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Local Pre-Hausdorff Constant Filter Convergence Spaces

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Abstract. The aim of this paper is to characterize local pre-Hausdorff constant filter convergence spaces and give some invariance properties of them.

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1. Introduction

Filters are introduced by H. Cartan [11, 12] in 1937 which can be seen as generalization of sequences. In 1948, Choquet [13] defined the concept of convergence of a filter. In 1954, Kowalsky [16] gave a filter description of convergence. In 1979, Schwarz [23] introduced the category ***ConFCO*** of constant filter convergence spaces and continuous maps.

In 1991, M. Baran [2] introduced a local pre-Hausdorff topological space (where a topological space X is local pre-Hausdorff for given any fixed point and any distinct point from this fixed point if there is a neighbourhood of one missing the other, then the two points have disjoint neighbourhoods). Pre-Hausdorff objects are used to characterize the decidable objects [19] in a topos [15], where an object X of $\mathcal{E}$, a topos, is said to be decidable if the diagonal is a complemented subobject [17]. Also, finite pre-Hausdorff spaces can also be described using the notion of a Borel field [22, 24]. Furthermore, local pre-Hausdorff objects are used to define various forms of each of local Hausdorff objects [2] and local T_3 objects, and local T_4 objects in arbitrary topological categories [9].

In this paper, we characterize local pre Hausdorff constant filter convergence spaces at point p and give some invariance properties of them.

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2. Preliminaries

Let A be a set, $F(A)$ set of all filters on A and K be a function from A to $P(F(A))$. If K satisfies the following two conditions, then (A, K) is called a constant filter convergence space.
(1) $[x] \in K$ for each $x \in A$, where $[x] = \{B \subset A : x \in B\}$.
(2) If $\alpha \subset \beta$ and $\alpha \in K$ implies $\beta \in K$ for any filter β on A.

A map $f : (A, K) \to (B, L)$ between constant filter convergence spaces is called continuous if and only if $\alpha \in K$ implies $f(\alpha) \in K$ (where $f(\alpha)$ denotes the filter generated by $\{f(D)|D \in K\}$ i.e., $f(\alpha) = \{U \subset X : D$ such that $f(D) \subset U\}$. The category of constant filter convergence spaces and continuous maps is denoted by ***ConFCO*** [23].

A functor $U : \mathcal{E} \to \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if U is concrete (i.e., faithful and amnestic (i.e., if $U(f) = id$ and f is an isomorphism, then $f = id$)), has small (i.e., sets) fibers, and for which every U-source has an initial lift or, equivalently, for which each U-sink has a final lift, see [1, 14, 21]. A topological functor $U : \mathcal{E} \to \mathcal{B}$ is said to be normalized if constant objects, i.e., subterminals, have a unique structure. Note also that U has a left adjoint called the discrete functor D. Recall, in [1, 21] that an object $X \in \mathcal{E}$ is discrete iff every map $U(X) \to U(Y)$ lift to map $X \to Y$ for each object $Y \in \mathcal{E}$. Note that the category ***ConFCO*** is normalized topological category.

We denote by $\alpha \cup \beta$ the smallest filter (proper or not) containing both α and β for filters α and β, i.e., $\alpha \cup \beta = \{W \subset A : U \cap V \subset W$ for some $U \in \alpha$ and $V \in \beta\}$.
2.1. A source $\{f_i : (A, K) \to (A_i, K_i), i \in I\}$ in ***ConFCO*** is initial iff $\alpha \in K$ precisely when $f_i(\alpha) \in K_i$ [18, 20].
2.2. A sink $\{f_i : (A_i, K_i) \to (A, K), i \in I\}$ is final if and only if $\alpha \in L$ implies there exists $\beta_i \in K_i$ such that $f_i(\beta_i) \subset \alpha, i \in I$ [20, 23].

3. Local Pre-Hausdorff Constant Filter Convergence Spaces

In this section, we give the characterization of pre-Hausdorff constant filter convergence spaces at a point p and give some invariance properties of them.

Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p [2], i.e., two disjoint copies of B identified at p, or in other words, the pushout of $p : 1 \to B$ along itself (where 1 is the terminal object in **Set**, the category of sets and functions). More precisely, if i_1 and $i_2 : B \to B \vee_p B$ denote the inclusion of B as the first and second factor, respectively, then $i_1 p = i_2 p$ is the pushout diagram [10]. A point x in $B \vee_p B$ will be denoted by $x_1(x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = p_2$.

The principal $p-$ axis map, $A_p : B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$, the skewed $p-$ axis map, $S_p : B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$, and the fold map at p, $\triangledown_p : B \vee_p B \to B$ is given by $\triangledown_p(x_i) = x$ for $i = 1, 2$ [2].

Definition 3.1. (X, τ) is called pre-Hausdorff space ($PreT_2$) [10] or [7] for each point x distinct from p, the set $\{x, p\}$ is not indiscrete, then the points x and p have disjoint neighborhoods.

Theorem 3.2. *Let (X, τ) be a topological space and $p \in X$. (X, τ) is $preT_2$ at p if and only if the initial topology induced by $A_p : X \vee_p X \to (X^2, \tau_*)$ and $S_p : X \vee_p X \to (X^2, \tau_*)$ are the same, where τ_* is product topology on X^2.*

Proof. The proof is given in [7]. □

Definition 3.3. Let $U : \mathcal{E} \to$ ***Set*** be a topological functor, X an object in $\mathcal{E}$ with $U(X) = B$ and p be a point in B. If the initial lift of the U-source $S_p : B\vee_p B \to U(X^2) = B^2$ and the initial lift of the U-source $A_p : B\vee_p B \to U(X^2) = B^2$ coincide, then X is said to be $PreT_2$ at p [2] or [6].

Theorem 3.4. *Let α_i, $i = 1, 2, 3$ be proper filter on B. If $\sigma = (\pi_1 A_p)^{-1}\alpha_1 \cup (\pi_2 A_p)^{-1}\alpha_2 \cup (\pi_2 S_p)^{-1}\alpha_3$, then*
(1) *σ is a proper filter if and only if either*
(a) $\alpha_2 \subset [p]$ and $\alpha_1 \cup \alpha_3$ is proper
or
(b) $\alpha_1 \subset [p]$ and $\alpha_1 \cup \alpha_3$ is proper.
(2) *If σ is a proper, then*

$$\pi_1 A_p\sigma = \begin{cases} [p], & \textit{if (a) fails} \\ \alpha_1 \cup \alpha_3, & \textit{if (b) fails} \\ [p] \cap (\alpha_1 \cup \alpha_3), & \textit{neither fails} \end{cases}$$

$$\pi_2 A_p\sigma = \begin{cases} \alpha_2 \cup \alpha_3, & \textit{if (a) fails} \\ [p], & \textit{if (b) fails} \\ [p] \cap (\alpha_2 \cup \alpha_3), & \textit{neither fails} \end{cases}$$

$$\pi_2 S_p\sigma = \begin{cases} \alpha_2 \cup \alpha_3, & \textit{if (a) fails} \\ \alpha_1 \cup \alpha_3, & \textit{if (b) fails} \\ \alpha_3 \cup (\alpha_1 \cap \alpha_2), & \textit{neither fails} \end{cases}$$

Proof. It is proved in [3].

□

Theorem 3.5. *Let α_i, $i = 1, 2, 3$ be proper filter on B. There exists a proper filter σ on $B \vee_p B$ such that $\pi_1 A_p\sigma = \alpha_1, \pi_2 A_p\sigma = \alpha_2$, and $\pi_2 S_p\sigma = \alpha_3$ if and only if*
***(1)** If (a) of Theorem 3.4 fails, then $\alpha_2 = \alpha_3$ and $\alpha_1 = [p]$.*
***(2)** If (b) of Theorem 3.4 fails, then $\alpha_1 = \alpha_3$ and $\alpha_2 = [p]$.*
***(3)** If neither (a) nor (b) of Theorem 3.4 fails, then $\alpha_1 \cap \alpha_2 = \alpha_3 \cap [p]$.*

Proof. It follows easily from Theorem 3.4.

