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An Examination Perpendicular Transversal Intersection of *IFRS* and *MFRS* in E^3

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Abstract. The surface-surface intersection (SSI) problem is very important subject in geometry. We examined perpendicular transversal intersection problems of eight Frenet ruled surfaces which are called *" Involutive Frenet ruledsurfaces (IFRS)* and *Mannheim Frenet ruled surfaces (MFRS)* ofa curve α, in terms of the Frenet apparatus of curve α. First using only one matrix and orthogonality conditions of the eight normal vector fields are given. Further perpendicular transversal intersection conditions and curves if there exist of eight $IFRS$ and $MFRS$ are examined.

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1. Introduction and Preliminaries

The surface-surface intersection (SSI) problems can be three types: parametric-parametric, implicit-implicit, parametric - implicit. The SSI is called transversal if the normal vectors of the surfaces are linearly independent. Also the SSI is called tangential if the normal vectors of the surfaces are linearly dependent, at the intersecting points. In transversal intersection problems, the tangent vector of the intersection curve can be found easily by the vector product of the normal vectors of the surfaces. Because of this, there are many studies related to the transversal intersection problems in literature on differential geometry. Also there are some studies about tangential intersection curve and its properties. Some of these studies are mentioned by Wu, Al essio and Costa in [13], using only the normal vectors of two regular surfaces, present an algorithm to compute the local geometric properties of the transversal intersection curve. Tangential intersection of two surfaces are examined in [1] too. We have already try generate a surface based on the other surface. The evolute and involute curves, Mannheim curves or Bertrand curves are the famous examples of the generated curve pairs. In the view of such information we have generate a new ruled surface based on the other ruled surface which are called as involutive $B-scrolls$, and the *involute* $\tilde{D}-scroll$. They are examined in [11] and [12] respectively. In this paper we consider special Frenet ruled surface, cause of their generators are the Frenet vector fields

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of a curve. The quantities $\left\{V_1, V_2, V_3, \tilde{D}, k_1, k_2\right\}$ are collectively Frenet-Serret apparatus of a curve α, where V_1, V_2, and V_3 are Frenet-Serret vector fields, k_1 and k_2 are first and second curvatures, respectively. Also

$$\tilde{D}(s) = \frac{k_2}{k_1}(s)V_1(s) + V_3(s)$$

is the modified Darboux vector field of α [5]. A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. Frenet ruled surface is one which can be generated by the motion of a Frenet vector of any curve in IE^3. The famous example of this situation can be seen in [3]. In this study tangent, normal, binormal, Darboux ruled surfaces of any curve are collectively named *"Frenet ruled surfaces (FRS) of the curve α"*. Before, in [7] we have an examination on the positions of Frenet ruled surfaces along Bertrand pairs according to their normal vector fields . Further we have some results on the positions of Frenet ruled surfaces along *involute-evolute curves* according to their normal vector fields in [11].

Definition 1.1 ([2]). In the Euclidean $3-$ *space*, let $\alpha(s)$ be the arclengthed curve. The equations

$$\begin{cases} \varphi_1(s,u_1) = \alpha(s) + u_1 V_1(s) \\ \varphi_2(s,u_2) = \alpha(s) + u_2 V_2(s) \\ \varphi_3(s,u_3) = \alpha(s) + u_3 V_3(s) \\ \varphi_4(s,u_4) = \alpha(s) + u_4 \tilde{D}(s) \end{cases}$$

are the parametrization of tangent ruled surface, normal ruled surface, binormal ruled surface, Darboux ruled surface couse of they are generated by the motion of tangent, normal, binormal, Darboux Frenet vector field of any curve, respectively, in IE^3. Collectively they are called Frenet ruled surfaces (FRS).

Theorem 1.2 ([11]). *In the Euclidean $3-$ space, let η_1, η_2, η_3, and η_4 be the normal vector fields of Frenet ruled surfaces $\varphi_1, \varphi_2, \varphi_3$ and φ_4, recpectively, along the curve α. They can be expressed by the following matrix;*

$$[\eta] = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a & 0 & b \\ c & d & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where

$$\begin{aligned} a &= \frac{-u_2 k_2}{\sqrt{(u_2 k_2)^2 + (1-u_2 k_1)^2}} \quad , \quad b = \frac{(1-u_2 k_1)}{\sqrt{(u_2 k_2)^2 + (1-u_2 k_1)^2}} \, , \\ c &= \frac{-u_3 k_2}{\sqrt{(u_3 k_2)^2 + 1}} \quad , \quad d = \frac{-1}{\sqrt{(u_3 k_2)^2 + 1}} . \end{aligned}$$

