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Discreteness of Spectrum of Normal Differential Operators for First Order

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Abstract. In this work under the condition $A^{-1} \in C_\infty(H)$, we investigate the discreteness of spectrum of normal extensions in detail. Later on, the asymptotical behavior of eigenvalues of any normal extension has been examined.

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1. Introduction

It is well-known that in the operator theory, for any differential operator the important questions are:
1. which differential expression; it is generated by in the corresponding functional space?
2. which boundary conditions, it is generated by?
3. which special class does it belong to?
4. in which cases its spectrum is discrete ?
(see [13]).

Remember that a densely defined closed operator N in any Hilbert space is called formally normal if $D(N) \subset D(N^*)$ and $\|Nf\| = \|N^*f\|$ for all $f \in D(N)$, where N^* is the adjoint to the operator N. If a formally normal operator has no formally normal extension, then it is called maximal formally normal operator. If a formally normal operator N satisfies the condition $D(N) = D(N^*)$, then it is called a normal operator [1].

Generalization of J. von Neumann's theory to the theory of normal extensions of formally normal operators in Hilbert space has been obtained by E. A. Coddington in [1]. The first results in the area of normal extension of unbounded formally normal operators in a Hilbert space are also due to Y. Kilpi [10, 11] and R. H. Davis [2]. Some applications of this theory to two-point regular type first order differential operators in Hilbert space of vector functions can be found in [9] (also see references therein).

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2. Statement of the Problem

In this work we consider the differential-operator expression given by

$$l(u) = (\alpha u)'(t) + Au(t),$$

in the weighted Hilbert space $L^2_\alpha(H,(a,\infty))$, where H is a Hilbert space, $a \in \mathbb{R}$, $\alpha : (a,\infty) \to (0,\infty)$, $\alpha \in C(a,\infty)$, $\frac{1}{\alpha} \in L^1(a,\infty)$, $A : D(A) \subset H \to H$ is a selfadjoint operator, $A \geq E$ and $E : H \to H$ is an identity operator. Connected with this differential expression one can construct the minimal and maximal operator in $L^2_\alpha(H,(a,\infty))$ (see [7]). In this case, it can be shown that the minimal operator is formally normal, but it is not maximal. The all normal extensions of the minimal operator and their spectrum have been studied in [8].

In this work under the condition $A^{-1} \in C_\infty(H)$, we investigate the discreteness of spectrum of normal extensions in detail. Later on, the asymptotical behavior of eigenvalues of any normal extension has been examined.

3. Asymptotical behavior of $s-$ number of inverse for normal extensions

In this section we will investigate discreteness of the spectrum and asymptotical behavior of singular numbers of normal extensions of minimal operator L_0 in $L^2_\alpha(H,(a,\infty))$.

Throughout the paper, it will be defined that

$$\delta(s,t) = \int_s^t \frac{1}{\alpha(\tau)} d\tau \ \text{ and } \ \delta = \delta(a,\infty).$$

Before of all we can give the following simple fact.

Theorem 3.1. *If $dim H < \infty$, then each normal extension L_W has a pure point spectrum and $s-$numbers of extensions L_W^{-1} have the same asymptotics*

$$s_n\left(L_W^{-1}\right) \sim \frac{\delta}{2n\pi}, \ \text{as } n \to \infty.$$

Now let prove us the following result.

Theorem 3.2. *If $A^{-1} \in \mathfrak{S}_\infty(H)$ and the operator L_W is any normal extension of minimal operator L_0, then $L_W^{-1} \in \mathfrak{S}_\infty(L^2_\alpha(H,(a,\infty)))$.*

Proof. Let L_W be any normal extension of the operator L_0 in $L^2_\alpha(H,(a,\infty))$.

It can be verified that

$$\begin{aligned} L_W^{-1} f(t) &= \frac{1}{\alpha(t)} e^{-A\delta(a,t)} \left(E - W^* e^{-A\delta}\right)^{-1} W^* \int_a^\infty e^{-A\delta(s,\infty)} f(s) ds \\ &+ \frac{1}{\alpha(t)} \int_a^t e^{-A\delta(s,t)} f(s) ds, \ f \in L^2_\alpha(H,(a,\infty)). \end{aligned}$$

Now we prove that if $A^{-1} \in \mathfrak{S}_\infty(H)$, then

$$Kf(t) := \frac{1}{\alpha(t)} \int_a^t e^{-A\delta(s,t)} f(s) ds \in \mathfrak{S}_\infty(L^2_\alpha(H,(a,\infty))).$$

Since $\dfrac{1}{\alpha} \in L^1(a,\infty)$, then from the absolute continuity property of Lebesgue integral for arbitrary $\epsilon > 0$ there exists $\tau > 0$ such that for any Lebesgue measurable set with $\lambda(e) < \tau$ it is true $\int_e \frac{1}{\alpha(s)} ds < \epsilon$ [12].

