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Darboux Vector and Stress Analysis of Equi-Affine Frame

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Abstract. A set of points that corresponds a vector of vector space constructed on a field is called an affine space associate with that vector space. We denote as affine 3-space A_3 associated with IR^3.
The first written sources that can be achieved about affine space curve theory are based on the 1890's when Ernesto Cesàro and Die Schon von Pirondini lived period. From that years to 2000's there are a some affine frames used in curve theory. One of them is equi-affine frame.
The grup of affine motions special linear transformation consist of volume preserving linear transformations denoted by and comprising diffeomorphisms of that preserve some important invariants such curvaures that in curve theory as well.
In this study, we separated the matrix representing affine frame as symmetric and antismmetric parts by using matrix demonstration of the equi-affine frame of a curve given in affine 3-space. By making use of antisymmetric part, we obtained the angular velocity vector which is also known as Darboux vector and then we expressed it in the form of linear sum of affine Frenet vectors.
On the other hand, by making use of symmetric part, we obtained the normal stresses and shear stress components of the stress on the frame of the curve in terms of the affine curvature and affine torsion. Thus we had the opportunity to be able to explane the distinctive geometric features of the affine curvature and affin torsion.
Lastly, we made stress analysis of a curve with constant affine curvature and affine torsion in affine 3-space as an example.

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1. Introduction

The grup of affine motions special linear transformation namely the group of equi-affine or unimodular transformations consists of volume preserving ($\det(a_{jk}) = 1$) linear transformations together with translation such that

$$x_j^* = \sum_{k=1}^{3} a_{jk}x_k + c_j \qquad j = 1, 2, 3$$

This transformations group denoted by $ASL(3, IR) := SL(3, IR) \times IR^3$ and comprising diffeomorphisms of IR^3 that preserve some important invariants such curvaures that in curve theory as well. An equi-affine group is also called a Euclidean group [3, 8–10].

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Let

$$\alpha : J \longrightarrow A_3$$

be a curve in A_3, where $J = (t_1, t_2) \subset IR$ is fixed open interval. Regularity of a curve in A_3 is defined as$\begin{vmatrix} \dot{\alpha} & \ddot{\alpha} & \dddot{\alpha} \end{vmatrix} \neq 0$ on J, where $\dot{\alpha} = d\alpha/dt$, etc. Then, we may associate the invariant parameter

$$\sigma(u) = \int_{t_1}^{t} \begin{vmatrix} \dot{\alpha} & \ddot{\alpha} & \dddot{\alpha} \end{vmatrix}^{1/6} dt \quad (\cdot = d/dt)$$

which is called the *affine arc length* of the curve. The coordinates of a curve are given by three linearly independent solutions of the equations

$$\alpha^{iv}(s) + \nu(s)\,\alpha''(s) + \omega(s)\,\alpha'(s) = 0$$

under the condition

$$\left|\alpha'(s), \alpha''(s), \alpha'''(s)\right| = 1$$

where $\nu(s)$ and $\omega(s)$ denote the affine curvatures. Some remarkable geometrical definitions of $\alpha''(s)$ and $\nu(s)$ were discovered by R. Weizenböck and G. Sannia and L. Berwald. E. Salkowski, W.Shells and A. Winternitz studied and gave some special futures both planar and space curves by using equi-affine frame. According to the analogy of Euclidean space theory, it would prefer to take the vectors $\{\alpha'(s), \alpha''(s), \alpha'''(s)\}$ for the moving triad of a curve $\alpha(s)$ in affine space, and then $\nu(s)$ and $\omega(s)$ are also called equi-affine curvature and equi-affine torsion [1,5,7,8,10].

We can decompose any square matrix Q uniquely as

$$Q = \frac{Q + Q^t}{2} + \frac{Q - Q^t}{2}$$

with symmetric part $\frac{Q+Q^t}{2}$ and with antisymmetric part $\frac{Q-Q^t}{2}$. If Q is anti-symmetric then symmetric part is zero matrix. According to Cayley's transformation

$$R = \left[I + \frac{Q - Q^t}{2}\right]^{-1} \left[I - \frac{Q - Q^t}{2}\right]$$

is a orthogonal (rotation) matrix which is $|R| = +1$ [2]. The stress tensor is a square symmetric matrix. In 2-dimensional space, stress matrix is given

$$\Psi = \begin{bmatrix} \sigma_X & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y \end{bmatrix}$$