Involute of a curve is very familiar offset curve. If the tangent vectors of α and $\alpha^*$are intersect orthogonally they are called evolute and involute curves, respectively. Let the quantities V_1^*, V_2^*, V_3^* and $\tilde{D}^*$ be collectively Frenet-Serret vector fields, k_1^*, and k_2^* be curvatures of the second curve α^*. Then we have the equalites $\left\langle V_1^*, V_1 \right\rangle = 0, V_2 = V_1^*$. For the evolute and involute curves.

$$\alpha^*(s) = \alpha(s) + (\sigma - s)V_1(s)$$

is the equation of involute of the curve α. The Frenet vectors of the *involute* α^*, based on the its evolute curve α [4] are

$$\begin{cases} V_1^* = V_2, \quad V_2^* = \frac{-k_1 V_1 + k_2 V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}, \quad V_3^* = \frac{k_2 V_1 + k_1 V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}} \\ \\ \tilde{D}^* = \frac{k_2}{\sqrt{k_1^2+k_2^2}} V_1 - \frac{k_1' k_2 - k_1 k_2'}{(k_1^2+k_2^2)^{\frac{3}{2}}} V_2 + \frac{k_1}{\sqrt{k_1^2+k_2^2}} V_3 \end{cases}$$

where $\tilde{D}^*$ is the modified Darboux vector of involute curve $\alpha^*$of an evolute curve α, based on the Frenet apparatus of evolute curve α.The first and second curvature of *involute* α^*, respectively, are

$$k_1^* = \frac{\sqrt{k_1^2 + k_2^2}}{(\sigma - s)k_1} \quad , \quad k_2^* = \frac{-k_2^2 \left(\frac{k_1}{k_2}\right)'}{(\sigma - s)k_1 \left(k_1^2 + k_2^2\right)}.$$

For more detail see in [4, 9].Mannheim curve was firstly defined as by A. Mannheim in 1878. A curve is called a Mannheim curve if and only if $\frac{k_1}{(k_1^2+k_2^2)}$ is a nonzero constant, k_1 is the curvature and k_2 is the torsion. Mannheim curve was redefined as; if the principal normal vector of first curve and binormal vector of second curve are linearly dependent, then first curve is called Mannheim curve, and the second curve is called Mannheim partner curve by Liu and Wang. As a result they called these new curves as Mannheim partner curves. For more detail see in [10].

Let $\alpha^{**} : I \to E^3$ be the $C^2-class$ differentiable curve with Frenet Apparatus $\left\{V_1^{**}(s^{**}), V_2^{**}(s^{**}), V_3^{**}(s^{**}), k_1^{**}, k_2^{**}\right\}$. If the principal normal vector V_2 of the curve α is linearly dependent on the binormal vector V_3^{**} of the curve α^{**}, then the pair $\{\alpha, \alpha^{**}\}$ is said to be Mannheim pair, then α is called a Mannheim curve and α^{**} is called Mannheim partner curve of α where $<(V_1, V_1^{**}) = \cos\theta$ and besides the equality $\frac{k_1}{k_1^2+k_2^2}$ = constant is known the offset property,for some non-zero constant. In [8] Mannheim offsets of ruled surfaces are defined and characterized. For some function λ^{**}, since V_2 and V_3 are linearly dependent. Mannheim partner curve has the following equation;

$$\alpha^{**}(s) = \alpha(s) - \lambda V_2(s)$$

where $\lambda = \frac{-k_1}{k_1^2+k_2^2}$. Frenet-Serret apparatus of Mannheim partner curve α^{**}, based in Frenet-Serret vectors of Mannheim curve α are

$$\begin{cases} V_1^{**} = cos\theta\, V_1 - sin\theta\, V_3, \quad V_2^{**} = sin\theta\, V_1 + cos\theta\, V_3, \quad V_3^{**} = V_2 \\ \tilde{D}^{**}(s) = \frac{k_1}{\lambda k_2}\frac{\cos^2\theta}{\dot{\theta}} V_1 + V_2 - \frac{k_1}{\lambda k_2}\frac{\cos\theta \sin\theta}{\dot{\theta}} V_3 \end{cases}$$

where $\tilde{D}^{**}$ is the modified Darboux vector of Mannheim partner $\alpha^{**}$of a Mannheim curve α. The curvature and the torsion have the following equalyties,

$$k_1^{**} = -\frac{d\theta}{ds^*} = \frac{\dot{\theta}}{\cos\theta} \quad , \quad k_2^{**} = \frac{k_1}{\lambda k_2}$$

we use dot to denote the derivative with respect to the arc length parameter of the curve α. Also $\frac{ds}{ds^{**}} = \frac{1}{\cos\theta}$, where $|\lambda|$ is the distance between the curves α and α^*. For more detail see in [8].

