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Araştırma Makalesi / Research Article

Another odd log-logistic logarithmic class of continuous distributions

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Abstract

In this work, a new class of continuous distributions is presented and the mathematical properties of the new distribution class is studied. We estimate the model parameters by the maximum likelihood method and assess its performance based on biases and mean squared errors in a simulation study framework. For the real data set, the special member of the new class provides a better fit than other models generated by other well-known families.

Keywords: *Odd Log-Logistic Family; Maximum Likelihood Estimation; Compound Class; Order Statistics.*

Sürekli dağılımların bir diğer odd log-logistik logaritmik sınıfı

Öz

Bu çalışmada, sürekli dağılımların yeni bir sınıfı sunulmuştur ve bu yeni dağılım sınıfının matematiksel özellikleri çalışılmıştır. Model parametreleri en çok olabilirlik tahmin yöntemi ile elde edilmiş ve bu tahmin edicilerin performansları yan ve hata kareler ortalamasına dayalı olarak bir simülasyon çalışması üzerinde gözlemlenmiştir. Gerçek bir seti için, yeni sınıfın özel bir üyesi diğer iyi bilinen dağılım sınıflarının üyelerinden daha iyi uyum sağlamıştır.

Anahtar sözcükler: *Odd log-logistik ailesi; En çok olabilirlik tahmini; Birleştirilmiş sınıf; Sıra istatistikleri.*

1. Introduction

Several continuous univariate models have been extensively used for modeling data in many areas such as insurance, economics, environmental sciences, engineering and biological studies. So, several new families of distributions have been constructed by extending common classes of continuous distributions. These new families of distributions give high flexibility by adding one "or more" parameters to the baseline distribution. Many odd log-logistic-G families can be cited by Alizadeh et al. (2015), Cordeiro et al. (2016a, b), Alizadeh et al. (2017), Brito et al. (2017), Cordeiro et al. (2017), Alizadeh et al. (2018) and Korkmaz et al. (2018), among others.

Let $G(x;\ \underline{\phi}) = G(x)$ be a baseline cumulative distribution function (cdf) and $\underline{\phi}$ be the $p \times 1$ vector of associated parameters. Recently, Gleaton and Lynch (2004, 2006 and 2010) introduced a class of distributions called the odd log-logistic family with one extra shape parameter $\alpha > 0$ defined by the cdf

$$H_{OLL-G}(x;\ \alpha,\ \underline{\phi}) = G(x;\ \underline{\phi})^{\alpha}\left[G(x;\ \underline{\phi})^{\alpha} + \bar{G}(x;\ \underline{\phi})^{\alpha}\right]^{-1}, \tag{1}$$

where $\bar{G}(x;\ \underline{\phi}) = \bar{G}(x) = 1 - G(x;\ \underline{\phi})$. In this paper, we introduce a new family of distributions called "another odd log-logistic logarithmic-G" (AOLLL-G) family. The cdf of this family is given by

$$F_{AOLLL-G}(x;\ \alpha,\ \beta,\ \underline{\phi}) = 1 - \left\{\left[\log(1-\beta)\right]^{-1}\log\left[1 - \frac{\beta\bar{G}(x,\ \underline{\phi})^{\alpha}}{G(x,\ \underline{\phi})^{\alpha} + \bar{G}(x,\ \underline{\phi})^{\alpha}}\right]\right\}, \tag{2}$$

where $G(x;\ \underline{\phi})$ is the baseline cdf depending on a parameter vector $\underline{\phi}$ and $\alpha > 0$ and $0 < \beta < 1$ are two additional shape parameters. For each baseline G, it includes odd log-logistic (AOLL-G) family by Gleaton and Lynch (2004 and 2006) and logarithmic-G family. Some special models are given in Table 1.

Table 1. Some special models.

α	β	$G(x,\ \underline{\phi})$	Reduced distribution
-	$\uparrow 1$	$G(x,\ \underline{\phi})$	OLL-G family [Gleaton and Lynch (2004 and 2006)]
1	-	$G(x,\ \underline{\phi})$	Logarithmic-G family baseline distribution
1	$\uparrow 1$	$G(x,\ \phi)$	$G(x,\ \phi)$

This paper is organized as follows. In Section 2, we define the new family. Some of its special cases are presented in Section 3. In Section 4, we derive some of its mathematical properties. Section 5 provides maximum likelihood estimation procedure for model parameters. In Section 6, a simulation study is performed to see the efficiency of maximum likelihood method. In Section 7, we illustrate the importance of the new family by means of an application to real data set. The paper is concluded in Section 8.

2. The new family and its motivation

The corresponding density function of (2) is given by

$$f_{AOLLL-G}(x;\ \alpha,\ \beta,\ \underline{\phi}) = \alpha\beta g(x,\ \underline{\phi})G(x,\ \underline{\phi})^{\alpha-1}\bar{G}(x,\ \underline{\phi})^{\alpha-1} \times \left\{\begin{array}{c} -\left[G(x,\ \underline{\phi})^{\alpha} + \bar{G}(x,\ \underline{\phi})^{\alpha}\right] \\ \times\left[G(x,\ \underline{\phi})^{\alpha} + (1-\beta)\bar{G}(x,\ \underline{\phi})^{\alpha}\right]\ \log(1-\beta) \end{array}\right\}^{-1}, \tag{3}$$

where $g(x;\ \underline{\phi})$ is the baseline pdf, $\alpha > 0$ and $0 < \beta < 1$. Equation (3) will be most tractable when the cdf $G(x)$ and the pdf $g(x)$ have simple analytic expressions. Hereafter, a random variable X with density function (3) is denoted by $X \sim$ AOLLL-G $(\alpha,\ \beta,\ \underline{\phi})$. Henceforward $G(x) = G(x;\ \underline{\phi})$ and $g(x) = g(x;\ \underline{\phi})$ and so on. A motivation of this family can be explained as follows: Suppose that a parallel system is made up of N components and the lifetimes of the components are independent and identically distributed (iid) random variables, denoted as $Z_1, \cdots,\ Z_N$, with common cdf (2). Then, the system fails as soon as the last component fails, namely the lifetime of the whole system is represented by $X=\min\{Z_1,\cdots,\ Z_N\}$. In many survival parallel systems, it is almost impossible to have a fixed number of components because some of them get lost or censored for various reasons. Therefore, we may assume that N is a discrete random variable. Suppose that N has the logarithmic distribution with, probability mass function given by