according to $\{X, Y\}$ coordinate system. The matrix Ψ consist of the three stress components σ_X, σ_Y and σ_{XY} which means stresses on X, on Y directions, and stresses on $\{X, Y\}$ planes respectively. Similarly, the matrix $\widetilde{\Psi}$

$$\widetilde{\Psi} = \begin{bmatrix} \sigma_X & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{XY} & \sigma_Y & \sigma_{YZ} \\ \sigma_{XZ} & \sigma_{YZ} & \sigma_Z \end{bmatrix}$$

consist of the six stress components $\sigma_X, \sigma_Y, \sigma_Z, \sigma_{XY}, \sigma_{XZ}$ and σ_{YZ} which means stresses parallel to X, parallel to Y, parallel to Z directions, and shear stress in Y direction on YZ plane, shear stress in Z direction on YZ plane, shear stress in Z direction on XZ plane respectively [4,6].

2. Equi-Affine Frame and Darboux Vector

A C^∞ map α from an interval I to IR^2 is called an equiaffine plane curve in IR^2 if $\begin{vmatrix} \alpha'(s) & \alpha''(s) \end{vmatrix} \neq 0$, and α is said to be parameterized by equiaffine arclength

parameter if $\begin{vmatrix} \alpha'(s) & \alpha''(s) \end{vmatrix} = 1$ for all $s \in I$. For an equiaffine plane curve parameterized by equiaffine arc-length parameter, the invariant equiaffine curvature defined by

$$k(s) = \begin{vmatrix} \alpha''(s) & \alpha'''(s) \end{vmatrix}$$

so for $T(s) := \alpha'(s)$ and $N(s) := \alpha''(s)$ which are called tangent and affine normal vectors, we have

$$\begin{bmatrix} T'(s) \\ N'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \end{bmatrix}. \tag{2.1}$$

K. Nomizu, and T. Sasaki [7], obtained some plane curves with constant curvature and gave the following important theorem.

Theorem 2.1. *An equiaffine plane curve $\alpha(s)$ with constant $k(s) := k$ is equiaffinely equivalent to one of the following curve*

i. *If $k = 0$,*

$$\alpha(s) = (s, \frac{1}{2}s^2)$$

that is a parabola $y = \frac{1}{2}x^2$,

ii. *If $k > 0$,*

$$\alpha(s) = (k^{-1/2}\sin(k^{-1/2}s), -k^{-1}\cos(k^{-1/2}s))$$

that is an ellipse $kx^2 + ky^2 = 1$

iii. *If $k < 0$,*

$$\alpha(s) = ((-k)^{-1/2}\sinh((-k)^{-1/2}s), (-k)^{-1}\cosh((-k)^{-1/2}s))$$

that is a hyperbola $kx^2 - ky^2 = 1$.

From (2.1), let

$$q = \begin{bmatrix} 0 & 1 \\ -k(s) & 0 \end{bmatrix},$$

then we can decompose q uniquely as $q = p + w$ with symmetric part

$$p = \frac{1}{2}\begin{bmatrix} 0 & 1-k(s) \\ 1-k(s) & 0 \end{bmatrix}$$

which means stress part and with antisymmetric part

$$w = \frac{1}{2}\begin{bmatrix} 0 & 1+k(s) \\ -(1+k(s)) & 0 \end{bmatrix}.$$

Also, by using Cayley's transformation we can find the rotation matrix r of equi-affine frenet motion such as

$$r = \frac{1}{(k+1)^2+4}\begin{bmatrix} -(k+1)^2+4 & -4(k+1) \\ 4(k+1) & -(k+1)^2+4 \end{bmatrix}$$

which is $|r| = +1$.

The matrix p consist of the three stress components $\sigma_T = \sigma_N = 0$ and $\sigma_{TN} = (1-k(s))/2$ which means stresses on tangent, on normal directions, and stresses on $\{T, N\}$ planes respectively. The three principal stresses are the eigen values of p which are the roots of $|\sigma I_2 - p| = 0$,

$$\sigma_{1,2} = \pm\frac{|1-k(s)|}{2}$$

and corresponding eigen vectors

$$v_{1,2} = (\mp 1, 1)$$

according to $\{T, N\}$. Thus we can give the following remark.

