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Solutions of Nonlinear Delay Differential Equations by Daftardar-Jafari Method

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Abstract. In this paper efficiency of Daftardar-Jafari Method (DJM) is compared with some other iterative methods, Adomian Decomposition Method (ADM) and Differential Transformation Method (DTM) in obtaining approximate solutions of nonlinear differential equations with proportional delay. As all these 3 methods consider solutions in series form it is important to know about their convergence rate to the exact solution. We analyse the n-term approximation of the series solutions of these 3 methods with 3 numerical examples to see if DJM is as good as ADM and DTM in solving nonlinear delay differential equations. The results show DJM approaches to the exact solution with fewer iterations compared with ADM and DTM.

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1. Introduction

Delay Differential Equations (DDE) have wide range of applications in science and engineering where the rate of change of the systems depends not only on their present stage but also on their history. Nonlinearity is also another key issue in describing almost all physical situations. There are methods commonly used to solve nonlinear problems like perturbation, linearization, Picard‘s method and Newton‘s method which may produce improper results due to simplifying the nonlinear model [1].

In 1980‘s Adomian Decomposition Method (ADM) was developed by George Adomian [1, 2]. In 1986 Zhou [11] used Differential Transformation Method (DTM) in electric circuit analysis. Both ADM and DTM have been shown to solve effectively and accurately large class of linear and nonlinear, deterministic and stochastic, ordinary or partial differential equations. In 2006 Daftardar-Gejji and Jafari [5] proposed an iterative method (DJM) to solve nonlinear functional equations.

Linear and nonlinear delay differential equations are studied by many researchers using the ADM and DTM. Evans and Raslan [6], and Ogunfiditimi [8] used ADM for solving linear and nonlinear delay differential equations. Cakir and Arslan [4] compared results of ADM and DTM on linear and nonlinear differential equations with proportional delay. Mirzaee and Latifi [7] used DTM for solving linear and nonlinear delay differential equations.

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Although Patade and Bhalekar [9] applied DJM to a nonlinear differential equation, to the author‘s knowledge there is not a research focused on DJM‘s application on nonlinear delay differential equations in literature.

In this paper we investigate the efficiency of DJM compared with ADM and DTM on initial value problems of ordinary nonlinear differential equations with proportional delay through 3 numerical examples.

The paper is organized as Section 2 to illustrate briefly the theory of DJM, ADM and DTM, Section 3 to show the application of these 3 methods to 3 numerical problems and analysis of the results, and Section 4 to give a discussion and conclusion.

2. Theory of methods

2.1. **Daftardar-jafari method, DJM.** DJM is an iterative method that can reach at solutions of nonlinear functional equations in series form. In most of the cases it is not practical to find all terms of the series that constitute the solutions. By taking into account the first n-terms of the series derived from DJM we obtain approximate solutions of the problems.

A general equation can be written as;

$$y = N(y) + g \tag{2.1}$$

where $\boldsymbol{N}$ is a nonlinear operator and $\boldsymbol{g}$ is any function. DJM decomposes the solution $\boldsymbol{y}$ into a series as;

$$y = \sum_{n=0}^{\infty} y_n \tag{2.2}$$

With the series expansion (2.2), as shown in [5], nonlinear operator N in (2.1) can be written as;

$$N(\sum_{n=0}^{\infty} y_n) = N(y_0) + \sum_{n=1}^{\infty}(N(\sum_{m=0}^{n} y_m) - N(\sum_{m=0}^{n-1} y_m)) \tag{2.3}$$

From (2.2) and (2.3), (2.1) can be re-written as;

$$y_0 + y_1 + \sum_{n=2}^{\infty} y_n = g + N(y_0) + \sum_{n=1}^{\infty}(N(\sum_{m=0}^{n} y_m) - N(\sum_{m=0}^{n-1} y_m))$$

Taking $y_0 = g$ and $y_1 = N(y_0)$ we get the following recurrence relation [5];

$$\begin{aligned} y_0 &= g \\ y_1 &= N(y_0) \\ &\vdots \\ y_{m+1} &= N(y_0 + y_1 + ... + y_m) - N(y_0 + y_1 + ... + y_{m-1}) \qquad m = 1, 2, ... \end{aligned} \tag{2.4}$$

This yields;

$$(y_1 + y_2 + ... + y_{m+1}) = N(y_0 + y_1 + ... + y_m) \qquad m = 1, 2, ...$$

From this we can write (2.1) as;

$$y = g + \sum_{n=1}^{\infty} y_n$$

It is shown in [3] that if N is a contraction then $y = \sum_{n=0}^{\infty} y_n$ is absolutely and uniformly convergent and converges to the unique $\boldsymbol{y}$ in view of Banach fixed point theorem.