$P(N=n)=-\beta^n/n\log(1-\beta),\quad n=1,\ 2,\ldots,\quad 0<\beta<1.$

Then the cdf of the life length of the whole system, X, is obtained as

$$
\begin{aligned}
F(x) &= \sum_{n=1}^{\infty} P(X\le x\mid N=n)P(N=n)\\
&= \sum_{n=1}^{\infty}\left[-\beta^n/n\log(1-\beta)\right]\left\{1-\left[1-\frac{G(x,\ \underline{\phi})^{\alpha}}{G(x,\ \underline{\phi})^{\alpha}+\bar{G}(x,\ \underline{\phi})^{\alpha}}\right]^{n}\right\}\\
&= 1-\left\{\left[\log(1-\beta)\right]^{-1}\log\left[1-\frac{\beta\bar{G}(x,\ \underline{\phi})^{\alpha}}{G(x,\ \underline{\phi})^{\alpha}+\bar{G}(x,\ \underline{\phi})^{\alpha}}\right]\right\}
\end{aligned}
$$

which is identical to (2). The hazard rate function (hrf) of X becomes

$$
\begin{aligned}
h(x;\ \alpha,\ \beta,\ \underline{\phi}) = -\alpha\beta g(x,\ \underline{\phi})G(x,\ \underline{\phi})^{\alpha-1}\bar{G}(x,\ \underline{\phi})^{\alpha-1}&\left[G(x,\ \underline{\phi})^{\alpha}+\bar{G}(x,\ \underline{\phi})^{\alpha}\right]^{-1}\\
&\times\left[G(x,\ \underline{\phi})^{\alpha}+(1-\beta)\bar{G}(x,\ \underline{\phi})^{\alpha}\right]^{-1}\\
&\times\left\{\log\left[1-\frac{\beta\bar{G}(x,\ \underline{\phi})^{\alpha}}{G(x,\ \underline{\phi})^{\alpha}+\bar{G}(x,\ \underline{\phi})^{\alpha}}\right]\right\}^{-1}.
\end{aligned}
\tag{4}
$$

3 Special AOLLL-G models

3.1 The AOLLL-normal distribution

We define a new model called AOLLL-normal (AOLLL-N) distribution from (3) by taking $G(x;\ \mu,\ \sigma)=\Phi\left(\frac{x-\mu}{\sigma}\right)$ and $g(x;\ \mu,\ \sigma)=\sigma^{-1}\phi\left(\frac{x-\mu}{\sigma}\right)$ with $\underline{\phi}=(\mu,\ \sigma)$, where $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively. Then, the new pdf can be obtained using (3), where $x\in\Re,\ \mu\in\Re,\ \ \alpha,\ \sigma>0, 0<\beta<1$. The pdf of the new model is denoted by AOLLL-N. We plot this pdf and its hrf in Figure 1. From Figure 1, we see that the pdf shapes of the AOLLL-N are skewed and bi-modal.

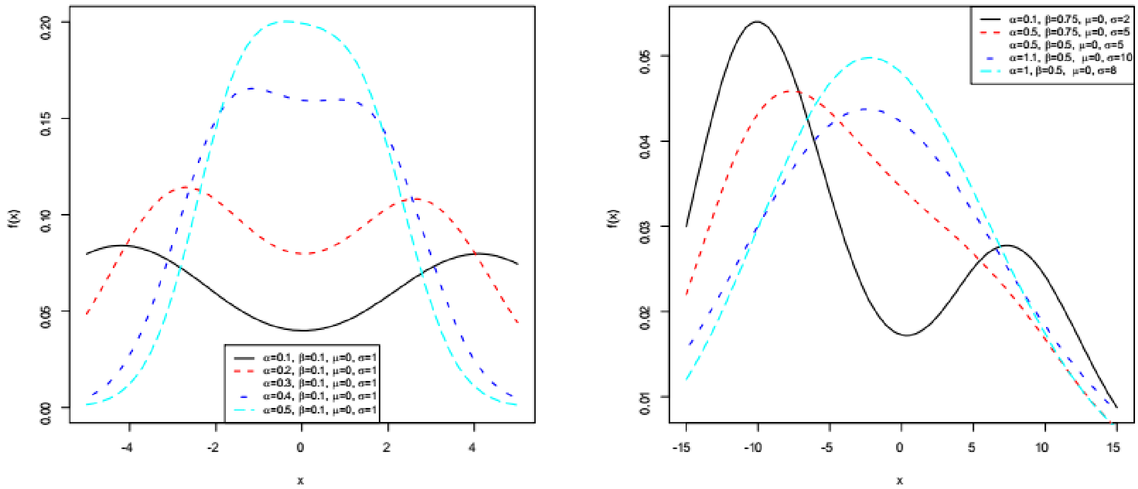

Figure 1. Plots of the pdf and hrf of the AOLLL-N distributions.

3.2 *The AOLLL-Weibull distribution*

We now consider the Weibull (W) distribution as a baseline distribution with pdf $g(x\ ;\ a,\ b)=ba^{b}x^{b-1}e^{-(ax)^{b}}$ and cdf $G(x\ ;\ a,\ \gamma)=1-e^{-(ax)^{b}}$. Then, the new pdf can be obtained via (3), where $x>0,\ \ \alpha,\ a,\ b>0, 0<\beta<1$. The new model is denoted by AOLLL-W and its pdf and hrf plots for selected parameter values are displayed in Figure 2. From Figure 2, we see that pdf of the new model have various important shapes, the pdf shapes are decreasing, unimodal, bi-modal, firstly decreasing then unimodal shaped. Also, its hrf shapes are increasing, decreasing, unimodal, bathtub and firstly unimodal and then increasing. So, we can say that new distribution can be useful for modelling various data sets.

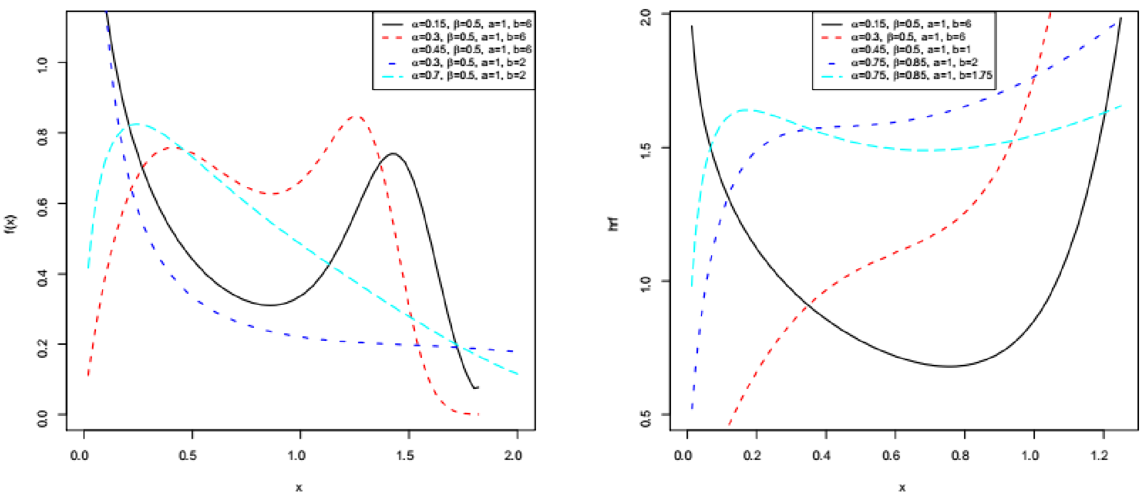

Figure 2. Plots of the pdf and hrf of the AOLLL-W distributions

4 Mathematical properties

4.1 *Quantile function*

The new family of distributions is easily simulated by inverting (2) as follows: if U has a uniform $U(0,\ 1)$ distribution, then