In order to prove $K \in \mathfrak{S}(L^2_\alpha(H,(a,\infty)))$ define the following operator:

$$K_\epsilon f(t) := \frac{1}{\alpha(t)} \int_a^{t-\tau/2} e^{-A\delta(s,t)} f(s)ds, \; f \in L^2_\alpha(H,(a,\infty)), \; \epsilon > 0,$$
$$K_\epsilon : L^2_\alpha(H,(a,\infty)) \to L^2_\alpha(H,(a,\infty)) \text{ for } \epsilon > 0.$$

In this case, for any $f \in L^2_\alpha(H,(a,\infty))$, we have that

$$\begin{aligned}
\|K_\epsilon f\|^2_{L^2_\alpha} &= \int_a^\infty \| \frac{1}{\alpha(t)} \int_a^{t-\tau/2} e^{-A\delta(s,t)} f(s)ds\|^2_H \alpha(t)dt \\
&\leq \int_a^\infty \frac{1}{\alpha(t)} \left(\int_a^{t-\tau/2} \|f(s)\|_H ds \right)^2 dt \\
&= \int_a^\infty \frac{1}{\alpha(t)} \left(\int_a^{t-\tau/2} \frac{\sqrt{\alpha(s)}}{\alpha(s)} \sqrt{\alpha(s)} \|f(s)\|_H ds \right)^2 dt \\
&\leq \int_a^\infty \frac{1}{\alpha(t)} \delta(a, t-\tau/2) \left(\int_a^{t-\tau/2} \alpha(s) \|f(s)\|^2_H ds \right) dt \\
&\leq \delta^2 \|f(s)\|^2_{L^2_\alpha(H,(a,\infty))} < \infty.
\end{aligned}$$

For each $\epsilon > 0$, the operator K_ϵ can be represented as follows

$$K_\epsilon f(t) := \frac{1}{\alpha(t)} \int_a^\infty K_\epsilon(t,s) f(s)ds,$$

where $f(t) \in L^2_\alpha(H,(a,\infty))$ and for each $(t,s) \in [a,\infty) \times [a,\infty)$,

$$K_\epsilon(t,s) = \begin{cases} e^{-A\delta(s,t)}, \; if \; a \leq s < t - \tau(\epsilon)/2, \\ 0, \; if \; t - \tau(\epsilon)/2 \leq s < \infty. \end{cases}$$

Since each dual $(t,s) \in [a,\infty) \times [a,\infty), \; a \leq s < t - \dfrac{\tau}{2}$, satisfies the following property:

$$Ae^{-A\delta(s,t)} \in B(H)$$

(Note that, $B(H)$ is the class of linear bounded operators in H (see [3])).

$$e^{-A\delta(s,t)} = Ae^{-A\delta(s,t)}A^{-1} \in \mathfrak{S}_\infty(H).$$

Now for any $\gamma > a$ define the following linear bounded operator in form

$$K^\gamma_\epsilon f(t) := \frac{1}{\alpha(t)} \int_a^\gamma K_\epsilon(t,s) f(s)ds,$$
$$K^\gamma_\epsilon : L^2_\alpha(H,(a,\infty)) \to L^2_\alpha(H,(a,\infty)).$$

Recall that this operator is compact (see [4]).

Since for every $t, s \in [a,\infty)$ such that $s < t, \; \delta(s,t) = \int_s^t \frac{1}{\alpha(s)} ds > 0$, then $\|K_\epsilon(t,s)\| \leq 1$.

On the other hand

$$
\begin{aligned}
\|(K_\epsilon - K_\epsilon^\gamma)f\|^2_{L^2_\alpha(H,(a,\infty))} &= \|\frac{1}{\alpha(t)}\int_\gamma^\infty K_\epsilon(t,s)f(s)ds\|^2_{L^2_\alpha(H,(a,\infty))} \\
&\leq \int_a^\infty \frac{1}{\alpha(t)}\left(\int_\gamma^\infty \|K_\epsilon(t,s)\|\frac{\sqrt{\alpha(s)}}{\alpha(s)}\sqrt{\alpha(s)}\|f(s)\|_H ds\right)^2 dt \\
&\leq \int_a^\infty \frac{1}{\alpha(t)}\left(\int_\gamma^\infty \frac{1}{\alpha(s)}ds\right)\left(\int_a^\infty \alpha(s)\|f(s)\|^2_H ds\right)dt \\
&= \delta\|f\|^2_{L^2_\alpha(H,(a,\infty))}\int_\gamma^\infty \frac{1}{\alpha(s)}ds.
\end{aligned}
$$

Hence $\|(K_\epsilon - K_\epsilon^\gamma)\| \leq \delta^{1/2}\left(\int_\gamma^\infty \frac{1}{\alpha(s)}ds\right)^{1/2}$ tends to 0 as γ tends to infinity.