The approximate solution of the $k-$term is shown as;

$$y \approx \sum_{n=0}^{k} y_n$$

2.2. **Adomian decomposition method, ADM.** ADM is an iterative method which can be applied to many linear and nonlinear (also stochastic and deterministic) differential (and algebraic) problems.

A general form of a deterministic differential equation can be written as;

$$L(y) + R(y) + N(y) = g$$

where L is the highest order linear differential operator ($L = \frac{d^n}{dt^n}$; with invertibility assumption), R is the remainder linear operator and N is the nonlinear operator and g is any function [1]. By applying the inverse operator L^{-1} we get the following;

$$L^{-1}(L(y)) = L^{-1}(g) - L^{-1}(R(y)) - L^{-1}(N(y)) \tag{2.5}$$

where for initial value problems the inverse operator L^{-1} is conveniently defined as the n-fold definite integral operator from 0 to t. ADM decomposes the solution into a series;

$$y = \sum_{n=0}^{\infty} y_n \tag{2.6}$$

Then

$$L^{-1}(L(y)) = L^{-1}(L(\sum_{n=0}^{\infty} y_n)) = L^{-1}(g) - L^{-1}(R(\sum_{n=0}^{\infty} y_n)) - L^{-1}(N(y))$$

As N is a nonlinear operator, $N(y)$ can not be evaluated as $N(y_0) + N(y_1) + ...$

ADM replaces $N(y)$ with a series of so called Adomian polynomials (A_n s) which are generated for the particular nonlinearity of the operator N.

Thus we have;

$$N(y) = \sum_{n=0}^{\infty} A_n$$

By the definition of L^{-1} and calculating y_0 which is the first term of the series y_n as the sum of $L^{-1}g$ and terms derived from the initial conditions, (2.6) can be written as follows;

$$y = \sum_{n=0}^{\infty} y_n = y_0 - L^{-1}(R(\sum_{n=0}^{\infty} y_n)) - L^{-1}(\sum_{n=0}^{\infty} A_n)$$

where A_n is formulated as follows [1];

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\Big[N(\sum_{i=0}^{\infty} \lambda^i y_i)\Big]_{\lambda=0} \qquad n = 0, 1, 2, ... \tag{2.7}$$

where N is the nonlinear term and λ is any parameter used for convenience.

Convergence of $\sum_{n=0}^{\infty} A_n$ is shown in [3] basing on the assumption that nonlinear operator $\boldsymbol{N}$ is a contraction in Banach space.

So that we reach at the following recursive relation;

$$\begin{aligned} y_1 &= -L^{-1}(R(y_0)) - L^{-1}(A_0) \\ y_2 &= -L^{-1}(R(y_1)) - L^{-1}(A_1) \\ &\vdots \\ y_n &= -L^{-1}(R(y_{n-1})) - L^{-1}(A_{n-1}) \end{aligned} \tag{2.8}$$

As y_0 is calculated from the initial conditions and A_n depends only on $(y_0, y_1, ..., y_n)$ we can find all y_n and A_n respectively.

In [3] it is shown that ADM and DJM generates series solution $y = \sum_{n=0}^{\infty} y_n$ converging to the same limit.

