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On the Control Invariants of Planar Bézier Curves for the Groups $M(2)$ and $SM(2)$

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Abstract. Let $G = M(2)$ be the group generated by all orthogonal transformations and translations of the 2-dimensional Euclidean space E_2 or $G = SM(2)$ be the subgroup of $M(2)$ generated by rotations and translations of E_2. In this paper, global G-invariants of plane Bézier curves in E_2 are introduced. Using complex numbers and the global G-invariants of a plane Bézier curves, for given two plane Bézier curves $x(t)$ and $y(t)$, evident forms of all transformations $g \in G$, carrying $x(t)$ to $y(t)$, are obtained. Similar results are given for plane polynomial curves.

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1. Introduction

Let E_2 be the 2-dimensional Euclidean space and $O(2)$ be the group of all orthogonal transformations of E_2. Put $SO(2) = \{g \in O(2) \mid detg = 1\}$, $M(2)=\{F : E_2 \to E_2 \mid Fx = gx + b,\ g \in O(2),\ b \in E_2\}$ and $SM(2) = \{F \in M(2) \mid detg = 1\}$.

In classic differential geometry, the following theorem is known in [14]:

"Let $x(t)$ and $y(t)$ be two curves in E_2. Then, $x(t)$ and $y(t)$ are equivalent if and only if the curvatures and speeds of $x(t)$ and $y(t)$ are equal."

In E_2, two different concepts of curvatures were defined: the signed curvature $\kappa_{\pm} = \frac{\left[x'(t)x^{(2)}(t)\right]}{<x'(t),x'(t)>^{\frac{3}{2}}}$ (see [4, p.64-66], [5, p.14-15], [6, p.25], [17, p.8]) and the curvature $\kappa(x) = \frac{\left\|\left[x'(t)x^{(2)}(t)\right]\right\|}{<x'(t),x'(t)>^{\frac{3}{2}}}$ (see [1, p.31]). The function $\kappa_{\pm}$ is $SM(2)$-invariant, but it is not $M(2)$-invariant. The function κ is $M(2)$-invariant. The signed curvature $\kappa_{\pm}$ is more used for investigation of curves in two dimensional classical differential geometry (see [4, p,64-66], [5, p.14-15]). Thus invariant theory of curves in the classical differential geometry was developed only for the group $SM(2)$. In addition, the method of orthogonal frame in the classical differential geometry give conditions only for the *local* $SM(2)$-equivalence of curves (see [13, p,9-19]).

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In [2], by using invariant parametrization of curves, the problem of G-equivalence of curves (that is nonparametric curves) was reduced to the problem of G-equivalence of paths (that is parametric curves) for $G = M(n), SM(n)$. Complete systems of global G-invariants of regular paths and regular curves in classical geometries were obtained in [2]. This approach was developed for curves in papers [9, 11, 16] and for vector fields in [7, 8].

In books ([4, Theorems 6.1 and 6.8], [5, p.136-137]) existence and uniqueness theorems for regular parametric curves (that is paths) in E_2 were obtained for the group $G = SM(2)$.

In [14], using differential invariants and Frenet frames of two curves, an isometry transformation which carrying a curve into another curve has been calculated.

In [15] , G-equivalence of two Bézier curves for groups $G = M(n)$ and $G = SM(n)$ without using differential invariants of Bézier curves in terms of control invariants of Bézier curves is proved. In this work, starting from the ideas in [15] we address how to compute explicitly an isometry transformation which carrying a Bézier curve into another Bézier curve in terms of control invariants of a Bézier curve for the groups $M(2)$ and $SM(2)$ without using differential invariants of Bézier curves.

2. Preliminaries

Let R be the field of real numbers and $\mathbb{C}$ be the field of complex numbers. The multiplication in $\mathbb{C}$ has the form $(a_1 + ia_2)(b_1 + ib_2) = (a_1b_1 - a_2b_2) + i(a_1b_2 + a_2b_1)$. We will consider element $a = a_1 + ia_2$ also in the form $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. For $a = a_1 + ia_2$, denote by P_a the matrix $\begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$ and consider P_a also as the transformation $P_a : \mathbb{C} \to \mathbb{C}$, where $P_a b = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 - a_2b_2 \\ a_1b_2 + a_2b_1 \end{pmatrix}$ for all $b = b_1 + ib_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{C}$. Then we have the equality

$$ab = P_a b. \tag{2.1}$$

for all $a, b \in \mathbb{C}$. Let $P(\mathbb{C})$ denote the set of all matrices P_a, where $a \in \mathbb{C}$. We consider on $P(\mathbb{C})$ the following standard matrix operations: the component-wise addition, a scalar multiplication and the multiplication of matrices. Then $P(\mathbb{C})$ is a field, where the unit element is the unit matrix. The following Propositions are known.