Remark 2.2. i. Throughout the planar equi-affine frame motion there are the sresses zero on tangent and on affine normal directions, also there are the shear stress $\frac{1-k(s)}{2}$ on $sp\{T, N\}$. Furthermore, two principal stresses $\sigma_{1,2} = \pm\frac{|1-k(s)|}{2}$ acts on principal axises $v_{1,2} = \mp T(s) + N(s)$, respectively.

ii. The only equi-affine plane curve $\alpha(s)$ whose shear strees is zero, is a hyperbola $-x^2 + y^2 = 1$.

Let $\alpha(s)$ be regular curve with affine arclenght parameter s. The vectors $\alpha'(s), \alpha''(s)$ and $\alpha'''(s)$ are called tangent, affine normal and binormal vectors respectively, and the planes $sp\{\alpha'(s), \alpha''(s)\}$, $sp\{\alpha'(s), \alpha'''(s)\}$ and $sp\{\alpha''(s), \alpha'''(s)\}$ are called osculating, rectifying and normal planes of the curve $\alpha(s)$. Thus the equi-affine frame given in the matrix form as

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_2(s) & -k_1(s) & 0 \end{bmatrix}\begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \tag{2.2}$$

is called equi-affine frame, where $k_1(s)$ and $k_2(s)$ are called affine curvature and affine torsion which are given as follows

$$\begin{aligned} k_1(s) &= \left| \begin{array}{ccc} \alpha'(s) & \alpha'''(s) & \alpha^{(iv)}(s) \end{array} \right| \\ k_2(s) &= -\left| \begin{array}{ccc} \alpha''(s) & \alpha'''(s) & \alpha^{(iv)}(s) \end{array} \right|. \end{aligned}$$

N. Hu obtained the some curves with constant curvature and gave the following important theorem by using Shengjin's formulae [5].

Theorem 2.3. *Any nondegenerate equiaffine space curve $\alpha(s)$ with constant equiaffine curvature $k_1(s) := k_1$, $k_2(s) := k_2$ is equiaffinely equivalent to one of the following curves:*

i. $\alpha(s) = (s, \frac{1}{2}s^2, \frac{1}{2}s^3)$

ii. $\alpha(s) = (e^{\rho s}, se^{\rho s}, -\frac{1}{18\sigma^5}e^{-2\rho s})$

iii. *If* $k_1 = 0$,

$$\alpha(s) = k_2(-k_2^{1/2}s, \sin(k_2^{1/2}s), \cos(k_2^{1/2}s))$$

if $k_1 \neq 0$,

$$\alpha(s) = \left(\frac{1}{2\rho_1\rho_2(9\rho_1^2+\rho_2^2)(\rho_1^2+\rho_2^2)}e^{-2\rho_1 s}, e^{\rho_1 s}\sin(\rho_2 s), e^{\rho_1 s}\cos(\rho_2 s)\right),$$

where $\rho = \left\{\frac{k_1}{2}\right\}^{1/3}$ *and*

$$\rho_1 = \frac{1}{6}\left\{\sqrt[3]{\frac{3(9k_1+\sqrt{12k_2^3+81k_1^2})}{2}} + \sqrt[3]{\frac{3(9k_1-\sqrt{12k_2^3+81k_1^2})}{2}}\right\},$$

$$\rho_2 = \frac{\sqrt{3}}{6}\left\{\sqrt[3]{\frac{3(9k_1+\sqrt{12k_2^3+81k_1^2})}{2}} - \sqrt[3]{\frac{3(9k_1-\sqrt{12k_2^3+81k_1^2})}{2}}\right\},$$

iv. *If* $k_1 = 0$,

$$\alpha(s) = -k_2^{1/2}(-k_2^{1/2}s, \sinh((-k_2)^{1/2}s), \cosh((-k_2)^{1/2}s)),$$

if $k_1 \neq 0$,

$$\alpha(s) = \left(\frac{1}{4\rho_3\rho_4(9\rho_3^2-\rho_4^2)(\rho_3^2-\rho_4^2)}e^{-2\rho_3 s}, e^{(\rho_3+\rho_4)s}, e^{(\rho_3-\rho_4)s}\right)$$

for which $k_2 < 0$, $\frac{27}{2}k_1(-3k_2)^{-3/2} \in (-1, 1)$,
where

$$\begin{aligned} \rho_3 &= \frac{1}{3}\sqrt{-3k_2}\cos\left(\frac{1}{3}\arccos\left(\frac{27}{2}k_1(-3k_2)^{-3/2}\right)\right), \\ \rho_4 &= \sqrt{-k_2}\sin\left(\frac{1}{3}\arccos\left(\frac{27}{2}k_1(-3k_2)^{-3/2}\right)\right), \end{aligned}$$