The approximate solution of the $k-$term is shown as;

$$y \approx \sum_{n=0}^{k} y_n$$

2.3. **Differential transformation method, DTM.** When $y(x)$ is an analytic function it can be expanded in Taylor series about a point $x = x_0$ as

$$y(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k y(x)}{dx^k}\bigg|_{x=x_0} (x - x_0)^k$$

Differential Transformation of $y(x)$, $Y(k)$, is defined as;

$$Y(k) = \frac{1}{k!} \frac{d^k y(x)}{dx^k}\bigg|_{x=x_0} \qquad k = 0, 1, 2, ...$$

Differential inverse transform of ***Y(k)*** is defined as follows;

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x - x_0)^k$$

The n terms approximation is shown as;

$$y(x) \approx \sum_{k=0}^{n} \frac{1}{k!} \frac{d^k y(x)}{dx^k}\bigg|_{x=x_0} (x - x_0)^k$$

This shows that concept of Differential Transformation is derived from the Taylor series expansion. The fundamental operations of DTM performed at $x = 0$ is shown in Table 1.

$OriginalFunction$	$TransformedFunction$
$y(x) = u(x) \mp v(x)$	$Y(k) = U(k) \mp V(k)$
$y(x) = cu(x)$	$Y(k) = cU(k)$
$y(x) = u(x)v(x)$	$Y(k) = \sum_{l=0}^{k} U(l)V(k-l)$
$y(x) = \frac{d^n u(x)}{dx^n}$	$Y(k) = \frac{(k+n)!}{k!} U(k+n)$
$y(x) = x^n$	$Y(k) = \delta(k-n) = \begin{cases} 1, & \text{if} \quad k = n \\ 0, & \text{if} \quad k \neq n \end{cases}$
$y(x) = e^{\lambda x}$	$Y(k) = \frac{\lambda^k}{k!}$
$y(x) = u(\frac{x}{a})$	$Y(k) = \frac{1}{a^k} U(k)$

Table 1. Most used Dif. Transform operators

3. Numerical Applications

In this section we consider numerical examples to compare the efficiency of DJM with ADM and DTM in solving nonlinear delay differential equations.

Example 1. Nonlinear proportional delay differential equation

$$\frac{d^3 y(x)}{dx^3} = -1 + 2y^2(\frac{x}{2}), \quad 0 \leq x \leq 1 \tag{3.1}$$

$$y(0) = 0, \qquad \frac{dy(0)}{dx} = 1, \qquad \frac{d^2y(0)}{dx^2} = 0$$

Exact solution is $y = Sin(x)$

Solving with DJM:
By taking 3-fold integration of both sides of (3.1) we get the standard form for DJM;

$$y(x) = x - \frac{x^3}{6} + 2\int_0^x \int_0^x \int_0^x y^2(\frac{x}{2})dxdxdx$$

Then, in accordance with (2.4) we have

$$y_0(x) = x - \tfrac{x^3}{6}$$

$$y_1(x) = 2\int_0^x \int_0^x \int_0^x (\tfrac{x}{2} - \tfrac{(\frac{x}{2})^3}{6})dxdxdx = 2(\tfrac{x^5}{240} - \tfrac{x^7}{10080} + \tfrac{x^9}{1161216})$$

$$y_2(x) = 2\left[\int_0^x \int_0^x \int_0^x \left[\left(\tfrac{x}{2} - \tfrac{(\frac{x}{2})^3}{6} + 2(\tfrac{(\frac{x}{2})^5}{240} - \tfrac{(\frac{x}{2})^7}{10080} + \tfrac{(\frac{x}{2})^9}{1161216})\right)^2 - \left(\tfrac{x}{2} - \tfrac{(\frac{x}{2})^3}{6}\right)^2\right] dxdxdx\right]$$

$$= -2(\tfrac{x^5}{240} - \tfrac{x^7}{10080} + \tfrac{x^9}{1161216}) + 2(\tfrac{x^5}{240} - \tfrac{x^7}{10080} + \tfrac{x^9}{725760} - \tfrac{x^{11}}{79833600} + \tfrac{1009x^{13}}{12752938598400}$$

$$-\tfrac{13x^{15}}{37456183296000} + \tfrac{83x^{17}}{81504654852096000} - \tfrac{x^{19}}{557491839188336640} + \tfrac{x^{21}}{7051943829356465356800})$$

$$y_3(x) = 2\left[\int_0^x \int_0^x \int_0^x \left[\left(y_0(\tfrac{x}{2}) + y_1(\tfrac{x}{2}) + y_2(\tfrac{x}{2})\right)^2 - \left(y_0(\tfrac{x}{2}) + y_1(\tfrac{x}{2})\right)^2\right] dxdxdx\right]$$

