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Darboux Vector and Stress Analysis of Centro-Affine Frame

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Abstract. A set of points that corresponds a vector of vector space constructed on a field is called an affine space associated with that vector space. We denote A_3 as affine 3-space associated with IR^3.
The first written sources that can be achieved about affine space curve theory are based on the 1890's when Ernesto Cesàro and Die Schon von Pirondini lived period. From that years to 2000's there are a some affine frames used in curve theory. One of them is centro-affine frame.
The grup of affine motions special linear transformation consist of volume preserving linear transformations denoted by and comprising diffeomorphisms of that preserve some important invariants such curvaures that in curve theory as well.
In this study, we separated the matrix representing affine frame as symmetric and antismmetric parts by using matrix demonstration of the centro-affine frame of a curve given in affine 3-space. By making use of antisymmetric part, we obtained the angular velocity vector which is also known as Darboux vector and then we expressed it in the form of linear sum of affine Frenet vectors.
On the other hand, by making use of symmetric part, we obtained the normal stresses and shear stress components of the stress on the frame of the curve in terms of the affine curvature and affine torsion. Thus we had the opportunity to be able to explane the distinctive geometric features of the affine curvature and affine torsion.
Lastly, we made stress analysis of a curve with constant affine curvature and affine torsion in affine 3-space as an example.

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1. Introduction

The grup of affine motions special linear transformation namely the group of equi-affine or unimodular transformations consists of volume preserving ($\det(a_{jk}) = 1$) linear transformations together with translation such that

$$x_j^* = \sum_{k=1}^{3} a_{jk} x_k + c_j \qquad j = 1, 2, 3$$

This transformations group denoted by $ASL(3, IR) := SL(3, IR) \times IR^3$ and comprising diffeomorphisms of IR^3 that preserve some important invariants such curvaures that in curve theory as well. An equi-affine group is also called a Euclidean group. If $c_j = 0$ then one obtains a centro-affine unimodular group of transformations.

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Let

$$\alpha : J \longrightarrow A_3$$

be a curve in A_3, where $J = (t_1, t_2) \subset IR$ is fixed open interval. Regularity of a curve in A_3 is defined as $\left| \begin{array}{ccc} \dot{\alpha} & \ddot{\alpha} & \dddot{\alpha} \end{array} \right| \neq 0$ on J, where $\dot{\alpha} = d\alpha/dt$, etc. Then, we may associate the invariant parameter

$$\sigma(t) = \int_{t_1}^{t} \left| \begin{array}{ccc} \dot{\alpha} & \ddot{\alpha} & \dddot{\alpha} \end{array} \right|^{1/6} dt \quad (\cdot = d/dt)$$

which is called the *affine arc length* of C and yields a representation $\alpha(t)$ of the curve. The fundamental theorem of curves in centroaffine geometry is obtained by Gardner and Wilkens. In that paper they used the classical Cartan approach to moving frames in order to find the formulation of the local rigidity theorem for curves that is amenable to direct application to problems in control theory [2,4,6–8]. Na Hu classified the centroaffine space curves with constant centroaffine curvatures, which are centroaffine homogeneous curves in IR^3.

We can decompose any square matrix Q uniquely as

$$Q = \frac{Q + Q^t}{2} + \frac{Q - Q^t}{2} \tag{1.1}$$

with symmetric part $\frac{Q+Q^t}{2}$ and with antisymmetric part $\frac{Q-Q^t}{2}$. If Q is anti-symmetric then symmetric part is zero matrix. According to Cayley's transformation

$$R = \left[I + \frac{Q - Q^t}{2} \right]^{-1} \left[I - \frac{Q - Q^t}{2} \right] \tag{1.2}$$

is a orthogonal (rotation) matrix which is $|R| = +1$ [1]. The stress tensor is a square symmetric matrix. In 2-dimensional space, stress matrix is given

$$\Psi = \begin{bmatrix} \sigma_X & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y \end{bmatrix}$$