From (2.2), let

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_2(s) & -k_1(s) & 0 \end{bmatrix}$$

then we can decompose Q uniquely as $Q = P + W$ with symmetric part

$$P = \begin{bmatrix} 0 & 1/2 & -k_2(s)/2 \\ 1/2 & 0 & (1-k_1(s))/2 \\ -k_2(s)/2 & (1-k_1(s))/2 & 0 \end{bmatrix}$$

and with antisymmetric part

$$W = \begin{bmatrix} 0 & 1/2 & k_2(s)/2 \\ -1/2 & 0 & (1+k_1(s))/2 \\ -k_2(s)/2 & -(1+k_1(s))/2 & 0 \end{bmatrix}.$$

Also, by using Cayley's transformation we can find the rotation matrix R of equi-affine frame motion such as

$$R = \frac{1}{k_1^2+k_2^2+2k_1+6}\begin{bmatrix} k_1^2-k_2^2+2k_1+4 & -2(k_1k_2+k_2+2) & 2(1+k_1-2k_2) \\ -2(k_1k_2+k_2-2) & -\left(k_1^2-k_2^2+2k_1-2\right) & -2(2+2k_1+k_2) \\ 2(1+k_1+2k_2) & 2(2+2k_1-k_2) & -\left(k_1^2+k_2^2+2k_1-4\right) \end{bmatrix}$$

which is $|R| = +1$. Thus we have the following theorem.

Theorem 2.4. *For any nondegenerate equiaffine space curve $\alpha(s)$, instantaneous rotatinon matrix is*

$$R = \frac{1}{k_1^2+k_2^2+2k_1+6}\begin{bmatrix} k_1^2-k_2^2+2k_1+4 & -2(k_1k_2+k_2+2) & 2(1+k_1-2k_2) \\ -2(k_1k_2+k_2-2) & -\left(k_1^2-k_2^2+2k_1-2\right) & -2(2+2k_1+k_2) \\ 2(1+k_1+2k_2) & 2(2+2k_1-k_2) & -\left(k_1^2+k_2^2+2k_1-4\right) \end{bmatrix}$$

and instantaneous rotation vector is

$$D = \frac{1+k_1(s)}{2}T(s) - \frac{k_2(s)}{2}N(s) + \frac{1}{2}B(s).$$

The matrix P consist of the six stress components $\sigma_T = \sigma_N = \sigma_B = 0$, $\sigma_{TN} = \frac{1}{2}, \sigma_{TB} = \frac{-k_2(s)}{2}$ and $\sigma_{NB} = \frac{1-k_1(s)}{2}$ which means stresses parallel to tangent, parallel to normal, parallel to binormal directions, and shear stress in $N-$direction on osculating plane, shear stress in $B-$direction on rectifying plane, shear stress in $B-$direction on normal plane, respectively. The three prinicipal stresses are the eigen values of P which are the roots of $|\sigma I_3 - P| = 0$,

$$\sigma^3 + \psi\sigma - \varphi = 0$$

where $\psi = -\frac{1}{4}\left\{k_2(s)^2 + (1-k_1(s))^2 + 1\right\}, \varphi = \frac{1}{4}k_2(s)(1-k_1(s))$. Let the roots be σ_1, σ_2 and σ_3 then

$$\begin{aligned} \sigma_1 &= \frac{\Gamma^2 - 12\psi}{6\Gamma} \\ \sigma_2 &= \frac{-\Gamma^2 + 12\psi + I\sqrt{3}\left(\Gamma^2 + 12\psi\right)}{12\Gamma} \\ \sigma_3 &= \frac{\Gamma^2 - 12\psi - I\sqrt{3}\left(\Gamma^2 - 12\psi\right)}{12\Gamma} \end{aligned} \tag{2.3}$$

where $\Gamma = \left\{108\varphi + 12\sqrt{12\psi^3 + 81\varphi^2}\right\}^{1/3}$.Thus, we can give the following theorem.

Theorem 2.5. *Throughout the equi-affine motion there are no stresses on tangent, on affine normal and on affine binormal directions, also there are three principal stresses σ_i given in (2.3) acts on corresponding principal axises. Additionally, throughout the motion there are shear stresses $\sigma_{TN} = \frac{1}{2}, \sigma_{NB} = \frac{-k_2(s)}{2}$ and $\sigma_{TB} = \frac{1-k_1(s)}{2}$.*

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