$$y_4(x) = 2\left[\int_0^x \int_0^x \int_0^x \left[\left(y_0(\tfrac{x}{2}) + y_1(\tfrac{x}{2}) + y_2(\tfrac{x}{2}) + y_3(\tfrac{x}{2})\right)^2 - \left(y_0(\tfrac{x}{2}) + y_1(\tfrac{x}{2}) + y_2(\tfrac{x}{2})\right)^2\right] dxdxdx\right]$$

$$\vdots$$

Solving with ADM:
With the choice of inverse operator

$$L^{-1} : \int_0^x \int_0^x \int_0^x (.)dxdxdx$$

in accordance with (2.5), equation (3.1) can be written as;

$$y(x) = x - \frac{x^3}{6} + 2L^{-1}y^2(\frac{x}{2})$$

By (2.5), from the initial conditions we see that $y_0(x) = x - \frac{x^3}{6}$, and the nonlinear term is $y^2(\frac{x}{2})$

Then, in accordance with (2.7) and (2.8) Adomian polynomials and y_n are calculated as follows;

$$A_0 = y_0^2(\tfrac{x}{2}) = (\tfrac{x}{2} - \tfrac{(\frac{x}{2})^3}{6})^2$$

$$y_1(x) = 2L^{-1}A_0 = 2(\tfrac{x^5}{240} - \tfrac{x^7}{10080} + \tfrac{x^9}{1161216})$$

$$A_1 = y_1(\tfrac{x}{2}).2y_0(\tfrac{x}{2}) = 4(\tfrac{x}{2} - \tfrac{x^3}{48})(\tfrac{x^5}{7680} - \tfrac{x^7}{1290240} + \tfrac{x^9}{594542592})$$

$$y_2(x) = 2L^{-1}A_1 = 2(\tfrac{x^9}{1935360} - \tfrac{x^{11}}{79833600} + \tfrac{101x^{13}}{2550587719680} - \tfrac{x^{15}}{19477215313920})$$

$$A_2 = 2y_2(\tfrac{x}{2}).y_0(\tfrac{x}{2}) + y_1^2(\tfrac{x}{2}) = 4(\tfrac{x^5}{7680} - \tfrac{x^7}{1290240} + \tfrac{x^9}{594542592})^2 + 4(\tfrac{x}{2} - \tfrac{x^3}{48}).$$

$$(\tfrac{x^9}{990904320} - \tfrac{x11}{163499212800} + \tfrac{101x^{13}}{20894414599618560} - \tfrac{x^{15}}{638229391406530560})$$

$$y_3(x) = 2\int_0^x\int_0^x\int_0^x A_2\,dx\,dx\,dx$$

$$\vdots$$

Solving with DTM:

Applying Differential Transformation at $x = 0$ in accordance with Table 1, we obtain;

$$(k+1)(k+2)(k+3)Y(k+3) = -\delta(k) + 2\sum_{l=0}^{k}\frac{1}{2^l}Y(l).\frac{1}{2^{k-l}}.Y(k-l)$$

From the initial conditions we get; $Y(0) = 0, Y(1) = 1, Y(2) = 0$

$$\begin{array}{ll} \text{For } k = 0; & 1.2.3.Y(3) = -1 + 2(Y(0))^2 \Rightarrow Y(3) = \frac{-1}{6} \\ k = 1; & Y(4) = 0 \\ k = 2; & Y(5) = \frac{1}{120} \\ k = 3; & Y(6) = 0 \\ k = 4; & Y(7) = \frac{1}{5040} \end{array}$$

With the definition of inverse transform, n=8 term approximate solution of (3.1) is obtained as;

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} Y(k)x^k \approx \sum_{k=0}^{n=8} Y(k)x^k \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 \end{aligned}$$

Comparison of Results of Example 1

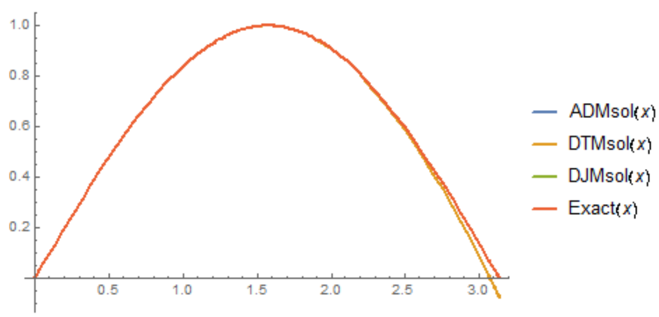
Results of DJM, ADM and DTM methods, with n=8 term approximation are listed in Table 2. Table 3 gives the Error Analysis of DJM, ADM and DTM compared with the exact solution.