□

Let (B, K) be a constant filter convergence space, $p \in B$, and $K_p = \{\alpha : \alpha \subset [p]\}$.

Theorem 3.6. *A constant filter convergence space (B, K) is $PreT_2$ at p if and only if the following conditions hold.*
(1) K_p is closed under finite intersection.
(2) For any $\alpha \in K_p$ and $\beta \in K$ if $\alpha \cup \beta$ is proper and $\beta \cap [p] \subset \alpha$, then $\beta \cap [p] \in K$.

Proof. Suppose (B, K) is $PreT_2$ at p and $\alpha, \beta \in K_p$. If we let $\alpha_1 = \alpha$, $\alpha_2 = \beta$, and $\alpha_3 = \alpha \cap \beta$ in Theorem 3.4, then $\alpha_1 \cup \alpha_3 = \alpha$ and $\alpha_2 \cup \alpha_3 = \beta$ are proper

$$\alpha_1, \alpha_2 \subset [p]$$

and

$$\alpha_1 \cap \alpha_2 = \alpha \cap \beta = \alpha_3 \cap [p]$$

Hence by Theorem 3.5 (3) there exists a proper filter σ on the wedge such that

$$\pi_1 A_p\sigma = \alpha$$

$$\pi_2 A_p\sigma = \beta$$

and

$$\pi_2 S_p\sigma = \alpha \cap \beta$$

Since $\pi_1 A_p\sigma, \pi_2 A_p\sigma \in K$ and (B, K) is $PreT_2$ at p, it follows that $\pi_2 S_p\sigma = \alpha \cap \beta \in K$.

We now show that condition (2) holds. Suppose that $\alpha \in K_p$ and $\beta \in K$ for which $\alpha \cup \beta$ is proper and $\beta \cap [p] \subset \alpha$. In Theorem 3.5, letting $\alpha_1 = \alpha, \alpha_2 = \beta \cap [p]$, and $\alpha_3 = \beta$. It follows that

$$\alpha_1, \alpha_2 \subset [p]$$

both

$$\alpha_1 \cup \alpha_2 = \alpha \cup (\alpha \cap \beta) = \alpha \cup \beta$$

and

$$\alpha_2 \cup \alpha_3 = \beta$$

are proper and

$$\alpha_1 \cap \alpha_2 = \alpha \cap (\beta \cap [p]) = \beta \cap [p] = \alpha_3 \cap [p]$$

Hence by Theorem 3.5(3) there exists a proper filter σ on the wedge such that

$$\pi_1 A_p \sigma = \alpha$$

$$\pi_2 A_p \sigma = \beta \cap [p]$$

and

$$\pi_2 S_p \sigma = \beta$$

Since $\pi_1 S_p \sigma \in K$, $\pi_2 S_p \sigma \in K$ and (B, K) is $PreT_2$ at p, it follows that $\pi_2 A_p \sigma = \beta \cap [p] \in K$. This shows (2) holds. Conversely, suppose conditions (1) and (2) hold. We show that (B, K) is $PreT_2$ at p. We first show that for any filter σ on the wedge if $\pi_1 S_p \sigma$ and $\pi_2 S_p \sigma$ are in K, then $\pi_2 A_p \sigma \in K$. Note that $\pi_1 A_p \sigma = \pi_1 S_p \sigma$. If σ is improper, then clearly $\pi_2 A_p \sigma \in K$. If σ is proper, then in case 1 of Theorem 3.5,

$$\pi_1 S_p \sigma = [p]$$

and

$$\pi_2 A_p \sigma = \pi_2 S_p \sigma$$

and consequently $\pi_2 A_p \sigma \in K$ since $\pi_2 S_p \sigma \in K$. In case 2 of Theorem 3.5 we have