$$Q(U)=G^{-1}\left\{\frac{\left\{1-\frac{1}{\beta}\left[1-(1-\beta)^{1-u}\right]\right\}^{\frac{1}{\alpha}}}{\left\{\frac{1}{\beta}\left[1-(1-\beta)^{1-u}\right]\right\}^{\frac{1}{\alpha}}+\left\{1-\frac{1}{\beta}\left[1-(1-\beta)^{1-u}\right]\right\}^{\frac{1}{\alpha}}}\right\} \tag{5}$$

has the density function (2). Although, we have stated that $\beta \in (0,\ 1)$, Equation (2) is still a cdf if $\beta < 0$. Hence, we can consider the new family for any $\beta < 1$.

4.2 Useful expansions

By using the power series

$$-\log(1-u)=\sum_{i=1}^{\infty} u^i / i \quad \text{for } |u|<1,$$

the cdf of the new model follows as

$$F(x)=1-\sum_{i=1}^{\infty}\beta^i \quad \bar{G}(x)^{\alpha i} / i\left[G(x)^{\alpha}+\left[1-G(x)\right]^{\alpha}\right]^i, \tag{6}$$

where $|\beta|<1$. Next, we obtain an expansion for $F(x)$. First, we use a power series for $\bar{G}(x)^{\alpha}$ ($\alpha > 0$ real) given by

$$\bar{G}(x)^{\alpha} = \sum_{k=0}^{\infty} a_k(\alpha) \quad G(x)^k, \tag{7}$$

where

$$a_k(\alpha)=(-1)^k \binom{\alpha}{k}. \tag{8}$$

For any real $\alpha > 0$, we consider the generalized binomial expansion

$$[1-G(x)]^{\alpha} = \sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k} \quad G(x)^k. \tag{9}$$

Also, we have

$$\frac{G(x)^{\alpha i}}{\left\{G(x)^{\alpha}+\left[1-G(x)\right]^{\alpha}\right\}^i} = \underbrace{\frac{\sum_{k=0}^{\infty} a_k(\alpha i) \quad G(x)^k}{\sum_{k=0}^{\infty} \ b_k \quad G(x)^k}}_{A},$$

where $b_k = h_k^*(\alpha,\ i)$ is defined in the Appendix. The ratio of the two power series can be expressed as

$$A=\sum_{k=0}^{\infty} c_k(\alpha,\ i) \quad G(x)^k, \tag{10}$$

where the coefficients $c_k(\alpha,\ i)$'s (for $k \geq 0$) are determined from the recurrence equation

$$c_k(\alpha,\ i) = \left(a_k(\alpha i) - b_0^{-1} \sum_{r=1}^{k} b_r \quad c_{k-r}(\alpha,\ i) \right) / b_0.$$

Then, we can write

$$F(x) = 1 - \sum_{k=0}^{\infty} b_k \quad H_k(x) = \sum_{k=0}^{\infty} d_k \quad H_k(x), \tag{11}$$

where

$$b_k = \sum_{i=1}^{\infty} \left[\beta^i c_k(\alpha,\ i) \right] / i, \ \ d_0 = 1 - b_0$$

and for $k \geq 1$, $d_k = b_k$. The pdf of X follows by differentiating (11) as

$$f(x) = \sum_{k=0}^{\infty} d_{k+1} \quad h_{k+1}(x), \tag{12}$$

where $h_{k+1}(x) = (k+1) \ \ G(x)^k g(x)$ is the exponentiated G (Exp-G) density function with power parameter $(k+1)$. Equation (12) reveals that the new density function is a linear combination of Exp-G densities for $|\beta| < 1$. Thus, some structural properties of the new family such as the ordinary and incomplete moments and generating function can be determined from well-established properties of the Exp-G distribution.

4.3 Moments

Let Y_k be a random variable with Exp-G distribution with power parameter $k+1$, i.e., with density $h_{k+1}(x)$. A first formula for the n^{th} ordinary moment of $X \sim$ AOLLL-G follows from (12) as

$$E(X^n) = \sum_{k=0}^{\infty} d_{k+1} \quad E(Y_k^n). \tag{13}$$

A second formula for $E(X^n)$ follows from (12) in terms of the G quantile function (qf) as

$$E(X^n) = \sum_{k=0}^{\infty} (k+1) \quad d_{k+1} \quad \tau(n,\ k), \tag{14}$$

$$E(X^n) = \sum_{k=0}^{\infty} (k+1) \quad d_{k+1} \quad \tau(n,\ k),$$

where $\tau(n,\ k) = \int_{-\infty}^{\infty} x^n \quad G(x)^k \quad g(x) dx = \int_0^1 Q_G(u)^n \quad u^k du$. For empirical purposes, the shape of many distributions can be usefully described by what we call the incomplete moments. These types of moments play an important role for measuring inequality, for example, income quantiles and Lorenz and Bonferroni curves, which depend upon the incomplete moments of a distribution. The n^{th} incomplete moment of X is calculated as

$$m_n(y) = E(X^n \mid X < y) = \sum_{k=0}^{\infty} (k+1) \quad d_{k+1} \quad \int_0^{G(y)} Q_G(u)^n u^k du. \tag{15}$$

The last integral can be computed for most G distributions.

