i}b_{i-j} = a_{i-j}b_0 +$$
$$a_{i-j-1}b_1 + \cdots + a_0b_{i-j}.$$
$$(BA)_{i-j} = \sum_{r=j}^{i} b_{i-r}a_{r-j} = b_{i-j}a_{j-j} + b_{i-j-1}a_{j+1-j} + \cdots + b_{i-i}a_{i-j} = b_{i-j}a_0 +$$
$$b_{i-j-1}a_1 + \cdots + b_0a_{i-j}.$$

Thus,

$$AB = BA.$$ □

So far, we established that $LTTM(n,\mathbb{R})$ is a commutative ring. Now let's turn attention to the other ring structures. Our next theorem disqualifies $LTTM(n,\mathbb{R})$ as an Integral Domain.

Theorem 3.3. *LTTM(n,ℝ) is not an Integral Domain.*

Proof. Consider the two nonzero matrices A and B from $LTTM(4,\mathbb{R})$.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & b & 0 & 0 \end{bmatrix} \text{ where } a, b \text{ and } c \text{ are non-zero real numbers.}$$

Then,

$$AB = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ □

4. Ideals

Let's first define a subset of $LTTM(n,\mathbb{R})$:

$$\mathbf{T_k} = \{A \in LTTM(n,\mathbb{R}) | A_i = 0 \text{ for } 0 \le i < k\}.$$

$\mathbf{T_k}$ is clearly a subgroup of $LTTM(n,\mathbb{R})$. We will next prove that each $\mathbf{T_k}$ is in fact an ideal of $LTTM(n,\mathbb{R})$. First, here is an example of the objects of $\mathbf{T_k}$.

Example 4.1.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & b & 0 & 0 \\ d & c & b & 0 \end{bmatrix} \in \mathbf{T_1}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & b & 0 & 0 \end{bmatrix} \in \mathbf{T_1}, \text{ and } I_1 \in \mathbf{T_1}.$$

But,

$$C = \begin{bmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & a & 0 \\ d & c & b & a \end{bmatrix} \notin \mathbf{T_1}.$$

Theorem 4.1. *$\mathbf{T_k}$ is an ideal of $LTTM(n,\mathbb{R})$.*

Proof. Let A be any matrix from $LTTM(n,\mathbb{R})$, and let $B \in \mathbf{T_k}$. Thus, $b_i = 0$ for all $0 \le i < k$.

$$\therefore (AB)_i = a_i b_0 + a_{i-1} b_1 + \cdots + a_0 b_i = 0 \quad \text{for all } \ 0 \le i < k.$$

Hence,

$$AB \in \mathbf{T_k}.$$ □

Furthermore, each $\mathbf{T_k}$ is a principal Ideal. In fact, $\mathbf{T_k}$ is generated by the matrix, I_k.

Theorem 4.2. *Each $\mathbf{T_k}$ is a principal Ideal of $LTTM(n,\mathbb{R})$.*

Proof. Consider $I_k \in \mathbf{T_k}$. Let B be any LTTM matrix.

$$(BI_k)_i = 0 \ \text{ for all } \ 0 \le i < k$$

$$(BI_k)_k = b_k a_0 + b_{k-1} a_1 + \cdots + b_0 a_k = b_0$$

$$(BI_k)_{k+1} = b_{k+1} a_0 + b_{k-1} a_1 + \cdots + b_1 a_k + b_0 a_{k+1} = b_1$$

$$\vdots$$

$$(BI_k)_{k+m} = b_{k+m} a_0 + b_{k+m-1} a_1 + \cdots + b_m a_k + \cdots + b_0 a_{k+m} = b_m$$

Thus,

$$(BI_k)_{n-1} = b_{n-k-1}$$

Therefore, for any $C \in \mathbf{T_k}$, if we define $b_i = c_{i+k}$ for all $i \le n-k-1$ then $BI_k = C$. Therefore, $\mathbf{T_k} = \langle I_k \rangle$. □

Example 4.2.

$$B = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ c_2 & c_1 & 0 & 0 \\ c_3 & c_2 & c_1 & 0 \\ b & c_3 & c_2 & c_1 \end{bmatrix} \quad \text{where} \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 \\ c_2 & c_1 & 0 & 0 \\ c_3 & c_2 & c_1 & 0 \end{bmatrix} \in \mathbf{T_1}.$$

Then,

$$BI_1 = C.$$

Another interesting result is that $LTTM(n, \mathbb{R})$ has no proper ideal with a non-zero main diagonal.