according to $\{X, Y\}$ coordinate system. The matrix Ψ consist of the three stress components σ_X, σ_Y and σ_{XY} which means stresses on X, on Y directions, and stresses on $\{X, Y\}$ planes respectively. Similarly, the matrix $\widetilde{\Psi}$

$$\widetilde{\Psi} = \begin{bmatrix} \sigma_X & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{XY} & \sigma_Y & \sigma_{YZ} \\ \sigma_{XZ} & \sigma_{YZ} & \sigma_Z \end{bmatrix}$$

consist of the six stress components $\sigma_X, \sigma_Y, \sigma_Z, \sigma_{XY}, \sigma_{XZ}$ and σ_{YZ} which means stresses parallel to X, parallel to Y, parallel to Z directions, and shear stress in Y direction on YZ plane, shear stress in Z direction on YZ plane, shear stress in Z direction on XZ plane normal respectively [3].

2. Centro-Affine Frame and Darboux Vector

A C^∞ map α from an interval I to IR^2 is called a nondegenerate centro-affine plane curve in IR^2 if $\left| \begin{array}{cc} \dot{\alpha}(s) & \ddot{\alpha}(s) \end{array} \right| \neq 0$, and α is said to be parameterized by centro-affine arclength

parameter if

$$\varepsilon(s) = \frac{\left| \begin{array}{cc} \dot{\alpha}(s) & \ddot{\alpha}(s) \end{array} \right|}{\left| \begin{array}{cc} \alpha(s) & \dot{\alpha}(s) \end{array} \right|} = \pm 1$$

for all $s \in I$. For a nondegenerate centro-affine plane curve parameterized by centro-affine arc-length parameter, the invariant centro-affine curvature defined by

$$\kappa(s) = \frac{\left| \begin{array}{cc} \alpha(s) & \alpha''(s) \end{array} \right|}{\left| \begin{array}{cc} \alpha(s) & \alpha'(s) \end{array} \right|}$$

so we have

$$\begin{bmatrix} \alpha'(s) \\ \alpha''(s) \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon(s) \\ 1 & \varepsilon(s)\kappa(s) \end{bmatrix} \begin{bmatrix} \alpha(s) \\ \alpha'(s) \end{bmatrix}.$$

N. Hu [5], obtained the position vectors of the some nondegenerate plane curves with constant curvature and gave the following important theorem.

Theorem 2.1. *A nondegenerate centro-affine plane curve $\alpha(s)$ with constant $k(s) := k$ is centro-affinely equivalent to one of the following curve*

i. *If $k = 0$ and $\varepsilon(s) = -1$,*

$$\alpha(s) = (\cosh(s), \sinh(s)),$$

if $k = 0$ and $\varepsilon(s) = +1$,

$$\alpha(s) = (\cos(s), \sin(s)),$$

ii. *If $k \neq 0$ and $\varepsilon(s) = -1$,*

$$\alpha(s) = \frac{1}{\lambda + \lambda^{-1}}(\lambda e^{-\lambda^{-1}s} + \lambda^{-1}e^{\lambda s}, -e^{-\lambda^{-1}s} + e^{\lambda s})$$

where $\lambda = \frac{1}{2}(k + \sqrt{k^2 + 4})$, and if $|k| > 2$ and $\varepsilon(s) = 1$,

$$\alpha(s) = \frac{1}{\lambda - \lambda^{-1}}(\lambda e^{\lambda^{-1}s} - \lambda^{-1}e^{\lambda s}, -e^{\lambda^{-1}s} + e^{\lambda s})$$

where $\lambda = \frac{1}{2}(k + \sqrt{k^2 - 4})$,

iii. *If $k = 2$ and $\varepsilon(s) = 1$,*

$$\alpha(s) = (e^s - se^s, se^s)$$

if $k = -2$ and $\varepsilon(s) = 1$,

$$\alpha(s) = (e^{-s} + se^{-s}, se^{-s})$$

and if $|k| < 2$ and $\varepsilon(s) = 1$,

$$\alpha(s) = (e^{as}\cos(bs) - ab^{-1}\sin(bs), -b^{-1}e^{as}\sin(bs))$$

where $a = \frac{k}{2}$, $b = \frac{1}{2}\sqrt{4 - k^2}$.