4.4 Generating function

Let $M_X(t)=E(e^{t\,X})$ be the moment generating function (mgf) of *X*. Then, we can write from (12) as

$$M_X(t)=\sum_{k=0}^{\infty} d_{k+1}\quad M_k(t), \tag{16}$$

where $M_k(t)$ is the mgf of Y_k. Hence, $M_X(t)$ can be determined from the Exp-G generating function. A second formula for $M(t)$ can be determined from (12) as

$$M(t)=\sum_{i=0}^{\infty}(k+1)\quad d_{k+1}\quad \zeta(t,\ k), \tag{17}$$

where

$$\zeta(t,k)=\int_0^1 \exp\left[tQ_G(u)\right]u^k du=\int_{-\infty}^{\infty}\exp\left[tx\right]G(x)^k\,dx.$$

4.5 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose that $X_1,\ldots,\ X_n$ is a random sample from the new class. Let $X_{i\,:\,n}$ denote the i^{th} order statistic. From equations (2) and (3), the pdf of $X_{i\,:\,n}$ can be written as

$$f_{i\,:\,n}(x)=K\quad \sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left[\sum_{r=0}^{\infty}d_{r+1}\quad (r+1)\quad G(x)^r\quad g(x)\right]\left[\sum_{k=0}^{\infty}d_k\quad G(x)^k\right]^{j+i-1},$$

where $K=n!/[(i-1)!\ \ (n-i)!])$ for a power series raised to a positive integer number. Following Gradshteyin and Ryzhik (2000) for power series raised, we obtain

$$\left[\sum_{k=0}^{\infty}d_k\quad G(x)^k\right]^{j+i-1}=\sum_{k=0}^{\infty}e_{j+i-1,\,k}\quad G(x)^k,$$

where

$e_{j+i-1,\,0}=d_0^{j+i-1}$ and, for $k\geq 1$,

$$e_{j+i-1,\,k}=\left(k\quad d_0\right)^{-1}\sum_{q=1}^{k}[q(j+i)-k]\quad d_q\quad e_{j+i-1,\,k-q}.$$

Setting $d_r^*=(r+1)d_{r+1}$ for multiplying two power series, we have

$$f_{i:n}(x) = K \sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} g(x)\left[\sum_{r=0}^{\infty} d_r^* \; G(x)^r\right]\left[\sum_{k=0}^{\infty} e_{j+i-1,\,k} \; G(x)^k\right]$$
$$= K \sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} g(x) \sum_{k=0}^{\infty} e_k^* \; G(x)^k,$$

where

$$e_k^* = \sum_{q=0}^{k} e_{j+i-1,\,q} \; d_{k-q}^*.$$

Hence, we can write

$$f_{i:n}(x) = \sum_{k=0}^{\infty} s_k \; h_{k+1}(x), \tag{18}$$

where (for $k \geq 0$)

$$s_k = \frac{k}{k+1}\sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} e_k^*.$$

Equation (18) is the main result of this section. It reveals that the pdf of the AOLLL-G order statistics is a linear combination of Exp-G densities when $-1<\beta<1$. So, several mathematical quantities of the new family order statistics can be obtained from those quantities of the Exp-G distribution.

4.6 Entropies

The Rényi entropy (Rényi, 1961), of a random variable with pdf $f(x)$ is defined by

$$I_R(\gamma) = (1-\gamma)^{-1}\log\left[\int_0^{\infty} f^{\gamma}(x)dx\right],$$

for $\gamma > 0$ and $\gamma \neq 1$. The Shannon entropy (Shannon, 1948) of a random variable X is defined by $E\left\{-\log\left[f(X)\right]\right\}$. It is the special case of the Rényi entropy when $\gamma \uparrow 1$. Direct calculation gives

$$E\left\{-\log\left[f(X)\right]\right\} = -\log\left\{\alpha\beta/\left[-\log(1-\beta)\right]\right\} - E\left\{\log\left[g(X;\ \underline{\phi})\right]\right\} + (1-\alpha)E\left\{\log\left[G(X;\ \underline{\phi})\right]\right\}$$
$$+(1-\alpha)\mathrm{E}\left\{\log\left[\bar{G}(X;\ \underline{\phi})\right]\right\} + 2E\left\{\log\left[G(X;\ \underline{\phi})^{\alpha} + \bar{G}(X;\ \underline{\phi})^{\alpha}\right]\right\}$$
$$+E\left[\log\left(1-\left\{\beta\bar{G}(X,\ \underline{\phi})^{\alpha}/\left[G(X,\ \underline{\phi})^{\alpha}+\bar{G}(X,\ \underline{\phi})^{\alpha}\right]\right\}\right)\right].$$

First, we define and compute

$$A(a_1,\ a_2,\ a_3,\ a_4;\ \alpha,\ \beta) = \int_0^1 \frac{u^{a_1}(1-u)^{a_2}}{[u^{\alpha}+(1-u)^{\alpha}]^{a_3}\left[1-\frac{\beta(1-u)^{\alpha}}{u^{\alpha}+(1-u)^{\alpha}}\right]^{a_4}}du.$$

By using the binomial expansion, we have (for $-1<\beta<1$)

$$A(a_1,\ a_2,\ a_3,\ a_4;\ \alpha,\ \beta)=\sum_{i,\ j=0}^{\infty}(-1)^{i+j}\binom{-a_4}{i}\binom{a_2+\alpha i}{j}\int_0^1 \beta^i \underbrace{\frac{u^{a_1+j}}{\left[u^{\alpha}+(1-u)^{\alpha}\right]^{a_3+i}}}_{A_{u_1}}du.$$

Further

$$A_{u_1}=\frac{\sum_{k=0}^{\infty}a_k^* u^k}{\sum_{k=0}^{\infty}b_k^* u^k}=\sum_{k=0}^{\infty}c_k^* u^k$$

where

$$a_k^*=\sum_{q=k}^{\infty}\binom{a_1+j}{q}\binom{q}{k}(-1)^{k+q},$$

$b_k^*=h_k^*(\alpha,\ a_3+i)$ is defined in Appendix and the coefficients c_k^* 's for $k\geq 0$ are obtained from the following recurrence equation

$$c_k^*=c_k^*(a_1,\ a_3,\ \alpha,\ i,\ j)=\left(a_k^*-b_0^{*-1}\sum_{r=1}^{k}b_r^*\ \ c_{k-r}^*\right)/\,b_0^*.$$

Then,

$$A(a_1,\ a_2,\ a_3,\ a_4;\ \alpha,\ \beta)=\sum_{i,\ j=0}^{\infty}\beta^i(-1)^{i+j}\binom{-a_4}{i}\binom{a_2}{j}\ c_k^*(a_1,\ a_3,\ \alpha,\ i,\ j)/\left(1+k\right).$$

After some algebraic manipulations, we obtain

$$E\left\{\log\left[G(X)\right]\right\}=\frac{\alpha\beta}{-\log(1-\beta)}\ \ \left\{\partial\left[A(\alpha+t-1,\ \alpha-1,\ 2,\ 1;\ \alpha,\ \beta)\,|_{t=0}\right]/\,\partial t\right\},$$

$$E\left\{\log\left[\bar{G}(X)\right]\right\}=\frac{\alpha\beta}{-\log(1-\beta)}\ \ \left\{\partial\left[A(\alpha-1,\ \alpha+t-1,\ 2,\ 1;\ \alpha,\ \beta)\,|_{t=0}\right]/\,\partial t\right\},$$

$$E\left\{\log\left\{G(X;\ \underline{\phi})^{\alpha}+\bar{G}(X;\ \underline{\phi})^{\alpha}\right\}\right\}=\frac{\alpha\beta}{-\log(1-\beta)}\ \ \left\{\partial\left[A(\alpha-1,\ \alpha-1,\ 2-t,\ 1;\ \alpha,\ \beta)\,|_{t=0}\right]/\,\partial t\right\},$$

$$E\left\{\log\left[1-\frac{\beta\bar{G}(X,\ \underline{\phi})^{\alpha}}{G(X,\ \underline{\phi})^{\alpha}+\bar{G}(X,\ \underline{\phi})^{\alpha}}\right]\right\}=\frac{\alpha\beta}{-\log(1-\beta)}\ \ \left\{\partial\left[A(\alpha-1,\ \alpha-1,\ 2,\ 1-t;\ \alpha,\ \beta)\,|_{t=0}\right]/\,\partial t\right\}.$$