Normal vector fields of *IFRS* and *MFRS*. In this section first, we give the Tangent, Normal, Binormal, Darboux Frenet ruled surfaces of the involute curve α^*. Further we write their parametric equations in terms of the Frenet apparatus of the involute-evolute curve curve α. Hence, they are called collectively *"Involutive Frenet ruled surfaces or IFRS of curve α"* as in the following way.

Definition 1.3 ([11]). In the Euclidean $3-space$, let $\alpha(s)$ be the arclengthed curve. The equations

$$\begin{cases} \varphi_1^*(s, v_1) = \alpha(s) + (\sigma - s)V_1(s) + v_1 V_2(s) \\ \varphi_2^*(s, v_2) = \alpha(s) + (\sigma - s)V_1(s) + v_2 \left(\frac{-k_1 V_1 + k_2 V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}\right) \\ \varphi_3^*(s, v_3) = \alpha(s) + (\sigma - s)V_1(s) + v_3 \left(\frac{k_2 V_1 + k_1 V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}\right) \\ \varphi_4^*(s, v_4) = \alpha(s) + (\sigma - s)V_1(s) \\ \qquad + v_4 \left(\frac{k_2}{\sqrt{k_1^2+k_2^2}} V_1 - \frac{k_1' k_2 - k_1 k_2'}{(k_1^2+k_2^2)^{\frac{3}{2}}} V_2 + \frac{k_1}{\sqrt{k_1^2+k_2^2}} V_3\right) \end{cases}$$

are the parametrization of the ruled surfaces which are called involutive tangent ruled surface ($ITRS$), involutive normal ruled surface ($INRS$), involutive binormal ruled surface ($IBRS$), involutive Darboux ruled surface ($IDRS$), respectively, couse of they are generated by the motion of tangent, normal, binormal, Darboux Frenet vector field of involute curve. respectively, in IE^3. They are called collectively " Involutive Frenet ruled surfaces (IFRS).

Theorem 1.4 ([11]). *In the Euclidean* $3-space$*, the normal vector fields* $\eta_1^*, \eta_2^*, \eta_3^*$,*and* η_4^* *of ruled surfaces* $\varphi_1^*, \varphi_2^*, \varphi_3^*$, *and* φ_4^**, recpectively, along the curve involute* α^**, can be expressed by the following matrix;*

$$[\eta^*] = \begin{bmatrix} \eta_1^* \\ \eta_2^* \\ \eta_3^* \\ \eta_4^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a^* & 0 & b^* \\ c^* & d^* & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}$$

where

$$a^* = \frac{-v_2 k_2^*}{\sqrt{\left(v_2 k_2^*\right)^2 + \left(1 - v_2 k_1^*\right)^2}} \quad , \quad b^* = \frac{\left(1 - v_2 k_1^*\right)}{\sqrt{\left(v_2 k_2^*\right)^2 + \left(1 - v_2 k_1^*\right)^2}} \, ,$$

$$c^* = \frac{-v_3 k_2^*}{\sqrt{\left(v_3 k_2^*\right)^2 + 1}} \quad , \quad d^* = \frac{-1}{\sqrt{\left(v_3 k_2^*\right)^2 + 1}}.$$

Also we give the Tangent, Normal, Binormal, Darboux Frenet ruled surfaces of the Mannheim partner α^{**} of curve α. Further we write their parametric equations in terms of the Frenet apparatus of the Mannheim curve α. Hence they are called collectively *" Mannheim Frenet ruled surfaces of curve (MFRS) of curve* α*"* as in the following way.

Definition 1.5 ([6]). In the Euclidean $3-space$, let $\alpha(s)$ be the arclengthed curve. The equations

$$\begin{cases} \varphi_1^{**}(s, w_1) = \alpha + w_1 cos\theta V_1 - \lambda V_2 - w_1 sin\theta V_3 \\ \varphi_2^{**}(s, w_2) = \alpha + w_2 sin\theta \, V_1 - \lambda V_2 + w_2 cos\theta V_3 \\ \varphi_3^{**}(s, w_3) = \alpha + w_3 V_2 - \lambda V_2 \\ \varphi_4^{**}(s, w_4) = \alpha + w_4 \frac{k_1 \cos\theta cos\theta}{\lambda\dot{\theta}k_2} V_1 + (w_4 - \lambda) V_2 \\ \qquad -w_4 \frac{k_1 \cos\theta sin\theta}{\lambda\dot{\theta}k_2} V_3 \end{cases}$$

are the parametrization of the ruled surfaces which are called Mannheim Tangent ruled surface (MTRS), Mannheim Normal ruled surface (MNRS), Mannheim Binormal ruled surface (MBRS), Mannheim Darboux ruled surface (MDRS), respectively, couse of they are generated by the motion of tangent, normal, binormal, Darboux Frenet vector field of Mannheim partner of any curve, respectively, in IE^3. They are called collectively as Mannheim Normal ruled surface, (MFRS).