	DJM	ADM	DTM
x=0.0	0	0	0
x=0.2	0.1986693308	0.1986693308	0.1986693308
x=0.4	0.3894183423	0.3894183423	0.3894183416
x=0.6	0.5646424734	0.5646424734	0.5646424457
x=0.8	0.7173560909	0.7173560909	0.7173557232
x=1.0	0.8414709848	0.8414709848	0.8414682539

Table 2. n=8-term approximation results

	DJM-Exact	ADM-Exact	DTM-Exact
x=0.0	0	0	0
x=0.2	$2.7755575616 \times 10^{-17}$	0	$-1.4103995749 \times 10^{-12}$
x=0.4	0	0	$-7.2134892504 \times 10^{-10}$
x=0.6	$-1.1102230246 \times 10^{-16}$	$-1.1102230246 \times 10^{-16}$	$-2.7680749692 \times 10^{-8}$
x=0.8	0	0	$-3.6772491962 \times 10^{-7}$
x=1.0	0	$1.1102230246 \times 10^{-16}$	$-2.7308396425 \times 10^{-6}$

Table 3. Error analysis



Example 2. Nonlinear proportional delay differential equation

$$\frac{d^2y(x)}{dx^2} = -y(x) + 5y^2(\frac{x}{2}), \quad x \geqslant 0 \tag{3.2}$$
$$y(0) = 1, \qquad \frac{dy(0)}{dx} = -2,$$

Exact solution is $y = e^{-2x}$

Solving with DJM:

By taking 2-fold integration of both sides of (3.2) we get the standard form for DJM;

$$y(x) = 1 - 2x - \int_0^x \int_0^x y(x)dxdx + 5\int_0^x \int_0^x y^2(\frac{x}{2})dxdx$$

In accordance with (2.4) we have

$$y_0(x) = 1 - 2x$$

In (3.2) we have a linear y term as well. So we set $y_1 = Ly_0 + Ny_0 = \int_0^x \int_0^x y_0(x)dxdx + 5\int_0^x \int_0^x y_0^2(\frac{x}{2})dxdx$,

and in accordance with (2.4) we have;

$y_{m+1} = L(y_0 + y_1 + ... + y_m) + N(y_0 + y_1 + ... + y_m) - N(y_0 + y_1 + ... + y_{m-1})$

which in this example is;

$$y_{m+1} = -\int_0^x\int_0^x(y_0(x)+y_1(x)...+y_m(x))dxdx + 5[\int_0^x\int_0^x(y_0(\tfrac{x}{2})+y_1(\tfrac{x}{2})+y_2(\tfrac{x}{2})+...+y_m(\tfrac{x}{2}))^2dxdx$$
$$-\int_0^x\int_0^x(y_0(\tfrac{x}{2})+y_1(\tfrac{x}{2})+...+y_{m-1}(\tfrac{x}{2}))^2dxdx]$$

So, we get;

$$y_1(x) = -\int_0^x\int_0^x(1-2x)dxdx + 5\int_0^x\int_0^x(1-2(\frac{x}{2}))^2dxdx = -\frac{x^2}{2}+\frac{x^3}{3}+5(\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{12})$$

$$y_2(x) = -\tfrac{x^4}{6}+\tfrac{x^5}{15}-\tfrac{x^6}{72}$$

$$+5(\tfrac{x^4}{12}-\tfrac{x^5}{15}+\tfrac{61x^6}{2880}-\tfrac{x^7}{192}+\tfrac{31x^8}{32256}-\tfrac{5x^9}{41472}+\tfrac{5x^{10}}{663552})$$

$$y_3(x) = -\int_0^x\int_0^x[(y_0(x)+y_1(x)+y_2(x))]\,dxdx$$

$$+5\left[\int_0^x\int_0^x\left(y_0(\tfrac{x}{2})+y_1(\tfrac{x}{2})+y_2(\tfrac{x}{2})\right)^2dxdx - \int_0^x\int_0^x\left(y_0(\tfrac{x}{2})+y_1(\tfrac{x}{2})\right)^2dxdx\right]$$
$$\vdots$$