Let

$$q = \begin{bmatrix} 0 & \varepsilon(s) \\ 1 & \varepsilon(s)k(s) \end{bmatrix}$$

then from (1.1), we can decompose q uniquely as $q = p + w$ with symmetric part and with antisymmetric part

$$p = \begin{bmatrix} 0 & \frac{\varepsilon(s)+1}{2} \\ \frac{\varepsilon(s)+1}{2} & \varepsilon(s)k(s) \end{bmatrix}, \; w = \begin{bmatrix} 0 & -\frac{(1-\varepsilon(s))}{2} \\ \frac{1-\varepsilon(s)}{2} & 0 \end{bmatrix}.$$

Also, from (1.2), we can find the rotation matrix r of centro-affine frame motion such as

$$r = \begin{bmatrix} \frac{\frac{\varepsilon(s)+1}{2}}{\frac{-2(\varepsilon(s)-1)}{\varepsilon(s)-3}} & \frac{\frac{2(\varepsilon(s)-1)}{\varepsilon(s)-3}}{\frac{\varepsilon(s)+1}{2}} \end{bmatrix}$$

which is $|r| = +1$. The matrix p consist of the three stress components $\sigma_\alpha = 0$, $\sigma_T = \varepsilon(s)k(s)$ and $\sigma_{\alpha T} = \frac{\varepsilon(s)+1}{2}$ which means stresses on pozition vector of α, on tangent directions, and stresses on $\{\alpha, T\}$ plane respectively. The three principal stresses are the eigen values of p which are the roots of $|\sigma I_2 - p| = 0$,

$$4\sigma^2 - 4\varepsilon(s)k(s)\sigma - (\varepsilon(s) + 1)^2 = 0$$

$$\sigma_{1,2} = \frac{\varepsilon(s)k(s) \mp \sqrt{k^2 + (\varepsilon(s) + 1)^2}}{2}$$

and the corresponding vectors are

$$v_1 = \frac{\varepsilon(s) + 1}{\varepsilon(s)k(s) - \sqrt{k^2 + (\varepsilon(s) + 1)^2}}\alpha(s) + T(s)$$

$$v_2 = \frac{\varepsilon(s) + 1}{\varepsilon(s)k(s) + \sqrt{k^2 + (\varepsilon(s) + 1)^2}}\alpha(s) + T(s).$$

Thus we can give the following remark.

Remark 2.2. Throughout the planar centro-affine frame motion there are the sresses zero on position vector and $\varepsilon(s)k(s)$ tangent directions and also there are the shear stress $\frac{\varepsilon(s)+1}{2}$ on $sp\{\alpha, T\}$. Furthermore, two principal stresses $\sigma_{1,2} = \frac{\varepsilon(s)k(s)\mp\sqrt{k^2+(\varepsilon(s)+1)^2}}{2}$ acts on principal axises $v_{1,2} = \frac{\varepsilon(s)+1}{\varepsilon(s)k(s)\mp\sqrt{k^2+(\varepsilon(s)+1)^2}}\alpha(s) + T(s)$, respectively.

Let $\alpha(s)$ be regular curve with affine arclenght parameter s. The vectors $\alpha'(s)$ and $\alpha''(s)$ are called tangent and affine normal vectors respectively. Thus

$$\begin{bmatrix} \alpha'(s) \\ \alpha''(s) \\ \alpha'''(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon(s) & k_1(s) & k_2(s) \end{bmatrix} \begin{bmatrix} \alpha(s) \\ \alpha'(s) \\ \alpha''(s) \end{bmatrix}. \tag{2.2}$$

is called centro-affine frame, where $k_1(s)$ and $k_2(s)$ are called centro-affine first and second curvatures which are given as follows