Theorem 1.6 ([6]). *In the Euclidean* $3-space$ *, the normal vector fields* $\eta_1^{**}, \eta_2^{**}, \eta_3^{**}$*, and* η_4^{**} *of ruled surfaces* $\varphi_1^{**}, \varphi_2^{**}, \varphi_3^{**}$*, and* φ_4^{**}*, recpectively, along the curve*
Mannheim partner α^**, can be expressed by the following matrix;*

$$[\eta^{**}] = \begin{bmatrix} \eta_1^{**} \\ \eta_2^{**} \\ \eta_3^{**} \\ \eta_4^{**} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a^{**} & 0 & b^{**} \\ c^{**} & d^{**} & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1^{**} \\ V_2^{**} \\ V_3^{**} \end{bmatrix}$$

where

$$a^{**} = \frac{-w_2 k_2^{**}}{\sqrt{\left(w_2 k_2^{**}\right)^2 + \left(1 - w_2 k_1^{**}\right)^2}} \quad , \quad b^{**} = \frac{\left(1 - w_2 k_1^{**}\right)}{\sqrt{\left(w_2 k_2^{**}\right)^2 + \left(1 - w_2 k_1^{**}\right)^2}} \, ,$$

$$c^{**} = \frac{-w_3 k_2^{**}}{\sqrt{\left(w_3 k_2^{**}\right)^2 + 1}} \quad , \quad d^{**} = \frac{-1}{\sqrt{\left(w_3 k_2^{**}\right)^2 + 1}}.$$

2. Perpendicular Taransversal IFRS and MFRS

In this section, using a matrix the sixteen positions of normal vector fields of eight IFRS and MFRS are examined. Further some interesting results are given, with simple matrices product and equality. It is trivial that, the product matrix of unit normal vector fields $[\eta^*] = \left[\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*\right]$ and $[\eta^{**}] = \left[\eta_1^{**}, \eta_2^{**}, \eta_3^{**}, \eta_4^{**}\right]$ of IFRS and MFRS, respectively, along the curve α is

$$[\eta^*]\,[\eta^{**}]^{\mathbf{T}} = \begin{bmatrix} \left\langle \eta_1^*, \eta_1^{**} \right\rangle & \left\langle \eta_1^*, \eta_2^{**} \right\rangle & \left\langle \eta_1^*, \eta_3^{**} \right\rangle & \left\langle \eta_1^*, \eta_4^{**} \right\rangle \\ \left\langle \eta_2^*, \eta_1^{**} \right\rangle & \left\langle \eta_2^*, \eta_2^{**} \right\rangle & \left\langle \eta_2^*, \eta_3^{**} \right\rangle & \left\langle \eta_2^*, \eta_4^{**} \right\rangle \\ \left\langle \eta_3^*, \eta_1^{**} \right\rangle & \left\langle \eta_3^*, \eta_2^{**} \right\rangle & \left\langle \eta_3^*, \eta_3^{**} \right\rangle & \left\langle \eta_3^*, \eta_4^{**} \right\rangle \\ \left\langle \eta_4^*, \eta_1^{**} \right\rangle & \left\langle \eta_4^*, \eta_2^{**} \right\rangle & \left\langle \eta_4^*, \eta_3^{**} \right\rangle & \left\langle \eta_4^*, \eta_4^{**} \right\rangle \end{bmatrix}.$$

Theorem 2.1. *The product matrix $[\eta^*]\,[\eta^{**}]^{\mathbf{T}}$ of the unit normal vector fields of $IFRS$ and $MFRS$, respectively, along the curve α is*