Solving with ADM:

$$L^{-1}:\int_0^x\int_0^x(.)dxdx$$

Applying L^{-1} to both sides of (3.2) we get;

$$y(x) = 1-2x-L^{-1}y(x)+5L^{-1}y^2(\frac{x}{2})$$

In accordance with (2.5), from the initial conditions we have $y_0(x) = 1-2x$ and the nonlinear term is $y^2(\frac{x}{2})$.

In accordance with (2.7) and (2.8) Adomian polynomials and y_n are calculated as follows;

$$A_0 = y_0^2(\tfrac{x}{2}) = (1-x)^2$$

$$y_1(x) = -L^{-1}y_0(x)+5L^{-1}A_0 = -\tfrac{x^2}{2}+\tfrac{x^3}{3}+5(\tfrac{x^2}{2}-\tfrac{x^3}{3}+\tfrac{x^4}{12})$$

$$A_1 = y_1(\tfrac{x}{2}).2y_0(\tfrac{x}{2}) = 2(1-x)(-\tfrac{x^2}{8}+\tfrac{x^3}{24}+5(\tfrac{x^2}{8}-\tfrac{x^3}{24}+\tfrac{x^4}{192}))$$

$$y_2(x) = -\tfrac{x^4}{6}+\tfrac{x^5}{15}-\tfrac{x^6}{72}+5(\tfrac{x^4}{12}-\tfrac{x^5}{15}+\tfrac{37x^6}{2880}-\tfrac{5x^7}{4032})$$

$$A_2 = 2y_2(\tfrac{x}{2}).y_0(\tfrac{x}{2})+y_1^2(\tfrac{x}{2}) = \left[-\tfrac{x^2}{8}+\tfrac{x^3}{24}+5(\tfrac{x^2}{8}-\tfrac{x^3}{24}+\tfrac{x^4}{192})\right]^2+2(1-x).$$

$$(-\tfrac{x^4}{96}+\tfrac{x^5}{480}-\tfrac{x^6}{4608}+5(\tfrac{x^4}{192}-\tfrac{x^5}{480}+\tfrac{37x^6}{184320}-\tfrac{5x^7}{516096}))$$
$$\vdots$$

Solving with DTM:
Applying Differential Transform at $x = 0$ in accordance with Table 1 we obtain the following;

$$(k+1)(k+2)Y(k+2) = -Y(k)+5\sum_{l=0}^{k}\frac{1}{2^k}.Y(l).Y(k-l)$$

From the initial conditions of (3.2) we get; $Y(0) = 1$, and $Y(1) = -2$

$$\begin{array}{rl} \text{For } k=0; & 1.2.Y(2) = -Y(0) + 5(Y(0))^2 \Rightarrow Y(2) = 2 \\ k=1; & Y(3) = -\frac{4}{3} \\ k=2; & Y(4) = \frac{2}{3} \\ k=3; & Y(5) = -\frac{4}{15} \\ k=4; & Y(6) = \frac{4}{45} \\ k=5 & Y(7) = -\frac{8}{315} \\ k=6 & Y(8) = \frac{2}{315} \end{array}$$

With the definition of inverse transform, n=8 term approximate solution is obtained as;

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} Y(k)x^k \approx \sum_{k=0}^{8} Y(k)x^k \\ &= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5 + \frac{4}{45}x^6 - \frac{8}{315}x^7 + \frac{2}{315}x^8 \end{aligned}$$