$$k_1(s) := -\frac{\left| \alpha(s) \quad \alpha''(s) \quad \alpha'''(s) \right|}{\left| \alpha(s) \quad \alpha'(s) \quad \alpha''(s) \right|}$$

$$k_2(s) := \frac{\left| \alpha(s) \quad \alpha'(s) \quad \alpha'''(s) \right|}{\left| \alpha(s) \quad \alpha'(s) \quad \alpha''(s) \right|}$$

and $\varepsilon(s) := \frac{\left| \dot\alpha \quad \ddot\alpha \quad \dddot\alpha \right|}{\left| \alpha \quad \dot\alpha \quad \ddot\alpha \right|} = \pm 1$. N. Hu obtained the position vectors of the some curves with constant curvature and gave the following important theorem by using Shengjin's formulae [5].

Theorem 2.3. *Any nondegenerate centro-affine space curve $\alpha(s)$ with constant centro-affine curvature $k_1(s) := k_1$, $k_2(s) := k_2$ is centro-affinely equivalent to one of the following curves:*

i. *If $k_1 = k_2 = 0$ and $\varepsilon(s) = -1$,*

$$\alpha(s) = (\sin(\sqrt{3}s/2)e^{s/2}, \cos(\sqrt{3}s/2)e^{s/2}, e^{-s})$$

ii. *If $k_1 = k_2 = -3$,*

$$\alpha(s) = (se^{-s}, e^{-s}, s^2e^{-s} + e^{-s})$$

iii. *If $A^2 + B^2 \neq 0$ and $\Delta = 0$,*

$$\alpha(s) = (e^{\sigma_1 s}, e^{\sigma_2 s}, se^{\sigma_1 s})$$

where $\sigma_1 = -\frac{B}{2A}, \sigma_2 = k_2 + \frac{B}{A}$,

iv. *If $A^2 + B^2 \neq 0$ and $\Delta > 0$,*

$$\alpha(s) = e^{k_2 s/3}(e^{\rho_1 s}\sin(\rho_2 s), e^{\rho_1 s}\cos(\rho_2 s), e^{-2\rho_1 s})$$

where

$$\rho_1 = \frac{1}{6}\left\{\sqrt[3]{-Ak_2 + \frac{3}{2}(-B + \Delta^{1/2})} + \sqrt[3]{-Ak_2 + \frac{3}{2}(-B - \Delta^{1/2})}\right\}$$

$$\rho_2 = \frac{1}{6}\left\{\sqrt[3]{-Ak_2 + \frac{3}{2}(-B + \Delta^{1/2})} - \sqrt[3]{-Ak_2 + \frac{3}{2}(-B - \Delta^{1/2})}\right\}$$

v. *If $A^2 + B^2 \neq 0$, and $\Delta < 0$,*

$$\alpha(s) = e^{k_2 s/3}(e^{-2\sigma_1 s}, e^{(\sigma_1+\sigma_2)s}, e^{(\sigma_1-\sigma_2)s})$$

where

$$\sigma_1 = \frac{1}{3}A^{1/2}\cos\left(\arccos\left(\frac{-Ak_2 - 3B}{2A^{3/2}}\right)/3\right)$$

$$\sigma_2 = \frac{1}{3}(3A)^{1/2}\sin\left(\arccos\left(\frac{-Ak_2 - 3B}{2A^{3/2}}\right)/3\right)$$

for which $A > 0$ and $\frac{-Ak_2-3B}{2A^{3/2}} \in (-1, 1)$, From (2.2), let

$$Q=\begin{bmatrix}0&1&0\\0&0&1\\\varepsilon(s)&k_1&k_2\end{bmatrix}$$

then from (1.1), we can decompose Q uniquely as $Q=P+W$, the symmetric part and the antisymmetric part

$$P=\begin{bmatrix}0&1/2&\varepsilon(s)/2\\1/2&0&(1+k_1)/2\\\varepsilon(s)/2&(1+k_1)/2&k_2\end{bmatrix},\ W=\begin{matrix}0&1/2&-\varepsilon(s)/2\\-1/2&0&(1-k_1)/2\\\varepsilon(s)/2&(k_1-1)/2&0\end{matrix}.$$