Then the simplest formula for the entropy of X is given by

$$
\begin{aligned}
\mathrm{E}\left\{-\log[f(X)]\right\} &= -\log\left\{\alpha\beta/\left[-\log(1-\beta)\right]\right\} - \mathrm{E}\left\{\log\left[g(X;\ \underline{\phi})\right]\right\} \\
&\quad -\frac{\alpha(1-\alpha)\beta}{\log(1-\beta)}\ \left\{\partial\left[A(\alpha+t-1,\ \alpha-1,\ 2,\ 1;\ \alpha,\ \beta)\,|_{t=0}\right]/\partial t\right\} \\
&\quad -\frac{\alpha(1-\alpha)\beta}{\log(1-\beta)}\ \left\{\partial\left[A(\alpha-1,\ \alpha+t-1,\ 2,\ 1;\ \alpha,\ \beta)\,|_{t=0}\right]/\partial t\right\} \\
&\quad -\frac{2\alpha\beta}{\log(1-\beta)}\ \left\{\partial\left[A(\alpha-1,\ \alpha-1,\ 2-t,\ 1;\ \alpha,\ \beta)\,|_{t=0}\right]/\partial t\right\} \\
&\quad -\frac{\alpha\beta}{\log(1-\beta)}\ \left\{\partial\left[A(\alpha-1,\ \alpha-1,\ 2,\ 1-t;\ \alpha,\ \beta)\,|_{t=0}\right]/\partial t\right\}.
\end{aligned}
$$

For the Rényi entropy, after some algebraic developments, we obtain an alternative expression

$$
I_R(\gamma) = \gamma\left(1-\gamma\right)^{-1}\log\left\{\alpha\beta/\left[-\log(1-\beta)\right]\right\} + \left(1-\gamma\right)^{-1}\log\left[B(\alpha,\ \beta,\ \gamma)\right],
$$

where

$$
B(\alpha,\ \beta,\ \gamma) = \int_0^1 \frac{g^{\gamma-1}\left[G^{-1}(u)\right]\ u^{(\alpha-1)\gamma}(1-u)^{(\alpha-1)\gamma}}{\left[u^\alpha+(1-u)^\alpha\right]^{2\gamma}\left[1-\frac{\beta\ (1-u)^\alpha}{u^\alpha+(1-u)^\alpha}\right]^\gamma}\,du.
$$

By using the binomial expansion, we have

$$
\begin{aligned}
B(\alpha,\ \beta,\ \gamma) &= \sum_{i,\ j=0}^{\infty}(-1)^{i+j}\beta^i\binom{-\gamma}{i}\ \binom{\gamma(\alpha-1)+\alpha\ \ i}{j} \\
&\quad \times\int_0^1\left\{g^{\gamma-1}\left[G^{-1}(u)\right]\right\}\ \underbrace{\frac{u^{\gamma(\alpha-1)+j}}{\left[u^\alpha+(1-u)^\alpha\right]^{2\gamma+i}}}_{A_{u_2}}\,du.
\end{aligned}
$$

Further,

$$
A_{u_2} = \frac{\sum_{k=0}^{\infty}a_{1,\ k}^{*}u^k}{\sum_{k=0}^{\infty}b_{1,\ k}^{*}u^k} = \sum_{k=0}^{\infty}c_{1,\ k}^{*}u^k,
$$

where

$$
a_{1,\ k}^{*} = \sum_{q=k}^{\infty}\binom{(\alpha-1)\gamma+j}{q}\binom{q}{k}(-1)^{k+q},
$$

$b_{1,\ k}^{*} = h_k^{*}(\alpha,\ 2\gamma+i)$ is defined in Appendix and the coefficients $c_{1,\ k}^{*}$'s for $k \geq 0$ are obtained from the following recurrence equation

$$
c_k^{*} = c_{1,\ k}^{*}(\gamma,\ \alpha,\ i,\ \ j) = \left(a_k^{*} - b_0^{*-1}\sum_{r=1}^{k}b_{1,\ r}^{*}\ \ c_{1,\ k-r}^{*}\right)/b_0^{*}.
$$

Finally

$$B(\alpha,\ \beta,\ \gamma)=\sum_{i,\ j,\ k=0}^{\infty} a_{i,\ j,\ k} E_{Y_k}\left\{g^{\gamma-1}\left(G^{-1}(Y)\right)\right\},$$

where

$$a_{i,\ j,\ k}=(-1)^{i+j}\beta^{i}c^{*}_{1,\ k}(\gamma,\ \alpha,\ i,\ j)\binom{-\gamma}{i}\binom{\gamma(\alpha-1)+\alpha\ \ i}{j}/\left(k+1\right)$$

and $Y_k \sim Beta(k+1,\ 1)$.

5 Estimation and Inference

Here, we determine the MLEs of the model parameters of the new family from complete samples. Let $x_1,\dots,\ x_n$ be the observed values from the new family of distributions with parameters $\alpha,\ \beta$ and $\underline{\phi}$. Let $\Theta=(\alpha,\ \beta,\ \underline{\phi})^T$ be the parameter vector. The total log-likelihood function for Θ is then given by

$$\begin{aligned}\ell_n=\ell_n(\Theta)=n\ \ \log\left\{\alpha\beta/\left[-\log(1-\beta)\right]\right\}+\sum_{i=1}^{n}\log\left[g(x_i;\ \underline{\phi})\right]+(\alpha-1)\sum_{i=1}^{n}\log\left[G(x_i;\ \underline{\phi})\bar{G}(x_i;\ \underline{\phi})\right]\\ -2\sum_{i=1}^{n}\log\left[G(x_i;\ \underline{\phi})^{\alpha}+\bar{G}(x_i;\ \underline{\phi})^{\alpha}\right]-\sum_{i=1}^{n}\log\left[G(x_i;\ \underline{\phi})^{\alpha}+(1-\beta)\bar{G}(x_i;\ \underline{\phi})^{\alpha}\right].\end{aligned} \tag{19}$$