Proposition 2.1. *The mapping $P : \mathbb{C} \to P(\mathbb{C})$, where $P : a \to P_a$ for all $a \in \mathbb{C}$, is an isomorphism of fields.*

For vectors $a = a_1 + ia_2, b = b_1 + ib_2 \in \mathbb{C}$, we put $< a, b >= a_1b_1 + a_2b_2$. Then $< a, b >$ is a bilinear form on E_2 and $< a, a >= a_1^2 + a_2^2$ is a quadratic form on E_2. Put $Q(a) =< a, a >$. We consider the field $\mathbb{C}$ also as the two-dimensional Euclidean space E_2 with the scalar product $< a, b >$. Then $||a|| = |a| = \sqrt{Q(a)}, \forall a \in \mathbb{C}$.

Proposition 2.2. (i) *Equalities $Q(a) = det(P_a)$, $Q(ab) = Q(a)Q(b)$, $|ab| = |a|\,|b|$, $Q(a) = det(P_a) =$ hold for all $a, b \in \mathbb{C}$.*
(ii) *Let $a = a_1 + ia_2 \in \mathbb{C}^*$. Then $det(P_a) = Q(a) > 0$.*

An endomorphism ψ of a vector space $\mathbb{C}$ is called an involution of the field $\mathbb{C}$ if $\psi(\psi(a)) = a$ and $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in \mathbb{C}$. For an element $a = a_1 + ia_2 \in \mathbb{C}$, we set $\overline{a} = a_1 - ia_2$.

Proposition 2.3. *The mapping $a \to \overline{a}$ is an involution of the field $\mathbb{C}$. In addition, for an arbitrary element $a = a_1 + ia_2 \in \mathbb{C}$, equalities $a + \overline{a} = 2a_1, < a, a >= a\overline{a} = a_1^2 + a_2^2 \in R$ hold.*

Proposition 2.4. *Let $x \in \mathbb{C}$. Then the element x^{-1} exists if and only if $Q(x) \neq 0$. In the case $Q(x) \neq 0$, equalities $x^{-1} = \frac{\overline{x}}{Q(x)}$ and $Q(x^{-1}) = \frac{1}{Q(x)}$ hold.*

Let $W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We will use W also for the writing of the element $\overline{z}$ in the form $\overline{z} = Wz$.

Proposition 2.5. *$Q(Wx) = Q(x)$ for all $x \in \mathbb{C}$ and $< Wx, Wy >=< x, y >$ for all $x, y \in \mathbb{C}$.*

Put $\mathbb{C}^* = \{z \in \mathbb{C} \mid Q(z) \neq 0\}$. $\mathbb{C}^*$ is a group with respect to the multiplication operation in the field $\mathbb{C}$. Let $a = a_1 + ia_2 \in \mathbb{C}^*$ that is $|a| \neq 0$. Put

$$P_a^+ = \begin{pmatrix} \frac{a_1}{|a|} & \frac{-a_2}{|a|} \\ \frac{a_2}{|a|} & \frac{a_1}{|a|} \end{pmatrix}.$$

Proposition 2.6. *Let $a = a_1 + ia_2 \in \mathbb{C}^*$. Then the equality $P_a = |a|\, P_a^+$ holds, where $P_a^+ \in SO(2)$.*

Proof. The equality $P_a = |a|\, P_a^+$ is obvious. Since $(\frac{a_1}{|a|})^2 + (\frac{a_2}{|a|})^2 = 1$, the implication $P_a^+ \in SO(2)$ follows from [3, p.161-162]. □

Put $S(\mathbb{C}^*) = \{z \in \mathbb{C} \mid Q(z) = 1\}$, $P(\mathbb{C}^*) = \{P_z \mid z \in \mathbb{C}^*\}$ and $P(S(\mathbb{C}^*)) = \{P_z \mid z \in S(\mathbb{C}^*)\}$. $S(\mathbb{C}^*)$ is a subgroup of the group $\mathbb{C}^*$ and $S(\mathbb{C}^*) = \left\{e^{i\varphi} \mid \varphi \in R\right\}$. Denote the set of all matrices $\{gW \mid g \in P(\mathbb{C}^*)\}$ by $P(\mathbb{C}^*)W$, where gW is the multiplication of matrices g and W.