$$\begin{bmatrix} 0 & -\frac{k_2 cos\theta - k_1 sin\theta}{m} a^{**} & \begin{array}{c} -\frac{k_2 cos\theta - k_1 sin\theta}{m} c^{**} \\ -\frac{k_1 cos\theta + k_2 sin\theta}{m} d^{**} \end{array} & \frac{k_1 cos\theta + k_2 sin\theta}{m} \\ -a^* & \begin{array}{c} b^* a^{**} \frac{k_2 cos\theta - k_1 sin\theta}{m} \\ -a^* b^{**} \end{array} & \begin{array}{c} \frac{k_2 cos\theta - k_1 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} b^* c^{**} \\ +\frac{k_1 cos\theta + k_2 sin\theta}{m} b^* d^{**} \end{array} & -a^* \\ -c^* & \begin{array}{c} d^* a^{**} \frac{-k_1 cos\theta - k_2 sin\theta}{m} \\ +c^* b^{**} \end{array} & \begin{array}{c} \frac{-k_1 cos\theta - k_2 sin\theta}{m} d^* c^{**} \\ +\frac{k_2 cos\theta - k_1 sin\theta}{m} d^* d^{**} \end{array} & d^* \frac{k_2 cos\theta - k_1 sin\theta}{m} \\ 0 & -\frac{-k_1 cos\theta - k_2 sin\theta}{m} a^{**} & \begin{array}{c} -\frac{-k_1 cos\theta - k_2 sin\theta}{m} c^{**} \\ -\frac{k_2 cos\theta - k_1 sin\theta}{m} d^{**} \end{array} & \frac{k_2 cos\theta - k_1 sin\theta}{m} \end{bmatrix}$$

where $m = \sqrt{k_1^2 + k_2^2} \neq 0$.

Proof. Let $[\eta^*] = [A^*]\,[V^*]$ and $[\eta^{**}] = [A^{**}]\,[V^{**}]$. Also

$$\begin{aligned} [\eta^*]\,[\eta^{**}]^{\mathbf{T}} &= [A^*]\,[V^*]\,([A^{**}]\,[V^{**}])^{\mathbf{T}} \\ &= [A^*]\left([V^*]\,[V^{**}]^{\mathbf{T}}\right)[A^{**}]^{\mathbf{T}} \\ &= [A^*] \begin{bmatrix} 0 & 0 & 1 \\ \frac{-k_1 cos\theta - k_2 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & \frac{k_2 cos\theta - k_1 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & 0 \\ \frac{k_2 cos\theta - k_1 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & \frac{k_1 cos\theta + k_2 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & 0 \end{bmatrix} [A^{**}]^T. \end{aligned}$$

Using the following matrix product form of Frenet vector fields of the involute curve α^*, and Mannheim partner curve α^{**}; we have

$$[V^*]\,[V^{**}]^T = \begin{bmatrix} 0 & 0 & 1 \\ \frac{-k_1 cos\theta - k_2 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & \frac{k_2 cos\theta - k_1 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & 0 \\ \frac{k_2 cos\theta - k_1 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & \frac{k_1 cos\theta + k_2 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & 0 \end{bmatrix}. \tag{2.1}$$

Hence

$$\begin{aligned} [\eta^*]\,[\eta^{**}]^{\mathbf{T}} &= \begin{bmatrix} 0 & 0 & -1 \\ a^* & 0 & b^* \\ c^* & d^* & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \frac{-k_1 cos\theta - k_2 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & \frac{k_2 cos\theta - k_1 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & 0 \\ \frac{k_2 cos\theta - k_1 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & \frac{k_1 cos\theta + k_2 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & 0 \end{bmatrix} [A^{**}]^{\mathbf{T}} \\ &= \begin{bmatrix} -\frac{k_2 cos\theta - k_1 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & -\frac{k_1 cos\theta + k_2 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & 0 \\ b^* \frac{k_2 cos\theta - k_1 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & b^* \frac{k_1 cos\theta + k_2 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & a^* \\ d^* \frac{-k_1 cos\theta - k_2 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & d^* \frac{k_2 cos\theta - k_1 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & c^* \\ -\frac{-k_1 cos\theta - k_2 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & -\frac{k_2 cos\theta - k_1 sin\theta}{(k_1^2+k_2^2)^{\frac{1}{2}}} & 0 \end{bmatrix} \begin{bmatrix} 0 & a^{**} & c^{**} & 0 \\ 0 & 0 & d^{**} & -1 \\ -1 & b^{**} & 0 & 0 \end{bmatrix} \end{aligned} \tag{2.2}$$

this product give us the result. □

The perpendicular transversal intersection of two surface, basically, can be examined by the position of their unit normal vector fields $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ and $\eta_1^{**}, \eta_2^{**}, \eta_3^{**}, \eta_4^{**}$. We can examine the sixteen positions of eight surfaces, basically, according to the position of their unit normal vector fields in a matrix. Since the equality of the last two matrices (2.1) and (2.2), we have sixteen interesting results according to the normal vector fields with the following theorems.