The log-likelihood function can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (19). We use the goodness of fit function in R and NLMixed procedure in SAS to obtain the MLEs. The components of the score function $U_n(\Theta)=\left(\partial\ell_n/\partial\alpha,\partial\ell_n/\partial\beta,\partial\ell_n/\partial\underline{\phi}\right)^T$ are

$$\begin{aligned}\frac{\partial\ell_n}{\partial\alpha}=-2\sum_{i=1}^{n}\frac{G(x_i;\ \underline{\phi})^{\alpha}\log\left[G(x_i;\ \underline{\phi})\right]+(1-\beta)\bar{G}(x_i;\ \underline{\phi})^{\alpha}\log\left[\bar{G}(x_i;\ \underline{\phi})\right]}{G(x_i;\ \underline{\phi})^{\alpha}+(1-\beta)\bar{G}(x_i;\ \underline{\phi})^{\alpha}}\\ -\sum_{i=1}^{n}\frac{G(x_i;\ \underline{\phi})^{\alpha}\log\left[G(x_i;\ \underline{\phi})\right]+\bar{G}(x_i;\ \underline{\phi})^{\alpha}\log\left[\bar{G}(x_i;\ \underline{\phi})\right]}{G(x_i;\ \underline{\phi})^{\alpha}+\bar{G}(x_i;\ \underline{\phi})^{\alpha}}\\ +n\alpha^{-1}+\sum_{i=1}^{n}\log\left[G(x_i;\ \underline{\phi})\bar{G}(x_i;\ \underline{\phi})\right],\end{aligned}$$

$$\frac{\partial\ell_n}{\partial\beta}=n\beta^{-1}+n\left[(1-\beta)\log(1-\beta)\right]^{-1}+\sum_{i=1}^{n}\frac{\bar{G}(x_i;\ \underline{\phi})^{\alpha}}{G(x_i;\ \underline{\phi})^{\alpha}+(1-\beta)\bar{G}(x_i;\ \underline{\phi})^{\alpha}}$$

and

$$\frac{\partial \ell_n}{\partial \underline{\phi}} = \sum_{i=1}^{n} \frac{g^{(\underline{\phi})}(x_i,\ \underline{\phi})}{g(x_i,\ \underline{\phi})} + (\alpha-1)\sum_{i=1}^{n} \frac{G^{(\underline{\phi})}(x_i,\ \underline{\phi})}{G(x_i,\ \underline{\phi})} + (1-\alpha)\sum_{i=1}^{n} \frac{G^{(\underline{\phi})}(x_i,\ \underline{\phi})}{\bar{G}(x_i,\ \underline{\phi})}$$
$$-2\alpha\sum_{i=1}^{n} \frac{G^{(\underline{\phi})}(x_i,\ \underline{\phi})\left[G(x_i;\ \underline{\phi})^{\alpha-1} - (1-\beta)\bar{G}(x_i;\ \underline{\phi})^{\alpha-1}\right]}{G(x_i;\ \underline{\phi})^{\alpha} + (1-\beta)\bar{G}(x_i;\ \underline{\phi})^{\alpha}},$$
$$-\alpha\sum_{i=1}^{n} \frac{G^{(\underline{\phi})}(x_i,\ \underline{\phi})\left[G(x_i;\ \underline{\phi})^{\alpha-1} - \bar{G}(x_i;\ \underline{\phi})^{\alpha-1}\right]}{G(x_i;\ \underline{\phi})^{\alpha} + \bar{G}(x_i;\ \underline{\phi})^{\alpha}}.$$

where $h^{(\underline{\phi})}(\cdot)$ means the derivative of the function h with respect to $\underline{\phi}$.

6 Simulation study

In here, we obtain the graphical results to see MLEs of the model parameters. We generate $N = 1000$ samples of size $n = 20,\ 21, \ldots,\ 500$ from AOLLL-W distribution with selected parameters values $\alpha = 2$, $\beta = 0.25$, $a = 2$ and $b = 0.5$. The random numbers generation is obtained by its quantile function. We also calculate the empirical means, standard deviations (sd), bias and mean square error (MSE) of the MLEs. The bias and MSE are calculated by (for $h = \alpha,\ \beta,\ a,\ b$)

$$Bias_h = N^{-1}\sum_{i=1}^{N}\left(\hat{h}_i - h\right) \text{ and } MSE_h = N^{-1}\sum_{i=1}^{N}\left(\hat{h}_i - h\right)^2$$

respectively. All computations are obtained by using optim-CG routine in R program. We give results of this simulation study in Figure 3. From Figure 3, we observe that when the sample size increases, the empirical means for all parameters approach to true parameter value whereas the all biases, sd's and MSEs decrease as expected.