Theorem 2.7. *(see [3, p.172]) The following equalities are hold:*

(i) $SM(2) = \{F : E_2 \to E_2 | F(x) = P_a x + b, a \in S(\mathbb{C}^*), b \in E_2, \forall x \in E_2\}$

(ii) $SM(2)W = \{F : E_2 \to E_2 | F(x) = P_a W(x) + b, a \in S(\mathbb{C}^*), b \in E_2, \forall x \in E_2\}$

(iii) $M(2) = SM(2) \cup SM(2)W$.

Proposition 2.8. (i) *Let $u, v \in \mathbb{C}$. Assume that $Q(u) \neq 0$. Then the element vu^{-1} exists, the following equalities hold: $vu^{-1} = \frac{<u,v>}{Q(u)} + i\frac{[uv]}{Q(u)}$ and*

$$P_{vu^{-1}} = \begin{pmatrix} \frac{<u,v>}{Q(u)} & -\frac{[uv]}{Q(u)} \\ \frac{[uv]}{Q(u)} & \frac{<u,v>}{Q(u)} \end{pmatrix}.$$

(ii) *Assume that $Q(u) \neq 0$. Then $det(P_{vu^{-1}}) = (\frac{<u,v>}{Q(u)})^2 + (\frac{[uv]}{Q(u)})^2 \neq 0$ if and only if $Q(v) \neq 0$.*

(iii) *The functions $Q(u)$, $< u, v >$ and $[uv]$ are $SO(2)$-invariant.*

Proof. The proof of this proposition is given in Theorem 2 in [10]. □

3. Control Invariants of Planar Bézier Curve

A planar Bézier curve is a parametric curve(or a I-path, where $I = [0, 1]]$) whose points $x(t)$ are defined by $x(t) = \sum_{i=0}^{m} p_i B_{i,m}(t)$, where the $p_i \in E_2$ are control points and $B_{i,m}(t)$ are Bernstein basis polynomials.(for more details, see [12].)

A planar polynomial curve is a parametric curve(or a I-path, where $I = [0, 1]]$) whose points $x(t)$ are defined by $x(t) = \sum_{i=0}^{m} a_i t^i$, where the $a_i \in E_2$ are monomial control points.(for more details, see [12, p.166].)

All polynomial curves can be represented in Bézier form. The following lemma is given in [12, p.166].

Lemma 3.1. *The following equalities*

$$\begin{cases} a_i = \dfrac{n!}{i!(n-i)!} \displaystyle\sum_{j=1}^{i} (-1)^{i-j} \dfrac{i!}{j!(i-j)!} (b_j - b_0), \\ b_i - b_0 = \displaystyle\sum_{j=1}^{i} \dfrac{i!(n-j)!}{n!(i-j)!} a_j \end{cases}$$

hold for all $i = 1, 2, \ldots, m$.

Let G be a one of the groups $O(2)$ and $SO(2)$.

Definition 3.2. A function $f(z_0, z_1, \ldots, z_m)$ of vectors $z_0, z_1, \ldots, z_m$ in E_2 will be called G-invariant if $f(Fz_0, Fz_1, \ldots, Fz_m) = f(z_0, z_1, \ldots, z_m)$ for all $F \in G$.
A G-invariant function $f(b_0, b_1, \ldots, b_m)$ of control points $b_0, b_1, \ldots, b_m$ of a Bézier curve $x(t) = \sum_{j=0}^{m} b_j B_{j,m}(t)$ will be called a control G-invariant of $x(t)$, where $B_{j,m}(t)$ are Bernstein basis polynomials. A G-invariant function $f(a_0, a_1, \ldots, a_m)$ of monomial control points $a_0, a_1, \ldots, a_m$ of a polynomial curve $x(t) = \sum_{j=0}^{m} a_j t^j$ will be called a monomial G-invariant of $x(t)$.