Theorem 2.2. *There are only two pairs of Frenet ruled surface which are always perpendicular transversal, these are $ITRS$; $MTRS$ of curve α and $IDRS$; $MTRS$ of curve α.*

Proof. According to matrices equality we can say easily that

$$\langle \eta_1^*, \eta_1^{**} \rangle = \langle \eta_4^*, \eta_1^{**} \rangle = 0$$

hence we have the proof. □

Theorem 2.3. *Involutive tangent ruled surface and Mannheim normal ruled surface of curve α have perpendicular normal vector fields and $w_2 k_2^{**} \neq 0$, so* $\tan\theta = \frac{k_2}{k_1}$.

Proof. Since $\left\langle \eta_1^*, \eta_2^{**} \right\rangle = (k_1 sin\theta - k_2 cos\theta)\, a^{**}$ and using the orthogonality condition $k_1 sin\theta - k_2 cos\theta = 0$, or since $w_2 k_2^{**} \neq 0$ we have $\tan\theta = \frac{k_2}{k_1}$. □

Theorem 2.4. *Involutive tangent ruled surface and Mannheim binormal ruled surface of curve α have perpendicular normal vector fields, if*

$$\tan\theta = \frac{k_1 k_2 (\lambda + w_3)}{-\lambda k_2^2 + w_3 k_1^2}.$$

Proof. Since $\left\langle \eta_1^*, \eta_3^{**} \right\rangle = (k_1 sin\theta - k_2 cos\theta)\, c^{**} - (k_1 cos\theta + k_2 sin\theta)\, d^{**}$ and under the orthogonality condition

$$(k_1 sin\theta - k_2 cos\theta)\, c^{**} - (k_1 cos\theta + k_2 sin\theta)\, d^{**} = 0$$

we have

$$\tan\theta = \frac{k_2 c^{**} + k_1 d^{**}}{k_1 c^{**} - k_2 d^{**}} = \frac{\lambda k_1 k_2 + w_3 k_1 k_2}{-\lambda k_2 k_2 + w_3 k_1 k_1}.$$ □

Theorem 2.5. *Involutive tangent ruled surface and Mannheim Darboux ruled surface of curve have perpendicular normal vector fields, if* $\tan\theta = -\frac{k_1}{k_2}$.

Proof. Since $\left\langle \eta_1^*, \eta_4^{**} \right\rangle = k_1 cos\theta + k_2 sin\theta$ and under the orthogonality condition $k_1 cos\theta + k_2 sin\theta = 0$, hence $\tan\theta = -\frac{k_1}{k_2}$. □

Theorem 2.6. *Involutive normal ruled surface and Mannheim tangent ruled surface of Bertrand curve α have not perpendicular normal vector fields, except $v_2 = 0$.*

Proof. Since $\left\langle \eta_2^*, \eta_1^{**} \right\rangle = -a^*$, and under the orthogonality condition

$$\frac{-v_2 k_2^*}{\sqrt{\left(v_2 k_2^*\right)^2 + \left(1 - v_2 k_1^*\right)^2}} = 0,$$

and $k_2^* \neq 0$ it is trivial. □

Theorem 2.7. *Involutive normal ruled surface and Mannheim normal ruled surface of Bertrand curve α have perpendicular normal vector fields along under the condition*

$$k_2 cos\theta - k_1 sin\theta = m \frac{-v_2 k_1 + v_2 k_1 w_2 \frac{\dot{\theta}}{\cos\theta}}{-w_2 k_1 1 + w_2 k_1 v_2 \frac{\sqrt{k_1^2 + k_2^2}}{(\sigma - s) k_1}}. \tag{2.3}$$

Proof. Since $\left\langle \eta_2^*, \eta_2^{**} \right\rangle = b^* a^{**} \frac{k_2 cos\theta - k_1 sin\theta}{m} - a^* b^{**}$, and under the orthogonality condition

$$b^* a^{**} \frac{k_2 cos\theta - k_1 sin\theta}{m} - a^* b^{**} = 0$$

we get

$$\begin{aligned} k_2 cos\theta - k_1 sin\theta &= m\frac{-v_2\frac{k_1}{\lambda k_2}\left(1 - w_2 k_1^{**}\right)}{-w_2\frac{k_1}{\lambda k_2}\left(1 - v_2 k_1^{*}\right)} \\ &= m\frac{-v_2 k_1\left(1 - w_2\frac{\dot{\theta}}{\cos\theta}\right)}{-w_2 k_1\left(1 - v_2\frac{\sqrt{k_1^2+k_2^2}}{(\sigma-s)k_1}\right)}. \end{aligned}$$

□

Theorem 2.8. *Involutive normal ruled surface and Mannheim binormal ruled surface have perpendicular normal vector fields for the value*

$$\tan\theta = \frac{-(\lambda + w_3)\,k_2 k_1}{\lambda k_2^2 - w_3 k_1^2 - v_2\lambda\frac{k_2^2 m}{(\sigma-s)k_1}}.$$