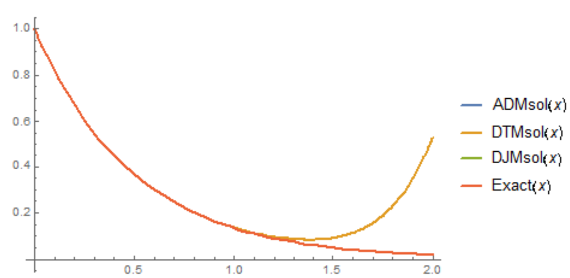
Comparison of Results of Example 2

	DJM	ADM	DTM
x=0.0	1	1	1
x=0.2	0.670320046	0.670320046	0.6703200467
x=0.4	0.4493289641	0.4493289641	0.4493293064
x=0.6	0.3011942119	0.3011942119	0.3012068937
x=0.8	0.201896518	0.201896518	0.2020594753
x=1.0	0.1353352832	0.1353352832	0.1365079365

Table 4. n=8-term approximation results

	DJM-Exact	ADM-Exact	DTM-Exact
x=0.0	0	0	0
x=0.2	$-1.110223025 \times 10^{-16}$	$-1.110223025 \times 10^{-16}$	$6.945193309 \times 10^{-10}$
x=0.4	$5.551115123 \times 10^{-17}$	0	$3.422954769 \times 10^{-7}$
x=0.6	$5.551115123 \times 10^{-17}$	$-5.551115123 \times 10^{-17}$	0.00001268180208
x=0.8	$3.053113318 \times 10^{-16}$	$-9.464651285 \times 10^{-15}$	0.0001629573069
x=1.0	$3.330669074 \times 10^{-16}$	$-4.770628337 \times 10^{-13}$	0.001172653271

Table 5. Error analysis



Example 3. Nonlinear proportional delay differential equation

$$\frac{d^2y(x)}{dx^2} = y(x) - \frac{8}{x^2}y^2(\frac{x}{2}), \quad x > 0 \tag{3.3}$$

$$y(0) = 1, \qquad \frac{dy(0)}{dx} = 1.$$

Exact solution is $y = xe^{-2x}$

Solving with DJM:
By taking 2-fold integration of both sides of (3.3) we get the standard form for DJM.

$y(x) = x + \int_0^x \int_0^x y(x) \; dxdx - \int_0^x \int_0^x \frac{8}{x^2} y^2(\frac{x}{2}) \; dxdx.$

We have;

$$y_0(x) = x.$$

With this y_0 and nonlinear term, the singularity at $x = 0$ disappears.

y_n, (n $\geq$ 1) s are as follows;

$y_1(x) = -x^2 + \frac{x^3}{6}$

$y_2(x) = \frac{x^4}{12} + \frac{x^5}{120} - 8(-\frac{x^3}{24} + \frac{x^4}{144} - \frac{x^5}{1920} + \frac{x^6}{69120})$

$y_3(x) = \frac{x^5}{60} - \frac{x^6}{216} + \frac{x^7}{3360} - \frac{x^8}{483840} - 8(\frac{x^4}{288} - \frac{17x^5}{11520} + \frac{7x^6}{25600} - \frac{709x^7}{23224329} + \frac{83x^8}{37158912} - \frac{31x^9}{318504960} + \frac{293x^{10}}{143327232000} - \frac{x^{11}}{77856768000} + \frac{x^{12}}{40360948531200})$

$$\vdots$$

Solving with ADM:

$$L^{-1} : \int_0^x \int_0^x (.)dxdxdx$$

$$y_0(x) = x$$

$$\text{nonlinear term: } \; y^2(\frac{x}{2})$$

With this y_0 and nonlinear term, the singularity at $x = 0$ disappears.
We get;

$$y(x) = x + L^{-1}y(x) - 8L^{-1}\left(\frac{1}{x^2}y^2(\frac{x}{2})\right)$$

Adomian polynomials and y_n, (n $\geq$ 1) s are as follows;

$$A_0 = \frac{x^2}{4}$$

$$y_1(x) = -x^2 + \frac{x^3}{6}$$

$$A_1 = -\frac{x^3}{4} + \frac{x^4}{48}$$

$$y_2(x) = \frac{x^3}{3} - \frac{7x^4}{72} + \frac{x^5}{120}$$

$$A_2 = 5\frac{x^4}{48} - \frac{19x^5}{1152} - \frac{x^7}{1290240} + \frac{x^6}{1440}$$

$$y_3(x) = -\frac{5x^4}{72} + \frac{67x^5}{2880} - \frac{37x^6}{10800} + \frac{x^7}{5040}$$

$$\vdots$$

Solving with DTM:

To remove the singularity at $x = 0$ we follow [10]. Equation can be written as

$$x^2 \frac{d^2y(x)}{dx^2} = x^2y(x) - 8y^2(\tfrac{x}{2}).$$

We can now apply the differential transform operations of Table 1;

$$\sum_{l=0}^{k} \delta(l-2)(k-l+1)(k-l+2)Y(k-l+2) = \sum_{m=0}^{k} \delta(m-2).Y(k-m) - 8\sum_{j=0}^{k} \frac{1}{2^k} Y(j).Y(k-j)$$

From the initial conditions we have $Y(0) = 0, Y(1) = 1$

$$\begin{array}{ll} \text{For } k = 2; & 1.2.Y(2) = Y(0) - 2 \Rightarrow Y(2) = -1 \\ k = 3; & Y(3) = \frac{1}{2} \\ k = 4; & Y(4) = \frac{-1}{6} \\ k = 5; & Y(5) = \frac{1}{24} \\ k = 6; & Y(6) = \frac{-1}{120} \\ k = 7; & Y(7) = \frac{1}{720} \end{array}$$

With the definition of inverse transform, n=8 term approximate solution is obtained as;

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} Y(k)x^k \approx \sum_{k=0}^{8} Y(k)x^k \\ &= x - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 + \frac{1}{24}x^5 - \frac{1}{120}x^6 + \frac{1}{720}x^7 \end{aligned}$$

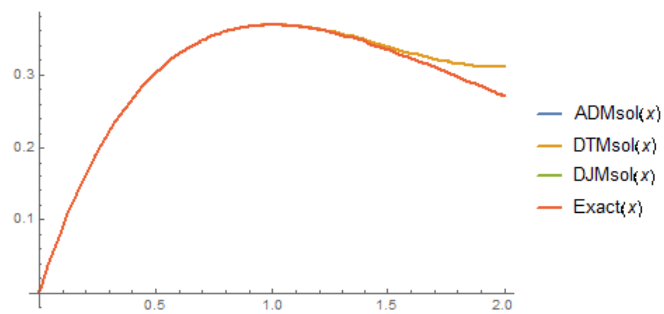
Comparison of Results of Example 3

	DJM	ADM	DTM
x=0.0	0	0	0
x=0.2	0.1637461506	0.1637461511	0.1637461506
x=0.4	0.2681280184	0.2681281422	0.2681280184
x=0.6	0.3292869817	0.3292869818	0.32929008
x=0.8	0.3594631713	0.3594631734	0.3594934044
x=1.0	0.3678794412	0.3678794591	0.3680555556

Table 6. n=8-term approximation results

	DJM-Exact	ADM-Exact	DTM-Exact
x=0.0	0	0	0
x=0.2	0	$2.72004641 \times 10^{-15}$	$4.955147404 \times 10^{-10}$
x=0.4	$-5.5511151123 \times 10^{-17}$	$2.545130773 \times 10^{-12}$	$1.238079664 \times 10^{-7}$
x=0.6	0	$1.325992649 \times 10^{-10}$	$3.098343584 \times 10^{-6}$
x=0.8	$-1.1110223025 \times 10^{-16}$	$2.127157339 \times 10^{-9}$	0.00003023315067
x=1.0	$-2.109423746 \times 10^{-15}$	$1.789524412 \times 10^{-8}$	0.0001761143841

Table 7. Error analysis



4. Conclusion

In 3 numerical examples of nonlinear delay differential equations it is seen that all 3 methods converge to exact solution very fast. When error analysis tables are examined we see DJM converges fastest and DTM converges slowest. However taking into consideration the requirement of solving $n-$fold integrals (according to the order of the differential equation) of ADM and DJM methods, we can say it is much more easier to increase the iteration number of DTM compared with the other methods which would definitely improve the accuracy of the DTM solution.
It is seen that DJM is a reliable alternative to ADM and DTM for solving nonlinear delay differential equations.

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