Definition 3.3. Bézier curves $x(t)$ and $y(t)$ in E_2 will be called G -equivalent and written $x \overset{G}{\sim} y$ if there exists $F \in G$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$.

Since Bézier curves can be introduced by control points, we will define the problem of G-equivalence of points in E_2.

Definition 3.4. m-uples $\{z_1, z_2, \ldots, z_m\}$ and $\{w_1, w_2, \ldots, w_m\}$ of vectors in E_2 will be called G-equivalent and written by $\{z_1, z_2, \ldots, z_m\} \overset{G}{\sim} \{w_1, w_2, \ldots, w_m\}$ if there exists $F \in G$ such that $w_j = Fz_j$ for all $j = 1, 2, \ldots, m$.

Example 3.5. Since $< g(u), g(v) >=< u, v >$ for all $g \in O(2)$, we obtain that the scalar product $< u, v >$ of points $u, v \in E_2$ is $O(2)$-invariant. Similarly, the function $f(u, v, w) =< u - w, v - w >$ is $M(2)$-invariant.

Example 3.6. Let $u_1, u_2, \ldots, u_m$ be points in E_2. We denote the the matrix of column-vectors $u_1, u_2, \ldots, u_m$ by $U = \|u_1 u_2 \ldots u_m\|$ and its determinant by $\det U$. Then $\det U$ is $SO(2)$-invariant.
In fact, $det \|gu_1 gu_2 \ldots gu_m\| = \det g \det U = \det U$ for all $g \in SO(2)$.

Example 3.7. Let $x(t)$ and $y(t)$ be Bézier curves of degrees of m and k, respectively. Assume that $x \overset{O(2)}{\sim} y$. Then $m = k$ that is the degree of a Bézier curve $x(t)$ is $O(2)$-invariant.

4. Equivalence of Planar Bézier Curves

Theorem 4.1. *Let* $x(t) = \sum_{j=0}^{m} a_j t^j = \sum_{j=0}^{m} p_j B_{j,m}(t)$ *and* $y(t) = \sum_{j=0}^{m} c_j t^j = \sum_{j=0}^{m} q_j B_{j,m}(t)$ *be Bézier curves in* E_2 *of degree m, where* $m \geq 1$. *Then,* $x(t) \overset{M(2)}{\sim} y(t) \Leftrightarrow x'(t) \overset{O(2)}{\sim} y'(t)$.

Proof. $\Rightarrow$. Assume that $x(t) \overset{M(2)}{\sim} y(t)$. Then there exists $F \in M(2)$ such that $y(t) = Fx(t), \forall t \in [0, 1]$. Then, there exist $g \in O(2)$ and $b \in E_2$ such that $y(t) = Fx(t) = gx(t) + b$. This equality implies $y'(t) = (Fx(t))' = gx'(t), \forall t \in [0, 1]$. That is, $x'(t) \overset{O(2)}{\sim} y'(t)$.

$\Leftarrow$. Assume that $x'(t) \overset{O(2)}{\sim} y'(t)$. Then there exists $g \in O(2)$ such that $y'(t) = gx'(t), \forall t \in [0, 1]$. Then we have $y'(t) = (gx(t))', \forall t \in [0, 1]$. This equality implies that $y'(t) - (gx(t))' = (y(t) - gx(t))' = 0, \forall t \in [0, 1]$. Then there exists $b \in E_2$ such that $y(t) = gx(t) + b, \forall t \in [0, 1]$. This means that $x(t) \overset{M(2)}{\sim} y(t)$. Proofs of other statements are given in [15, Theorem 1]. □

Theorem 4.2. *Let* $x(t) = \sum_{j=0}^{m} a_j t^j = \sum_{j=0}^{m} p_j B_{j,m}(t)$ *and* $y(t) = \sum_{j=0}^{m} c_j t^j = \sum_{j=0}^{m} q_j B_{j,m}(t)$ *be Bézier curves in* E_2 *of degree m, where* $m \geq 1$. *Then,* $x(t) \overset{SM(2)}{\sim} y(t) \Leftrightarrow x'(t) \overset{SO(2)}{\sim} y'(t)$.