Proof. Since $\left\langle \eta_2^*, \eta_3^{**}\right\rangle = (k_2 cos\theta - k_1 sin\theta)\, b^* c^{**} + b^* d^{**}\,(k_1 cos\theta + k_2 sin\theta)$ and under the orthogonality condition

$$(k_2 cos\theta - k_1 sin\theta)\, b^* c^{**} + b^* d^{**}\,(k_1 cos\theta + k_2 sin\theta) = 0$$

we have

$$\begin{aligned} \tan\theta &= \frac{-w_3 k_2 k_2^{**} - k_1}{k_2 - w_3 k_2^{**} k_1 - v_2 k_2 k_1^{*}} \\ &= \frac{-w_3\frac{k_1}{\lambda k_2} k_2 - k_1}{k_2 - w_3\frac{k_1}{\lambda k_2} k_1 - v_2 k_2\frac{\sqrt{k_1^2+k_2^2}}{(\sigma-s)k_1}}. \end{aligned}$$

□

Theorem 2.9. *Involutive normal ruled surface and Mannheim Darboux ruled surface have not perpendicular normal vector fields, except $v_2 = 0$.*

Proof. Since $\left\langle \eta_2^*, \eta_4^{**}\right\rangle = -a^*$, and under the orthogonality condition

$$\frac{-v_2 k_2^*}{\sqrt{\left(v_2 k_2^*\right)^2 + \left(1 - v_2 k_1^*\right)^2}} = 0$$

since $k_2^* \neq 0$ it is trivial. □

Theorem 2.10. *Involutive binormal ruled surface and Mannheim tangent ruled surface of curve α have not perpendicular normal vector fields, except $w_3 = 0$, or $k_2^{**} = 0$.*

Proof. Since $\left\langle \eta_3^*, \eta_1^{**}\right\rangle = -c^* = -\frac{-w_3 k_2^{**}}{\sqrt{\left(w_3 k_2^{**}\right)^2+1}}$ and under the orthogonality condition $-w_3 k_2^{**} \neq 0$. it is trivial. □

Theorem 2.11. *Involutive binormal ruled surface and Mannheim normal ruled surface of curve α have perpendicular normal vector fields for*

$$k_1 cos\theta + k_2 sin\theta = \frac{\lambda k_2^3 v_3\left(1 - w_2\dot{\theta}\right)}{m k_1^2(\sigma - s) w_2 \cos\theta}\left(\frac{k_1}{k_2}\right)'$$

Proof. Since $\left\langle \eta_3^*, \eta_2^{**}\right\rangle = d^* a^{**}\frac{-k_1 cos\theta - k_2 sin\theta}{m} + c^* b^{**}$ and under the orthogonality condition

$$d^* a^{**}\frac{-k_1 cos\theta - k_2 sin\theta}{m} + c^* b^{**} = 0$$

we have

$$\begin{aligned} k_1 cos\theta + k_2 sin\theta &= m\frac{-1 + w_2 k_1^{**}}{w_2 k_2^{**}} v_3 k_2^* \\ &= m\left(\frac{-1 + w_2 \frac{\dot{\theta}}{\cos\theta}}{w_2 \frac{k_1}{\lambda k_2}}\right) \frac{-k_2^2 \left(\frac{k_1}{k_2}\right)' v_3}{(\sigma - s) k_1 \left(k_1^2 + k_2^2\right)}. \end{aligned}$$

□

Theorem 2.12. *Involutive binormal ruled surface and Mannheim binormal ruled surface have perpendicular normal vector fields for the value;*

$$\tan\theta = \frac{w_3 k_1^2 - \lambda k_2^2}{-(\lambda + w_3) k_2 k_1}.$$

Proof. Since $\left\langle \eta_3^*, \eta_3^{**} \right\rangle = d^* d^{**} (k_2 cos\theta - k_1 sin\theta) - (k_1 cos\theta + k_2 sin\theta) d^* c^{**}$ and under the orthogonality condition

$$d^{**} (k_2 cos\theta - k_1 sin\theta) - (k_1 cos\theta + k_2 sin\theta) c^{**} = 0$$

we have

$$\begin{aligned} (d^{**} k_2 - c^{**} k_1) cos\theta &= sin\theta \\ \tan\theta &= \frac{-k_2 + w_3 k_2^{**} k_1}{-k_1 - w_3 k_2^{**} k_2}. \end{aligned}$$

□

Theorem 2.13. *Involutive binormal ruled surface and Mannheim Darboux ruled surface of curve α have perpendicular normal vector fields, if* $\tan\theta = \frac{k_2}{k_1}$.