$$\pi_2 A_p \sigma = [p]$$

and

$$\pi_2 S_p \sigma = \pi_1 S_p \sigma$$

and thus, $\pi_2 S_p \sigma \in K$. In case 3 of Theorem 3.5, we have in particular

$$\pi_1 S_p \sigma \subset [p]$$

$$\pi_1 S_p \sigma \cup \pi_2 S_p \sigma$$

is proper and

$$\pi_1 A_p \sigma \supset \pi_2 S_p \sigma \cap [p]$$

Hence, by the condition (2) with $\alpha = \pi_1 A_p \sigma$ and $\beta = \pi_2 S_p \sigma$, we get $\pi_2 S_p \sigma \cap [p] \in K$ and consequently, $\pi_2 A_p \sigma \in K$ since

$$\pi_2 A_p \sigma \supset \pi_2 S_p \sigma \cap [p]$$

and

$$\pi_2 S_p \sigma \cap [p] \in K$$

It remains to show that if $\pi_1 A_p \sigma$ and $\pi_2 A_p \sigma$ are in K, then $\pi_2 S_p \sigma \in K$. In case 1 of Theorem 3.5, we have $\pi_1 A_p \sigma = [p]$ and $\pi_2 S_p \sigma = \pi_2 A_p \sigma$, and consequently $\pi_2 S_p \sigma \in K$ since $\pi_2 A_p \sigma \in K$. In case 2 of Theorem 3.5 we have $\pi_2 A_p \sigma = [p] \in K$. In case 3 of Theorem 3.5, we have in particular,

$$\pi_1 A_p \sigma \subset [p]$$

$$\pi_2 A_p \sigma \subset [p]$$

and

$$\pi_1 A_p \sigma \cap \pi_2 A_p \sigma = \pi_2 S_p \sigma \cap [p]$$

Hence by condition (1), we get $\pi_1 A_p \sigma \cap \pi_2 A_p \sigma \in K$ and consequently $\pi_2 S_p \sigma \in K$. Thus, by Definition 3.3, (B, K) is $PreT_2$. □

Theorem 3.7. *(1) If a constant filter convergence space (B,K) is preT_2 at p and $M \subset B$ with $p \in M$, then M is preT_2 at p.*

(2) For all $i \in I$ and $p_i \in B_i$, (B_i, K_i) preT_2 at p_i if and only if $(B = \prod_{i\in I} B_i, K)$ is preT_2 at p, where K is the product structure on B and $p = (p_1, p_2, ...)$.

(3) If (B_i, K_i) T_1 at p_i for all $i \in I$ and $p_i \in B_i$, then $(B = \coprod_{i\in I} B_i, K)$ is preT_2 at (i, p), where K is the coproduct structure on B and $(i, p) \in B$.

Proof. (1) Let K_M be the initial structure on M induced by the inclusion map $i : M \subset B$ and $\alpha, \beta \in (K_M)_p$. By 2.1, $i(\alpha), i(\beta) \in K$ and by Theorem 3.6, $i(\alpha \cap \beta) \in K$ and by 2.1, $\alpha \cap \beta \in K_M$.
Suppose $\alpha \in (K_M)_p$ and $\beta \in K_M$ for which $\alpha \cup \beta$ is proper and $\beta \cap [p] \subset \alpha$. It follows from 2.1 that

$$i(\alpha), i(\beta) \in K_p, i(\alpha) \cup i(\beta) = i(\alpha \cup \beta)$$

is proper and

$$i(\beta) \cap i([p]) = i(\beta \cap [p]) \subset i(\alpha)$$

By Theorem 3.6, $i(\beta \cap [p]) \in K$ and by 2.1, $\beta \cap [p] \in K_M$. Hence, (M, K_M) is *preT_2* at p.