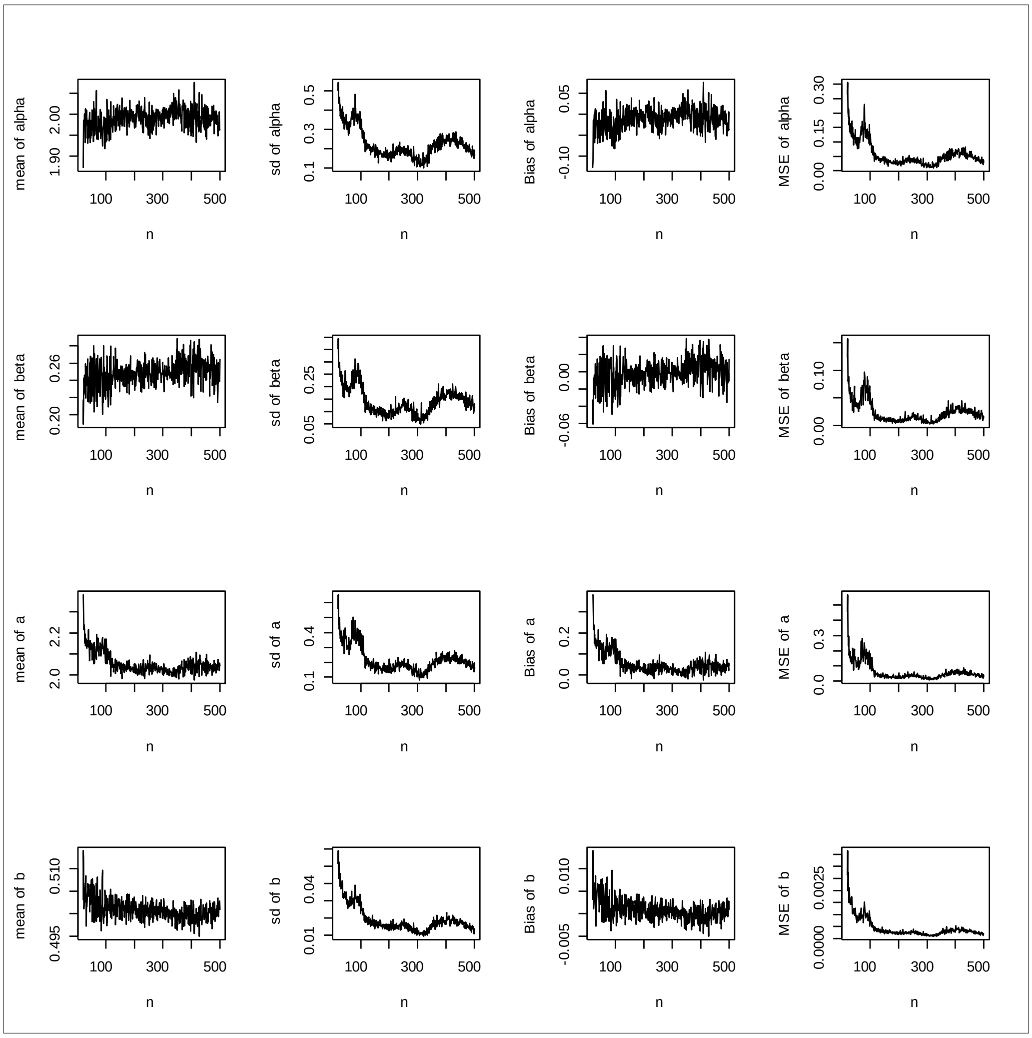

Figure 3. Simulation results for the AOLLL-W distribution.

7 An Application

In this section, we illustrate the flexibility of the AOLLL-W distribution on the real data set. We also compare this model with the beta Weibull (BW) model by Famoye et al. (2005), Kumaraswamy Weibull (KwW) model by Cordeiro, et al. (2010), odd log-logistic Weibull (OLL-W) model by Cruz et al. (2017) and generalized odd log-logistic Weibull (GOLL-W) model by Cordeiro et al. (2017). The cdfs of these models are given by (for $x>0,\ \alpha,\ \beta,\ a,\ b>0$):

$$F_{BW}(x)=B^{-1}(\alpha,\ \beta)\ \int_0^{1-\exp\left(-(bx)^a\right)}\omega^{\alpha-1}(1-\omega)^{\beta-1}\,d\omega \quad ,$$

$$F_{KwW}(x)=1-\left[1-\left(1-e^{-(bx)^a}\right)^{\alpha}\right]^{\beta}$$

$$F_{GOLL-W}(x)=\left(1-e^{-(bx)^{a}}\right)^{\alpha\beta}\left[\left(1-e^{-(bx)^{a}}\right)^{\alpha}+\left(1-\left(1-e^{-(bx)^{a}}\right)^{\beta}\right)^{\alpha}\right]^{-1}$$

and

$$F_{OLL-W}(x)=\left(1-e^{-(bx)^{a}}\right)^{\alpha}\left[\left(1-e^{-(bx)^{a}}\right)^{\alpha}+e^{-\alpha(bx)^{a}}\right]^{-1},$$

where $\mathrm{B}(\alpha,\ \beta)$ is the complete beta function. To compare AOLLL-W model with above models process has been done under the estimated log-likelihood values $\hat{\ell}$, Akaike Information Criteria (AIC), Cramer von Mises (W^{*}) and Anderson-Darling (A^{*}) goodness of-fit statistics for all distribution models. We note that The AIC is by given by $AIC=-2\widehat{\ell}+2p$ where p is the number of the estimated model parameters and n is sample size. The W^{*} and A^{*} statistics have been described as

$$W^{*}=\sum_{i=1}^{n}\left\{\hat{F}\left(x_{(i)}\right)-[i-0.5]\ \ n^{-1}\right\}^{2}+(12n)^{-1}$$

and

$$A^{*}=-\sum_{i=1}^{n}\left[(2i-1)\ \ n^{-1}\right]\ \left[\ln\hat{F}\left(x_{(i)}\right)+\ln\hat{\bar{F}}\left(x_{(n+1-i)}\right)\right]-n.$$

It can be seen for W^{*} and A^{*} statistics Chen and Balakrishnan (1995) and Evans et al. (2008). In general, it can be chosen as the best model which has the smaller the values of the AIC, W^{*} and A^{*} statistics and the larger the values of $\hat{\ell}$. All computations are performed by the maxLike routine in the R programme. The real data set is the stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all had failed. This data set was studied by Andrews and Herzberg (1985) and Cooray and Ananda (2008). The data are: 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.1, 0.1, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.2, 0.23, 0.24, 0.24, 0.29 , 0.34, 0.35, 0.36, 0.38, 0.4, 0.42, 0.43, 0.52, 0.54, 0.56, 0.6, 0.6, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.8, 0.8, 0.83, 0.85, 0.9, 0.92, 0.95, 0.99, 1, 1.01, 1.02, 1.03, 1.05, 1.1, 1.1, 1.11, 1.15, 1.18, 1.2, 1.29, 1.31, 1.33, 1.34, 1.4, 1.43, 1.45, 1.5, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.8, 1.8, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.2, 4.69, 7.89. In the applications, the information about the hazard shape can help in selecting a model. For this aim, a device called the total time on test (TTT) plot (Aarset, 1987) is useful. The TTT plot is obtained by plotting $T\left(\frac{r}{n}\right)$ against r/n where $T\left(\frac{r}{n}\right)=\left[\sum_{i=1}^{n}y_{(i)}+(n-r)\ \ y_{(r)}\right]/\sum_{i=1}^{n}y_{(i)}$, $r=1,\ldots,\ n$ and y_{i} are the order statistics of the sample. It is convex shape for decreasing hrf and is concave shape for increasing hrf. The TTT plot for the kevlar data in Figure 4 deals with convex-concave-convex shaped.

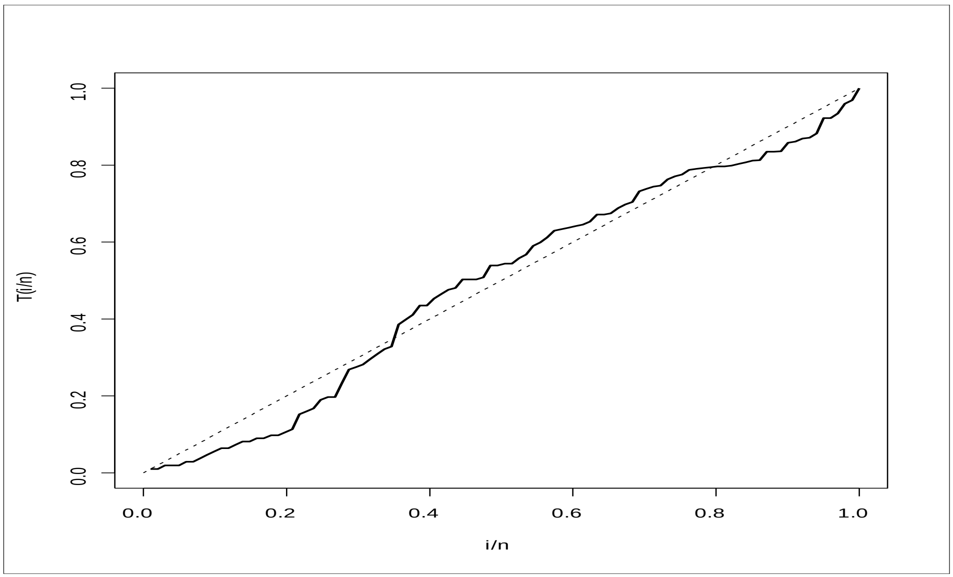

Figure 4. TTT plot for the Kevlar data

The MLEs of all models parameters, $\hat{\ell}$, their standard errors, AIC, W^* and A^* statistics are given in Table mle. As it can be seen from Table MLE, the AOLLL-W model could be chosen as the best model among the fitted models under the comparing statistics.