Proof. Since $\left\langle \eta_3^*, \eta_4^{**} \right\rangle = d^* (k_2 cos\theta - k_1 sin\theta)$ and under the condition

$$k_2 cos\theta - k_1 sin\theta = 0$$

and $d^* \neq 0$. □

Theorem 2.14. *Involutive Darboux ruled surface and Mannheim normal ruled surface of Bertrand curve α have perpendicular normal vector fields*, *if*
$\tan\theta = -\frac{k_1}{k_2}$.

Proof. Since $\left\langle \eta_4^*, \eta_2^{**} \right\rangle = (k_1 cos\theta + k_2 sin\theta) a^{**}$ and under the orthogonality condition there is not a real value of $a^{**} \neq 0$, hence we get $k_1 cos\theta + k_2 sin\theta = 0$. □

Theorem 2.15. *Involutive Darboux ruled surface and Mannheim binormal ruled surface of curve α have perpendicular normal vector fields, for the value*

$$\tan\theta = \frac{w_3 k_1^2 - \lambda k_2^2}{(\lambda + w_3) k_1 k_2}.$$

Proof. Since $\left\langle \eta_4^*, \eta_3^{**} \right\rangle = (k_1 cos\theta + k_2 sin\theta) c^{**} - (k_2 cos\theta - k_1 sin\theta) d^{**}$ and under the orthogonality condition

$$(k_1 cos\theta + k_2 sin\theta) c^{**} - (k_2 cos\theta - k_1 sin\theta) d^{**} = 0,$$

we have

$$\tan\theta = \frac{c^{**} k_1 - d^{**} k_2}{d^{**} k_1 + c^{**} k_2} = \frac{-w_3 \frac{k_1}{\lambda k_2} k_1 + k_2}{-k_1 - w_3 \frac{k_1}{\lambda k_2} k_2}.$$ □

Theorem 2.16. *Involutive Darboux ruled surface and Mannheim Darboux ruled surface of curve α have perpendicular normal vector fields, if* $\tan\theta = \frac{k_2}{k_1}$.

Proof. Since $\left\langle \eta_4^*, \eta_4^{**} \right\rangle = (k_2 cos\theta - k_1 sin\theta)$ and under the orthogonality condition

$$(k_2 cos\theta - k_1 sin\theta) = 0$$

we have the proof. □

Corollary 2.17. *The perpendicular intersection conditions of eigth Frenet ruled surfaces which are called $IFRS$ and $MFRS$ are given as in one table*

$$\begin{bmatrix} \langle , \rangle & \eta_T^{**} & \eta_N^{**} & \eta_B^{**} & \eta_D^{**} \\ \eta_T^* & 0 & \tan\theta = \frac{k_2}{k_1} & \tan\theta = \frac{k_1 k_2(\lambda + w_3)}{-\lambda k_2^2 + w_3 k_1^2} & \tan\theta = \frac{-k_1}{k_2} \\ \eta_N^* & \neq 0 & \begin{array}{c} k_2 cos\theta - k_1 sin\theta \\ = m\frac{v_2 k_1\left(1 - w_2 \frac{\dot{\theta}}{\cos\theta}\right)}{w_2 k_1\left(1 - v_2 k_1^*\right)} \end{array} & \begin{array}{c} \tan\theta \\ = \frac{(\lambda + w_3) k_2 k_1}{\lambda k_2\left(w_3 \frac{k_1}{\lambda k_2} k_1 - k_2 + v_2 k_2 \frac{m}{(\sigma - s) k_1}\right)} \end{array} & \neq 0 \\ \eta_B^* & \neq 0 & \begin{array}{c} k_1 cos\theta + k_2 sin\theta \\ = \frac{\lambda k_2^3 v_3 \left(1 - w_2 \dot{\theta}\right)\left(\frac{k_1}{k_2}\right)'}{m k_1^2 (\sigma - s) w_2 \cos\theta} \end{array} & \tan\theta = \frac{-w_3 k_1^2 + \lambda k_2^2}{(\lambda + w_3) k_1 k_2} & \tan\theta = \frac{k_2}{k_1} \\ \eta_D^* & 0 & \tan\theta = \frac{-k_1}{k_2} & \tan\theta = \frac{w_3 k_1^2 - \lambda k_2^2}{(\lambda + w_3) k_1 k_2} & \tan\theta = \frac{k_2}{k_1} \end{bmatrix}.$$

Proof. It is trivial, using the $[\eta^*]\,[\eta^{**}]^{\mathbf{T}} = 0$. □

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