Proof. It is similar to proof of Theorem 4.1. □

In [15], The following theorems are given as follows:

Theorem 4.3. *Let* $x(t) = \sum_{j=0}^{m} a_j t^j = \sum_{j=0}^{m} p_j B_{j,m}(t)$ *and* $y(t) = \sum_{j=0}^{m} c_j t^j = \sum_{j=0}^{m} q_j B_{j,m}(t)$ *be Bézier curves in* E_2 *of degree m, where* $m \geq 1$. *Then following conditions are equivalent:*

(*i*) $x(t) \overset{M(2)}{\sim} y(t)$
(*ii*) $\{p_0, p_1, \ldots, p_m\} \overset{M(2)}{\sim} \{q_0, q_1, \ldots, q_m\}$
(*iii*) $\{p_1 - p_0, p_2 - p_0, \ldots, p_m - p_0\} \overset{O(2)}{\sim} \{q_1 - q_0, q_2 - q_0, \ldots, q_m - q_0\}$
(*iv*) $\{a_1, a_2, \ldots, a_m\} \overset{O(2)}{\sim} \{c_1, c_2, \ldots, c_m\}$

Theorem 4.4. *Let* $x(t) = \sum_{j=0}^{m} a_j t^j = \sum_{j=0}^{m} p_j B_{j,m}(t)$ *and* $y(t) = \sum_{j=0}^{m} c_j t^j = \sum_{j=0}^{m} q_j B_{j,m}(t)$ *be Bézier curves in* E_2 *of degree m,where* $m \geq 1$. *Then following four conditions are equivalent:*

(*i*) $x \overset{SM(2)}{\sim} y$
(*ii*) $\{p_0, p_1, \ldots, p_m\} \overset{SM(2)}{\sim} \{q_0, q_1, \ldots, q_m\}$
(*iii*) $\{p_1 - p_0, p_2 - p_0, \ldots, p_m - p_0\} \overset{SO(2)}{\sim} \{q_1 - q_0, q_2 - q_0, \ldots, q_m - q_0\}$
(*iv*) $\{a_1, a_2, \ldots, a_m\} \overset{SO(2)}{\sim} \{c_1, c_2, \ldots, c_m\}$

Remark 4.5. In Theorems 4.3 and 4.4, we have considered the problem of G-equivalence of polynomial curves in the case $m \geq 1$. For the case $m = 0$, the problem of G-equivalence of polynomial curves $x(t) = a_0$ and $y(t) = c_0$ reduces to the problem of G-equivalence of vectors a_0 and c_0 in E_2. For groups $M(2)$ and $SM(2)$, it is obvious that $a_0 \overset{M(2)}{\sim} c_0$ and $a_0 \overset{SM(2)}{\sim} c_0$ for all a_0 and c_0 in E_2. In what follows, $m \geq 1$. The case $m = 0$ is easily considered.

Theorem 4.6. *(i) Let $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ be two polynomial curves in E_2 of degree m,where $m \geq 1$ such that $x(t) \overset{SM(2)}{\sim} y(t)$. Then, the following equalities hold.*

$$\begin{cases} <a_j, a_k> = <c_j, c_k>, \\ \qquad [a_{i_1} a_{i_2}] = [c_{i_1} c_{i_2}] \end{cases} \tag{4.1}$$

for all $i_1, i_2 = 1, \ldots, m; 1 \leq i_1 < i_2 \leq m; j, k = 1, 2, \ldots, m; j \leq k$.

(ii) Conversely, if $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ are two polynomial curves in E_2 of degree m, where $m \geq 1$ such that the equalities (4.1) hold, then $x(t) \overset{SM(2)}{\sim} y(t)$. Moreover, there exists the unique $F \in SM(2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$. In this case, $Fx(t) = U_1 x(t) + b$, where $U_1 \in SO(2)$ and U_1 has the following form

$$U_1 = \begin{pmatrix} \frac{<a_m, c_m>}{Q(a_m)} & -\frac{[a_m c_m]}{Q(a_m)} \\ \frac{[a_m c_m]}{Q(a_m)} & \frac{<a_m, c_m>}{Q(a_m)} \end{pmatrix}, \tag{4.2}$$

and $b = y(t) - U_1 x(t) \in E_2$. U_1 and b do not depend on $t \in T$.