Table 2. MLEs, standard erros of the estimates (in parentheses), $\hat{\ell}$, *AIC*, W^* and A^* statistics.

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{a}$	$\widehat{b}$	$-\hat{\ell}$	AIC	A^*	W^*
AOLLL-W	1.0520	-18.1865	0.7106	2.8107	101.7842	211.5684	0.7087	0.1187
	(0.2562)	(6.8255)	(0.1097)	(0.6321)				
GOLL-W	1.1651	0.6119	1.1100	0.6195	102.7667	213.5335	0.9353	0.1563
	(0.9266)	(0.8888)	(0.4325)	(0.9775)				
KwW	0.7358	0.2288	1.0252	3.7984	102.6115	213.2231	0.8577	0.1388
	(0.1818)	(0.0274)	(0.0214)	(0.0196)				
BW	0.7119	0.2188	1.0753	3.5333	102.2966	212.5933	0.7924	0.1280
	(0.1183)	(0.0244)	(0.0332)	(0.3835)				
OLL-W	0.8893		1.0396	1.0194	102.8435	211.6869	1.0140	0.1813
	(0.1946)		(0.1285)	(0.1773)				

The plots of the fitted pdfs, cdfs and hrfs of all models are displayed in Figures 5-7. These plots also show that the AOLLL-W model has the best fitting to these data compared to the other models. The fitted hrf of the AOLLL-W model provides better fitting than other models.

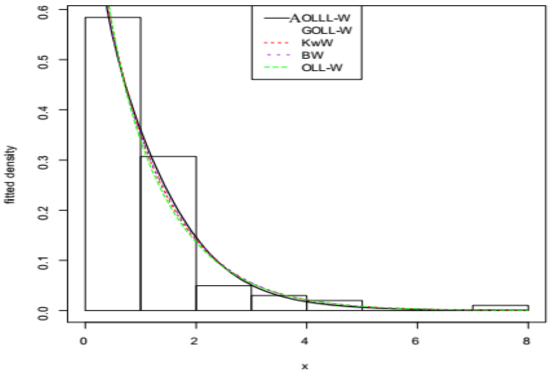

Figure 5. Fitted pdfs for the data set.

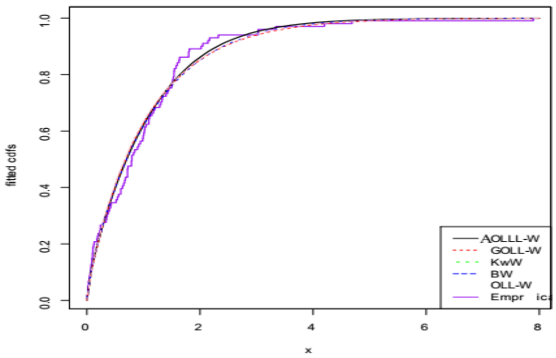

Figure 6. Fitted cdfs for the data set.

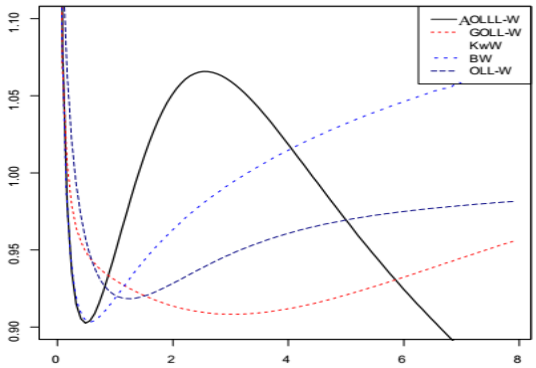

Figure 7. Fitted hrfs for the data set.

8 Conclusions

In this work, we present a new class of distributions called another odd log-logistic logarithmic-G (AOLLL-G) family of distributions. The mathematical properties of this new family are provided. The model parameters are estimated by the maximum likelihood estimation method and the observed information matrix is determined. Simulation results to assess the performance of the maximum likelihood estimators are discussed. It is shown that a special case of the new class can provide a better fit than other models generated by well-known families.

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Appendix: Useful power series

It can be seen Gradshteyn and Ryzhik (2000) for following equations.

By expanding z^{λ} in Taylor series, we can write

$$z^{\lambda}=\sum_{k=0}^{\infty}(\lambda)_k\,(z-1)^k\,/\,k!=\sum_{i=0}^{\infty}f_i\quad z^i, \tag{A1}$$

where

$$f_i=f_i(\lambda)=\sum_{k=i}^{\infty}(\lambda)_k\,(-1)^{k-i}\binom{k}{i}/\,k!$$

and $(\lambda)_k=\lambda(\lambda-1)\ldots(\lambda-k+1)$ is the descending factorial. Further, we obtain an expansion for $[G(x)^a+\overline{G}(x)^a]^c$. We can write

$$[G(x)^a+\overline{G}(x)^a]=\sum_{j=0}^{\infty}t_j\quad G(x)^j,$$

where

$$t_j=t_j(a)=a_j(a)+(-1)^j\binom{a}{j}.$$

Then, using (A1), we have

$$[G(x)^a+\overline{G}(x)^a]^c=\sum_{i=0}^{\infty}f_i\left(\sum_{j=0}^{\infty}t_j\quad G(x)^j\right)^i,$$

where $f_i=f_i(c)$. Finally, we obtain

$$[G(x)^a+\overline{G}(x)^a]^c=\sum_{j=0}^{\infty}h_j(a,\ c)\quad G(x)^j,$$

where $h_j(a,\ c)=\sum_{i=0}^{\infty}\quad f_i\quad m_{i,\ j}$ and for $i\geq 0$

$$m_{i,\ j}=(j\quad t_0)^{-1}\sum_{m=1}^{j}[m(j+1)-j]\ \ t_m\quad m_{i,\ j-m}\ \text{(for } j\geq 1\text{) and } m_{i,\ 0}=t_0^i.$$