That is $K_\epsilon^\gamma \to K_\epsilon$ as $\gamma \to \infty$ in operator norm. It is well known in operator theory that $K_\epsilon \in \mathfrak{S}(L^2_\alpha(H,(a,\infty)))$ ([4]) . Then, we have the following

$$
\begin{aligned}
\|(K - K_\epsilon)f\|^2_{L^2_\alpha(H,(a,\infty))} &= \|\frac{1}{\alpha(t)}\int_{t-\tau/2}^t e^{-A\delta(s,t)}f(s)ds\|^2_{L^2_\alpha(H,(a,\infty))} \\
&= \int_a^\infty \|\frac{1}{\alpha(t)}\int_{t-\tau/2}^t e^{-A\delta(s,t)}f(s)ds\|^2_H \alpha(t)dt \\
&\leq \int_a^\infty \frac{1}{\alpha(t)}\left(\int_{t-\tau/2}^t \|e^{-A\delta(s,t)}\|\|f(s)\|_H ds\right)^2 dt \\
&\leq \int_a^\infty \frac{1}{\alpha(t)}\left(\int_{t-\tau/2}^t \frac{\sqrt{\alpha(s)}}{\alpha(s)}\sqrt{\alpha(s)}\|f(s)\|_H ds\right)^2 dt \\
&\leq \int_a^\infty \frac{1}{\alpha(t)}\delta(t-\tau/2,t)\left(\int_{t-\tau/2}^t \alpha(s)\|f(s)\|^2_H ds\right)dt \\
&\leq \int_a^\infty \frac{1}{\alpha(t)}\delta(t-\tau/2,t)\left(\int_a^\infty \alpha(s)\|f(s)\|^2_H ds\right)dt \\
&= \int_a^\infty \frac{1}{\alpha(t)}\delta(t-\tau/2,t)\|f\|^2_{L^2_\alpha(H,(a,\infty))} \\
&\leq \epsilon\delta\|f\|^2_{L^2_\alpha(H,(a,\infty))},\ f \in L^2_\alpha(H,(a,\infty)),
\end{aligned}
$$

that is,

$$\|K_\epsilon - K\| \leq \epsilon^{1/2}\delta^{1/2},$$

therefore, K_ϵ converges to K as ϵ tends to 0.

Again it is well known that $K \in \mathfrak{S}_\infty(L^2_\alpha(H,(a,\infty)))$ ([4]).

Thus the representation of L_W implies that $L_W^{-1} \in \mathfrak{S}_\infty(L^2_\alpha(H,(a,\infty)))$.

Corollary 3.3. *Let L_W be any normal extension of the minimal operator L_0 and let $\lambda \in \rho(L_W)$. Then $R_\lambda(L_W) \in \mathfrak{S}_\infty(L^2_\alpha(H,(a,\infty)))$.*

One can easily deduce it by the following relation

$$R_\lambda(L_W) = L_W^{-1} - \lambda R_\lambda(L_W) L_W^{-1}.$$

Using the same method in the proof of Theorem 3.2, we obtain the following results.

Corollary 3.4. *If $A^{-1} \in \mathfrak{S}_p(H),\ p \geq 1$ and L_W is any normal extension of L_0, then $L_W^{-1} \in \mathfrak{S}_p(L^2_\alpha(H,(a,\infty)))$.*

Furthermore, from the representation of resolvent $R_\lambda(L_W)$ of the operator