Also, from (1.2) we can find the rotation matrix R of centro-affine frame motion such as

$$R=\frac{1}{(k_1)^2-2k_1+7}\begin{bmatrix}(k_1)^2-2k_1+4&-2(\varepsilon k_1-\varepsilon+2)&2(1-k_1+2\varepsilon)\\-2(\varepsilon k_1-\varepsilon-2)&-(k_1)^2+2k_1+3&2(-2+2k_1+\varepsilon)\\-2(-1+k_1+2\varepsilon)&2(2-2k_1+\varepsilon)&-(k_1)^2+2k_1+\varepsilon\end{bmatrix}$$

which is $|R|=+1$. Thus we have the following theorem.

Theorem 2.4. *For any nondegenerate space curve $\alpha(s)$, instantaneous rotatinon matrix is*

$$R=\frac{1}{(k_1)^2-2k_1+7}\begin{bmatrix}(k_1)^2-2k_1+4&-2(\varepsilon k_1-\varepsilon+2)&2(1-k_1+2\varepsilon)\\-2(\varepsilon k_1-\varepsilon-2)&-(k_1)^2+2k_1+3&2(-2+2k_1+\varepsilon)\\-2(-1+k_1+2\varepsilon)&2(2-2k_1+\varepsilon)&-(k_1)^2+2k_1+\varepsilon\end{bmatrix}$$

and instantaneous rotation vector is

$$D=\frac{1}{2}(k_1-1)\alpha-\frac{1}{2}\varepsilon T-\frac{1}{2}N.$$

On the other hand, the matrix P consist of the six stress components $\sigma_\alpha=\sigma_T=\sigma_N=k_2$, $\sigma_{\alpha T}=\frac{1}{2}$, $\sigma_{\alpha N}=\frac{\varepsilon(s)}{2}$ and $\sigma_{TN}=\frac{1+k_1}{2}$ which means stresses parallel to position vector, parallel to tangent, parallel to normal directions, and shear stresses on $\{\alpha,T\}$ plane, shear stress on $\{\alpha,N\}$ plane and shear stress on $\{T,N\}$ plane, respectively. The three prinicipal stresses are the eigen values of P which are the roots of $|\sigma I_3-P|=0$,

$$\sigma^3+\psi\sigma-\varphi=0$$

where $\psi=-\frac{1}{4}\left\{2+(1+k_1)^2+4k_2\right\}$, $\varphi=\frac{\varepsilon(s)(1+k_1)-k_2}{4}$. Let the roots be σ_1,σ_2 and σ_3 then

$$\begin{aligned}\sigma_1&=\frac{\Gamma^{2/3}-12\psi}{6\Gamma^{1/3}}\\\sigma_2&=\frac{-\Gamma^{2/3}+12\psi+I\sqrt{3}\left(\Gamma^{2/3}+12\psi\right)}{12\Gamma^{1/3}}\\\sigma_3&=\frac{\Gamma^{2/3}-12\psi-I\sqrt{3}\left(\Gamma^{2/3}-12\psi\right)}{12\Gamma^{1/3}}\end{aligned}\tag{2.3}$$

where $\Gamma=108\varphi+12\sqrt{12\psi^3+81\varphi^2}$. Thus, we can give the following theorem.

Theorem 2.5. *Throughout the centro-affine frenet motion there are no stresses on position vector, on tangent and there is the stress k_2 on normal directions. Also there are three principal stresses σ_i given in (2.3) acts on corresponding principal axises. Additionally, throughout the motion there are shear stresses $\sigma_{\alpha T}=\frac{1}{2}$, $\sigma_{\alpha N}=\frac{\varepsilon(s)}{2}$ and $\sigma_{TN}=\frac{1+k_1}{2}$.*

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