Proof. (i) It follows from [15, Corollary 1].

(ii) Assume that the equalities (4.2) hold. From [15, Corollary 1], we have $x(t) \overset{SM(2)}{\sim} y(t)$. Then, there exist $F \in SM(2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$. In this case, $y(t) = Fx(t) = U_1 x(t) + b$, where $U_1 \in SO(2)$ and $b \in E_2$.

Since $x(t)$ and $y(t)$ are two polynomial curves of degree $m \geq 1$, m^{th} order derivatives of $x(t)$ and $y(t)$ are $x^{(m)}(t) = m!a_m$ and $y^{(m)}(t) = m!c_m$,respectively. The equality $y(t) = Fx(t) = U_1 x(t) + b$ implies $y^{(m)}(t) = U_1 x^{(m)}(t)$. Using this equality, we have $c_m = U_1 a_m$ such that $U_1 \in SO(2)$, where $U_1 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Using the equality $c_m = U_1 a_m$, we obtain $a = \frac{<a_m, c_m>}{Q(a_m)}$ and $b = \frac{[a_m c_m]}{Q(a_m)}$.

Prove the uniqueness of $U_1 \in SO(2)$ and uniqueness of b satisfying the condition $y(t) = U_1 x(t) + b$. Assume that $K \in SO(2)$ and $b_1 \in E_2$ such that $y(t) = Kx(t) + b_1$. This equality implies $y^{(m)}(t) = Kx^{(m)}(t)$. Then by (2.1), Proposition 2.1 and Theorem 2.7, there exists the unique $u \in S(\Omega^*)$ such that $K = P_u$. Hence we have $y^{(m)}(t) = P_u x^{(m)}(t)$. By (2.1), we obtain $y^{(m)}(t) = ux^{(m)}(t)$. Since $Q(x^{(m)}(t)) = Q(m!a_m) = m!Q(a_m) \neq 0$, $y^{(m)}(t) = ux^{(m)}(t)$ implies that $u = y^{(m)}(t)(x^{(m)}(t))^{-1} = g$. Hence $P_u = P_g = U_1$. The uniqueness of U_1 is proved. Then $b = y(t) - U_1 x(t) = y(t) - Kx(t) = b_1$. Hence the uniqueness of b is proved.

So, the proof is completed. □

The following corollary is given in [12, Corollary 1.7].

Corollary 4.7. *Let $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ be Bézier curves in E_2 of degree m, where $m \geq 1$. Then, the r^{th} derivative of $x(t)$ is*

$$x^{(r)}(t) = \sum_{j=0}^{m-r} p_j^r B_{j,m-r}(t),$$

where

$$p_i^r = m(m-1)\ldots(m-r+1) \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} p_{i+j},$$

for all $i = 0, \ldots, m$

Theorem 4.8. *(i) Let $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be two Bézier curves in E_2 of degree m,where $m \geq 1$ such that $x(t) \overset{SM(2)}{\sim} y(t)$. Then, the following equalities hold.*

$$\begin{cases} < p_j - p_0, p_k - p_0 >=< q_j - q_0, q_k - q_0 >, \\ \quad [p_{i_1} - p_0 \ p_{i_2} - p_0] = [q_{i_1} - q_0 \ q_{i_2} - q_0] \end{cases} \tag{4.3}$$

for all $i_1, i_2 = 1, \ldots, m; 1 \leq i_1 < i_2 \leq m; j, k = 1, 2, \ldots, m; j \leq k$.

(ii) Conversely, if $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ are two Bézier curves in E_2 of degree m, where $m \geq 1$ such that the equalities (4.3) hold, then $x(t) \overset{SM(2)}{\sim} y(t)$. Moreover, there exists the unique $F \in SM(2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$. In this case, $Fx(t) = V_1 x(t) + b$, where $V_1 \in SO(2)$ and V_1 has the following form

$$V_1 = \begin{pmatrix} \frac{<p_0^m, q_0^m>}{Q(p_0^m))} & -\frac{[p_0^m q_0^m]}{Q(p_0^m)} \\ \frac{[p_0^m q_0^m]}{Q(p_0^m)} & \frac{<p_0^m, q_0^m>}{Q(p_0^m)} \end{pmatrix}, \tag{4.4}$$

and $b = y(t) - V_1 x(t) \in E_2$. V_1 and b do not depend on $t \in T$.