$$\begin{aligned} R_\lambda(L_W)f(t) &= \frac{1}{\alpha(t)} e^{(\lambda E - A)\delta(a,t)} \left(W - e^{(\lambda E - A)\delta}\right)^{-1} \int_a^\infty e^{(\lambda E - A)\delta(s,\infty)} f(s)ds \\ &+ \frac{1}{\alpha(t)} \int_a^t \exp\left((\lambda E - A)\delta(s,t)\right) f(s)ds,\ f \in H,\ \lambda \in \rho(L_W) \\ R_\lambda(L_{W_1}) - R_\lambda(L_{W_2}) &= \frac{1}{\alpha(t)} e^{(\lambda E - A)\delta(a,t)} \left[\left(W_1 - e^{(\lambda E - A)\delta}\right)^{-1} - \left(W_2 - e^{(\lambda E - A)\delta}\right)^{-1}\right] \\ &\times \int_a^\infty e^{(\lambda E - A)\delta(s,\infty)} f(s)ds \\ &= -\frac{1}{\alpha(t)} e^{(\lambda E - A)\delta(a,t)} \left(W_1 - e^{(\lambda E - A)\delta}\right)^{-1} (W_1 - W_2) \left(W_2 - e^{(\lambda E - A)\delta}\right)^{-1} \\ &\times \int_a^\infty e^{(\lambda E - A)\delta(s,\infty)} f(s)ds,\ f \in H, \end{aligned}$$

we also have the following assertion.

Corollary 3.5. *Let $L_{W_1},\ L_{W_2}$ be two normal extensions of the minimal operator L_0 in $L^2_\alpha(H,(a,\infty))$ and $\lambda \in \rho(L_{W_1}) \cap \rho(L_{W_2})$. Then we have*

$$R_\lambda(L_{W_1}) - R_\lambda(L_{W_2}) \in \mathfrak{S}_p(L^2_\alpha(H,(a,\infty))),\ 1 \leq p,$$

if and only if

$$W_1 - W_2 \in \mathfrak{S}_p(H),\ p \geq 1.$$

Now we will present a result on the structure of the spectrum of the normal extension of the minimal operator L_0.

Theorem 3.6. *If $A^{-1} \in \mathfrak{S}_\infty(H)$ and L_W is any normal extension of the minimal operator L_0 in $L^2_\alpha(H,(a,\infty))$, then the spectrum of L_W has the form*

$$\sigma(L_W) = \left\{\lambda_n(A) - \frac{i}{\delta}\left(arg\lambda_n\left(W^* exp\left(-A\delta\right)\right) + 2k\pi\right), n \in \mathbb{N}; k \in \mathbb{Z}\right\}.$$

Proof. Theorem 4.1 in [8] implies that

$$\sigma(L_W) = \left\{\lambda \in \mathbb{C} : \lambda = \delta^{-1}(ln|\mu|^{-1} + 2k\pi i - iarg\mu),\ k \in \mathbb{Z},\ \mu \in \sigma\left(W^* exp\left(-A\delta\right)\right)\right\}.$$

Since $A^{-1} \in \mathfrak{S}_\infty(H)$, then

$$W^* e^{-A\delta} = W^* \left(A e^{-A\delta}\right) A^{-1} \in \mathfrak{S}_\infty(H).$$

Because for any eigenvector $x_\lambda \in H$ corresponding to the eigenvalue $\lambda \in \sigma_p\left(W^* e^{-A\delta}\right)$ we have

$$W^* e^{-A\delta} x_\lambda = \lambda\left(W^* e^{-A\delta}\right) x_\lambda.$$

In this case since $\overline{\lambda} \in \mathbb{C}$ is an eigenvalue of the adjoint operator to $W^* e^{-A\delta}$, that is, of the operator $e^{-A\delta}W$ with the same eigenvector x_λ in H, then the last relation implies

$$e^{-A\delta} W W^* e^{-A\delta} x_\lambda = \lambda\left(W^* e^{-A\delta}\right) e^{-A\delta} W x_\lambda = \lambda\left(W^* e^{-A\delta}\right) \overline{\lambda\left(W^* e^{-A\delta}\right)} x_\lambda,$$

namely,

$$e^{-2A\delta}x_\lambda = |\lambda\left(W^* e^{-A\delta}\right)|^2 x_\lambda.$$

Hence

$$|\lambda\left(W^* e^{-A\delta}\right)|^2 = \lambda e^{-2A\delta} = e^{-2\lambda(A)\delta}$$

that is,

$$|\mu| = |\lambda\left(W^* e^{-A\delta}\right)| = e^{-\lambda(A)\delta}$$

and from this relation we have

$$ln|\mu| = -\lambda(A)\delta.$$

Thus

$$\sigma(L_W) = \left\{\lambda \in \mathbb{C} : \lambda = \lambda_n(A) - \frac{i}{\delta}\left(arg\lambda_n\left(W^* e^{-A\delta}\right) + 2k\pi\right), n \in \mathbb{N}; k \in \mathbb{Z}\right\}.$$

Now we can prove the main theorem of this section.