Theorem 4.3. *$LTTM(n, \mathbb{R})$ has no proper ideal with non-zero main diagonal.*

Proof. Suppose such an Ideal (I^*) exists and let A be in I^* such that $a_0 \neq 0$. Let B be a diagonal matrix such that $b_0 = \frac{1}{a_0}$. Then,

$$(AB)_i = a_i b_0 \text{ for } 0 \leq i < n-1 \quad \therefore (AB)_0 = 1 \text{ and } (AB)_i = a_i^* = \frac{a_1}{a_0}.$$

Denote $AB = A^*$.
For any given matrix C in $LTTM(n, \mathbb{R})$, one can construct a matrix D also in $LTTM(n, \mathbb{R})$ such that $DA^* = C$. That is,

$$(DA^*)_0 = d_0 = c_0,$$

$$(DA^*)_1 = d_1 + d_0 a_1^* = c_1, \cdots$$

$$(DA^*)_{n-1} = d_{n-1} + d_{n-2} a_1^* + \cdots + d_0 a_{n-1}^* = c_{n-1}$$

Therefore, for any C in $LTTM(n, \mathbb{R})$, there is D such that $DA^* = C$.
Thus,

$$I^* = LTTM(n, \mathbb{R}).$$ □

In fact, all ideals of $LTTM(n, \mathbb{R})$ are in the form of $\mathbf{T_k}$.

Theorem 4.4. *Any ideal of $LTTM(n, \mathbb{R})$ is in the form of $\mathbf{T_k}$.*

Proof. Suppose some I^* contained in $\mathbf{T_1}$. Then, there must be some A in I^* such that A is also in $\mathbf{T_k}$ for some $1 \leq k \leq n-1$. Now, let's choose the smallest k such that this is true.
Thus, $a_k \neq 0$.
Consider $I_k \in \mathbf{T_k}$ where $(I_k)_i = 1$ for $i = k$ and 0 otherwise. There is ar least one matrix B in $LTTM(n, \mathbb{R})$ such that $BA = I_k$. That is:

$$(BA)_i = 0 \text{ for } 0 \leq i < k$$

$$(BA)_k = b_k a_0 + \cdots + b_0 a_k = 1 \quad \therefore \; b_0 = \frac{1}{a_k}$$

$$(BA)_{k+1} = b_{k+1} a_0 + \cdots + b_1 a_k + b_0 a_{k+1} = 0 \quad \therefore \; b_1 = -\frac{b_0 a_{k+1}}{a_k}$$

$$\vdots$$

$$(BA)_{k+m} = b_{k+m} a_0 + \cdots + b_0 a_{k+m} = 0 \quad \therefore \; b_m = -\frac{b_{m-1} a_{k+1} + \cdots + b_0 a_{k+m}}{a_k}$$

$$\vdots$$

$$(BA)_{n-1} = b_{n-1} a_0 + \cdots + b_0 a_{n-1} = 0 \quad \therefore \; b_{n-k-1} = -\frac{b_{n-k-2} a_{k+1} + \cdots + b_0 a_{n-1}}{a_k}$$

Hence, $I_k \in I^*$. Therefore, $I^* = \mathbf{T_k}$ due to $I^* = \mathbf{T_k} = \langle I_k \rangle$. Thus, there are no other ideals. □

In light of what we know now about the ideals of $LTTM(n, \mathbb{R})$, we can safely conclude that $\mathbf{T_1}$ is the only maximal ideal of $LTTM(n, \mathbb{R})$. Moreover, $\mathbf{T_1}$ is the only prime ideal.

Theorem 4.5. *$\mathbf{T_1}$ is the only prime ideal.*

Proof. Suppose A and B are LTTM matrices, and suppose $AB \in \mathbf{T_1}$. Then,

$$a_0 b_0 = 0 \text{ meaning } a_0 \text{ or } b_0 = 0.$$

Therefore $A \in \mathbf{T_1}$ or $B \in \mathbf{T_1}$.
Now suppose A and $B \in \mathbf{T_{k\text{-}1}}$ for $1 < k$ where $a_{k-1} \neq 0$ and $b_{k-1} \neq 0$.
We get:

$$(AB)_i = a_i b_1 + \cdots + a_1 b_i = 0 \text{ for all } 0 \leq i \leq k-1.$$

Thus, $AB \in \mathbf{T_k}$.
But,

$$A \notin \mathbf{T_k} \text{ and } B \notin \mathbf{T_k}.$$

□

5. Conclusion

We shared a few algebraic structures of LTTM matrices. First, we provided the basic ring structure, and later we discussed the ideals of LTTM. Latter results, we believe, have not been reported before, and will be off interest to the researchers and scientists alike.

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