Proof. It follows from Theorem 4.6 and Corollary 4.7. □

Lemma 4.9. *For all vectors y_1, y_2, z_1, z_2 in E_2, the equality $[y_1 y_2][z_1 z_2] = det\| < y_i, z_k > \|_{i,k=1}^2$ holds.*

Proof. A proof of this lemma is given in [9, Lemma 13]. □

Theorem 4.10. *(i) Let $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ be two polynomial curves in E_2 of degree m,where $m \geq 2$ such that $x(t) \overset{M(2)}{\sim} y(t)$. Then, the following equalities hold.*

$$< a_j, a_k >=< c_j, c_k >, \tag{4.5}$$

for all $j, k = 1, 2, \ldots, m; j \leq k$.

(ii) Conversely, if $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ are two polynomial curves in E_2 of degree m, where $m \geq 2$ such that the equalities (4.5) hold, then $x(t) \overset{M(2)}{\sim} y(t)$. Moreover, there exist the unique $F \in M(2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$. In this case, the following cases exist:

(ii.1) $[a_{i_1} a_{i_2}] = [c_{i_1} c_{i_2}]$,

(ii.2) $[a_{i_1} a_{i_2}] = -[c_{i_1} c_{i_2}]$.

In the case (ii.1), $Fx(t) = U_1 x(t) + b_1$, where $U_1 \in SO(2), b_1 \in E_2$. Here U_1 and b_1 have the forms (4.2) and $b_1 = y(t) - U_1 x(t)$, resp.

In the case (ii.2), $Fx(t) = (U_2 W)x(t) + b_2$, where $U_2 \in SO(2), b_2 \in E_2$. Here U_2 and b_2 have the forms

$$U_2 = \begin{pmatrix} \frac{<Wa_m, c_m>}{Q(Wa_m)} & -\frac{[Wa_m c_m]}{Q(Wa_m)} \\ \frac{[Wa_m c_m]}{Q(Wa_m)} & \frac{<Wa_m, c_m>}{Q(Wa_m)} \end{pmatrix}, \tag{4.6}$$

and $b_2 = y(t) - (U_2 W)x(t) \in E_2$, resp.

The matrices U_i and the constants b_i do not depend on $t \in [0, 1]$ for $i = 1, 2$.

Proof. (i) It follows from [15, Theorem 4].

(ii) Assume that the equalities (4.5) hold. Then, from [15, Theorem 4], we have $x(t) \overset{M(2)}{\sim} y(t)$. That is, there exist $F \in M(2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$.

Applying Lemma 4.9 to vectors $y_1 = a_{i_1}, y_2 = a_{i_2}, z_1 = a_{i_3}, z_2 = a_{i_4}$ for all $1 \leq i_1 < i_2 \leq m$ and $1 \leq i_3 < i_4 \leq m$, we obtain

$$[a_{i_1} a_{i_2}]\,[a_{i_3} a_{i_4}] =< a_{i_1}, a_{i_3} >< a_{i_2}, a_{i_4} > - < a_{i_1}, a_{i_4} >< a_{i_2}, a_{i_3} > . \tag{4.7}$$

Since the inner products in this equality are $O(2)$-invariants, $[a_{i_1} a_{i_2}]\,[a_{i_3} a_{i_4}]$ are $O(2)$-invariants.

The equalities (4.5) and (4.7) imply the equalities $[a_{i_1} a_{i_2}] = [c_{i_1} c_{i_2}]$ and $[a_{i_1} a_{i_2}] = -[c_{i_1} c_{i_2}]$. Using the equality (4.5) and the equality $[a_{i_1} a_{i_2}] = [c_{i_1} c_{i_2}]$ imply the equalities (4.1). Then, by Theorem (4.6), the unique $U_1 \in SO(2)$ and the unique $b_1 \in E_2$ exist such that $y(t) = U_1 x(t) + b_1$ for all $t \in [0, 1]$. Here U_1 and b_1 have the forms (4.2) and $b_1 = y(t) - U_1 x(t)$, resp.