Theorem 3.7. *If $A^{-1} \in \mathfrak{S}_\infty(H)$ and $\lambda_n(A) \sim cn^\alpha,\ 0 < c,\ \alpha < \infty$, then $L_W^{-1} \in \mathfrak{S}_\infty(L^2_\alpha(H,(a,\infty)))$ and*

$$s_n(L_W^{-1}) \sim dn^{-\beta},\ 0 < d < \infty,\ \beta = \frac{\alpha}{1+\alpha}.$$

Proof. Since $A^{-1} \in \mathfrak{S}_\infty(H)$, then by Theorem 3.2 $L_W^{-1} \in \mathfrak{S}_\infty(L^2_\alpha(H,(a,\infty)))$.

Firstly, note that if **N** is any normal compact operator in any Hilbert space H, then for the $s-$number of the operator **N**, we have

$$s(\mathbf{N}) = |\lambda(\mathbf{N})|.$$

from [4]. Therefore,

$$\begin{aligned} s_m(L_W^{-1}) &= |\lambda_m(L_W^{-1})| = |\lambda_m(L_W)|^{-1} \\ &= |\lambda_n(A) - \frac{i}{\delta}\left(arg\lambda_n\left(W^* e^{-A\delta}\right) + 2k\pi\right)|^{-1} \\ &= |\lambda_n(A) - \frac{i}{\delta}\left(\delta_n + 2k\pi\right)|^{-1}, \end{aligned}$$

where, $m = m(n,k) \in \mathbb{N},\ n \in \mathbb{N},\ k \in \mathbb{Z},\ \delta_n = arg\lambda_n\left(W^* e^{-A\delta}\right)$.

It is clear that for each $n \in \mathbb{N},\ 0 \le \delta_n < 2\pi$.

Denote by $\mathbb{N}(\lambda; T)$ the cardinality of the set $\{n : |\lambda_n(T)| \le |\lambda|\}$; that is,

$$\mathbb{N}(\lambda; T) := \sum_{0 \le |\lambda_n(T)| \le |\lambda|} 1,\ \lambda \in \mathbb{C},$$

is the number of eigenvalues of the some linear closed operator T in any Hilbert space with modules of eigenvalues less than or equal to $|\lambda|$. This function takes values in the set of nonnegative integers and in case where T is unbounded it is nondecreasing and tends to $+\infty$ as $|\lambda|$ to ∞.

It is easy to see that

$$|\lambda(L_W)| = \left[c^2 n^{2\alpha} + \frac{1}{\delta^2}(\delta_n + 2k\pi)^2\right]^{1/2},$$

where $n \in \mathbb{N},\ k \in \mathbb{Z}$.

Since $0 \le \delta_n < 2\pi$ for each $n \in \mathbb{N}$, then from the last equality we have

$$\left[c^2 n^{2\alpha} + \frac{4\pi^2}{\delta^2}k^2\right]^{1/2} \le |\lambda(L_W)| \le \left[c^2 n^{2\alpha} + \frac{4\pi^2}{\delta^2}(k+1)^2\right]^{1/2},\ n \in \mathbb{N},\ k \in \mathbb{Z}.$$

Therefore

$$|\lambda(L_W)| \sim \sqrt{c^2 n^{2\alpha} + h^2 k^2},\ n \in \mathbb{N},\ k \in \mathbb{Z},$$

where $h = \dfrac{2\pi}{\delta}$.

On the other hand, we note that $\left(c^2n^{2\alpha} + h^2k^2\right)^{1/2}$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$, are modules of eigenvalues of the periodical boundary condition, for the Dirichlet problem, i.e.,

$$|\lambda(L_E)| = \left(c^2n^{2\alpha} + h^2k^2\right)^{1/2}, \ n \in \mathbb{N}, \ k \in \mathbb{Z}.$$

In another word, asymptotical behavior of the modules of eigenvalues of each normal extension L_W and Dirichlet extension is the same, that is,

$$|\lambda_m(L_W)| \sim |\lambda_m(L_E)| \text{ as } m \to \infty.$$

Using the method established in [5] or [6] (in our case $k \in \mathbb{Z}$) can be found that

$$|\lambda_m(L_E)| \sim pm^{\frac{\alpha}{\alpha+1}}, \ m \to \infty, \ 0 < p < \infty.$$

On the other hand, since

$$s_m(L_E^{-1}) \sim |\lambda_m(L_E^{-1})|, \ m \to \infty,$$

then following result holds

$$s_m(L_W^{-1}) \sim |\lambda_m(L_W^{-1})| \sim |\lambda_m(L_E^{-1})| \sim dm^{-\frac{\alpha}{\alpha+1}}, \text{ as } m \to \infty, \ 0 < d < \infty,$$

which completes the proof.

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