(2) Suppose that $(B = \prod_{i\in I} B_i, K)$ is *preT_2* at p. Since each (B_i, K_i) is isomorphic to a subspace of (B, K), it follows from part (1) that (B_i, K_i) is *preT_2* at p_i for all $i \in I$ and $p_i \in B_i$.
Suppose that (B_i, K_i) is *preT_2* at p_i for all $i \in I$, $p_i \in B_i$ and $\alpha, \beta \in K_p$, where $p = (p_1, p_2, ...)$. By 2.1, $\pi_i(\alpha), \pi_i(\beta) \in ((K_i)_{p_i}$ for all $i \in I$. Since (B_i, K_i) is *preT_2* at p_i for all $i \in I$, by Theorem 3.6,

$$\pi_i(\alpha) \cap \pi_i(\beta) = \pi_i(\alpha \cap \beta) \in (K_i)_{p_i}$$

and by 2.1, $\alpha \cap \beta \in K_p$.
Suppose that $\alpha \in K_p$ and $\beta \in K$ with $\alpha \cup \beta$ is proper and $\beta \cap [p] \subset \alpha$. For all $i \in I$

$$\pi_i(\alpha) \in (K_i)_{p_i}, \pi_i(\beta) \in K_i$$

$\pi_i(\alpha \cup \beta)$ is proper and

$$\pi_i(\beta \cap [p]) \subset \pi_i(\alpha).$$

By Theorem 3.5,

$$\pi_i(\beta \cap [p]) = \pi_i(\beta) \cap \pi_i[p] = \pi_i(\beta) \cap [p_i] \in K_i$$

since (B_i, K_i) is *preT_2* at p_i for all $i \in I$ and by 2.1, $\beta \cap [p] \in K$. Hence, (B, K) is *preT_2* at p.

(3) Suppose that (B_i, K_i) *preT_2* at p_i for all $i \in I$, $p_i \in B_i$, $(B = \coprod_{i\in I} B_i, K)$, where K is the coproduct structure on B and $(i, p) \in B$.
If $\alpha, \beta \in K_{(i,p)}$, then by 2.2, there exist $\delta, \gamma \in (K_i)_{p_i}$ such that $i(\delta) \subset \alpha$ and $i(\gamma) \subset \beta$. Note that

$$i(\delta \cap \gamma) = i(\delta) \cap i(\gamma) \subset \alpha \cap \beta.$$

Since (B_i, K_i) is *preT_2* at p_i, by Theorem 3.6, $\delta \cap \gamma \in (K_i)_{p_i}$ and by 2.2, $\alpha \cap \beta \in K_{(i,p)}$.
Suppose that $\alpha \in K_{(i,p)}$ and $\beta \in K$ with $\alpha \cup \beta$ is proper and $\beta \cap [(i, p)] \subset \alpha$. Then there exist $\delta \in (K_i)_{p_i}$ and $\gamma \in K_i$ such that $i(\delta) \subset \alpha$ and $i(\gamma) \subset \beta$. Note that

$$i(\delta) \cup i(\gamma) = i(\delta \cup \gamma)$$

is proper and

$$i(\gamma \cap [p_i]) = i(\gamma) \cap [(i, p)] \subset \alpha$$

implies $\delta \cup \gamma$ is proper and

$$\gamma \cap [p_i] \subset \delta.$$

Since (B_i, K_i) is *preT_2* at p_i for all $i \in I$, by Theorem 3.6, $\gamma \cap [p_i] \in (K_i)_{p_i}$ and by 2.2, $\beta \cap [(i, p)] \in K_{(i,p)}$. Hence, by Theorem 3.6, (B, K) is *preT_2* at p.

□

Remark 3.8. In a topological category over the category of sets, $preT_2$ at p could be only indiscrete objects [4] (a topological category is the category of stack convergence spaces [23]), they could be all objects of the category [5] (a topological category is the category of constant limit convergence spaces [5]), and they could be only discrete objects [8] (a topological category is the category of local filter convergence spaces [23])

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