Now, consider the polynomial curve $Wx(t)$. Since the inner products in the equality (4.5) are $O(2)$-invariants, we have $< Wa_j, Wa_k >=< a_j, a_k >=< c_j, c_k >$ for all $j, k = 1, 2, \dots, m;\ j \le k$. Using $detW = -1$ and the equality $[a_{i_1} a_{i_2}] = -[c_{i_1} c_{i_2}]$, we obtain $[Wa_{i_1} Wa_{i_2}] = (detW)[a_{i_1} a_{i_2}] = (-1)(-[c_{i_1} c_{i_2}]) = [c_{i_1} c_{i_2}]$.

Then the following equalities hold:

$$< Wa_j, Wa_k >=< c_j, c_k >,$$
$$[Wa_{i_1} Wa_{i_2}] = [c_{i_1} c_{i_2}].$$

Then, by Theorem 4.6, the unique $U_2 \in SO(2)$ and the unique $b_2 \in E_2$ exist such that $y(t) = U_2(Wx(t)) + b_2 = (U_2W)x(t) + b_2$ for all $t \in [0, 1]$. Here U_2 and b_2 have the forms (4.6) and $b_2 = y(t) - (U_2W)x(t)$, resp.

The matrices U_i and the constants b_i do not depend on $t \in [0, 1]$ for $i = 1, 2$.

Let $F \in M(2)$ such that $y(t) = Fx(t)$. Prove that $Fx(t) = U_1x(t) + b_1$ or $Fx(t) = (U_2W)x(t) + b_2$. Let $y(t) = Fx(t) = Cx + b$ for some $C \in O(2)$ and some $b \in E_2$. Then $C \in SO(2)$ or $C \in SO(2)W$. Assume that $C \in SO(2)$. Then, by the uniqueness in Theorem 4.6, $C = U_1$ and $b = b_1$. Assume that $C \in SO(2)W$. Then C has the form $C = DW$, where $D \in SO(2)$. We have $y(t) = (DW)x(t) + b = D(Wx(t)) + b$. Hence paths $y(t)$ and $Wx(t)$ are $SM(2)$-equivalent. By the uniqueness in Theorem 4.6, $D = U_2$. This implies $b = b_2$.

□

Theorem 4.11. *(i) Let $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be two Bézier curves in E_2 of degree m,where $m \ge 2$ such that $x(t) \overset{M(2)}{\sim} y(t)$. Then, the following equalities hold.*

$$< p_j - p_0, p_k - p_0 >=< q_j - q_0, q_k - q_0 >, \tag{4.8}$$

for all $j, k = 1, 2, \dots, m;\ j \le k$.

(ii) Conversely, if $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ are two polynomial curves in E_2 of degree m, where $m \ge 2$ such that the equalities (4.8) hold, then $x(t) \overset{M(2)}{\sim} y(t)$. Moreover, there exist the unique $F \in M(2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$. In this case, the following cases exist:

(ii.1) $[p_{i_1} - p_0\ p_{i_2} - p_0] = [q_{i_1} - q_0\ q_{i_2} - q_0]$,

(ii.2) $[p_{i_1} - p_0\ p_{i_2} - p_0] = -[q_{i_1} - q_0\ q_{i_2} - q_0]$.

In the case (ii.1), $Fx(t) = V_1x(t) + b_1$, where $V_1 \in SO(2), b_1 \in E_2$. Here V_1 and b_1 have the forms (4.4) and $b_1 = y(t) - V_1x(t)$, resp.

In the case (ii.2), $Fx(t) = (V_2W)x(t) + b_2$, where $V_2 \in SO(2), b_2 \in E_2$. Here V_2 and b_2 have the forms

$$V_2 = \begin{pmatrix} \frac{<Wp_0^m, q_0^m>}{Q(Wp_0^m))} & -\frac{[Wp_0^m q_0^m]}{Q(Wp_0^m)} \\ \frac{[Wp_0^m q_0^m]}{Q(Wp_0^m)} & \frac{<Wp_0^m, q_0^m>}{Q(Wp_0^m)} \end{pmatrix},$$

and $b_2 = y(t) - (V_2W)x(t)$, resp.

The matrices V_i and the constants b_i do not depend on $t \in [0, 1]$ for $i = 1, 2$.

Proof. It follows from Theorems 4.8 and 4.10. □

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