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On Surrogate Dual Search Method for Minimum-Cost Flow Problems

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Abstract. In this paper, we study on surrogate dual formulations which generate relaxations by assembling multiple constraints into a single surrogate constraint. Similar to the Lagrangian dual search methods for integer programming, the conventional surrogate dual method utilizes an auxiliary linear programming problem for updating the multiplier vector. The technique enlarges the feasible region of the original (primal) problem and provides a lower bound for the optimal objective value. This bound is tighter than the Lagrangian lower bound. In case there exists a duality gap, the conventional surrogate dual search method fails to find the optimal solutions of the primal problem. In order to eliminate this issue, nonlinear $p-$norm surrogate constraint methods can be used. To illustrate how we choose the initial multiplier vector or the parameter p, we argue on minimum-cost flow problems, in which we find the feasible flow from the source nodes to the sink nodes with minimum cost. Some integer programming problems, such as transportation problems, transshipment problems, assignment problems, shortest path problems (with or without time windows), and maximal flow problems can be seen those type of problems. Furthermore, we consider arrangements to solve those network problems which cannot be solved with the conventional surrogate dual method.

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Keywords: Surrogate dual search, Lagrangian dual search, duality gap, integer programming, minimum-cost flow problems.

1. Introduction

Consider the following primal integer programming problem:

$$\begin{array}{lll} (P) & \min & f(x) \\ & \text{s.t.} & g_i(x) \leq b_i \quad i = 1, 2, \ldots, m, \text{ (major constraints)} \\ & & x \in X, \text{ (discrete constraints)} \end{array}$$

where $X \subseteq \mathbb{Z}^n$ is a finite set of integer points. Define S to be the feasible region of the Problem (P):

$$S = \{x \in X \mid g_i(x) \leq b_i, \quad i = 1, 2, \ldots, m\},$$

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where g_i's can be linear or nonlinear.

By moving the constraints into the objective function, we generate Lagrange relaxation:

$$\begin{array}{ll} \min & f(x) + \mu^T(g(x) - b) \\ \text{s.t.} & x \in X, \end{array}$$

where $g(x) = (g_1(x), \ldots, g_m(x))^T$, $b = (b_1, \ldots, b_m)^T$and Lagrange multipliers vector $\mu = (\mu_1, \ldots, \mu_m)^T \geq 0$. For branch and bound procedures, it provides a lower bound on the optimal objective value of the Problem (P).

Similarly, by assembling multiple constraints into a single surrogate constraint, we generate surrogate relaxation:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & \mu^T(g(x) - b) \leq 0, \\ & x \in X, \end{array}$$

where surrogate multipliers vector $\mu = (\mu_1, \ldots, \mu_m)^T \geq 0$. The technique enlarges the feasible region S and also provides a lower bound on the optimal objective value of the Problem (P).

Suppose that an algorithm generates a nondecreasing sequence of lower bounds as $f_1 \leq f_2 \leq \ldots \leq f_k \leq \ldots \leq f^*$ for the Problem (P), we need to find the tightest relaxation, i.e., we should maximize f_k to be able to find the saddle point. One can define the corresponding Lagrangian dual problem:

$$\begin{array}{l} \max\limits_{\mu \geq 0} \theta_L(\mu) \\ \text{such that} \\ \theta_L(\mu) = \min\limits_{x} \left\{ f(x) + \mu^T(g(x) - b) : x \in X \right\} \end{array}$$

by generating a new objective function which absorbs the original constraints of the primal problem (P). In a similar manner, one can define the corresponding surrogate dual problem as:

$$\begin{array}{l} \max\limits_{\mu \geq 0} \theta_S(\mu) \\ \text{such that} \\ \theta_S(\mu) = \min\limits_{x} \left\{ f(x) : \mu^T(g(x) - b) \leq 0, x \in X \right\} \end{array}$$

by replacing the original constraints of the primal problem (P) with one surrogate constraint. The main idea of using a less restrictive (relaxed) model is to reduce in size and/or complexity relative to the original model, to get infeasible but near-optimal solutions, as in linear programming relaxation in integer programming. The complicating constraints can be relaxed by aggregating them with the help of non-negative surrogate multipliers.

Especially, we focus on minimum-cost flow problems which give the feasible flow from source nodes to sink nodes with minimum cost. Let us now review the relevant literature which consists of applications of surrogate relaxations to minimum-cost flow problems. Glover et al. [9] investigated relaxation strategies on large-scale manpower planning problem. In [13], the authors addressed some relationships between Lagrangian and surrogate bounds, and also provided some applications to minimum-cost flow problems. In [37, 38], the authors used surrogate constraints for network optimization problems. Kroon and Ruhe [15] presented a procedure based on surrogate constraints and parametric flows for a minimum-cost flow problem with simple additional constraints and integrality demand. Rogers et al. [31] surveyed application areas of surrogate modeling in terms of clustering methodology. Ruhe [32] demonstrated an interval scheduling problem and its solution via surrogate constraints and parametric programming as an application to network flow problems. Chu and Beasley [6] presented a heuristic approach based on genetic algorithm and surrogate relaxations for solving multidimensional knapsack problems. For other applications of the knapsack problems using surrogate relaxations, see also [14, 21, 27]. Narciso and Lorena [26] presented combined Lagrangian and surrogate relaxations to the problem of finding the optimal assignment of tasks to agents with maximum profit. The authors applied the same technique to the traveling salesman problem in [19]. For the minimum network synthesis problem, Wynants [40] took into consideration a decomposition scheme based on the surrogate principle which was named the block surrogate relaxation. Cappanera et al. [4] used both relaxations on the decomposition of environmental facility location problem. In [7], the authors examined scatter search method using surrogate relaxation for the bi-criteria multi-dimensional knapsack problem. Chen and Pinto [5] applied Lagrangian/surrogate relaxation method to chemical supply chain problem. Nagih and Soumis [24] proposed a procedure based on both Lagrangian and surrogate dualities to relax the resource constraints of the shortest path problem. Yu et al. [41] considered large-scale inventory routing problems which were solved by using a combined method of Lagrangian relaxation and surrogate subgradient methods. Molina et al. [22] studied a distribution and lot-sizing problem with the use of the Lagrangian/surrogate heuristic. Ablanedo-Rosas and Rego [1] introduced some surrogate constraint normalization rules for the set covering problems. For other applications of the set covering and minimum spanning tree problems with surrogate modeling, the reader may refer to [2, 8, 11, 20] and [28], respectively. Jain et al. [12] gave a new method for calculating the upper bounds on solutions of some binary integer programs with application to market split problems. Riley et al. [30] considered a cross-parametric relaxation method which combines surrogate and Lagrangian relaxations coupled with Lagrangian based subgradient search to generate good surrogate constraints. Shen et al. [34, 35] solved crude oil distribution planning problems by using surrogate subgradient method. Nassiffe et al. [25] developed Lagrangian and surrogate relaxations to compute bounds in order to use in a greedy heuristic for a energy consumption model. Monabbati [23] examined surrogate semi-Lagrangian relaxation for uncapacitated facility location problem. A literature survey of location models can be found in [29] and [33]. Robust models related to uncertainty are excluded from the scope of this work.

The main contribution of this framework is the application-based discussion of choosing the appropriate parameters. Our secondary goal in this paper is to determine relaxation strategies for reformulating some small size network models. The rest of this paper is organized as follows. In the next section, some preliminary information on surrogate duality is given by the aid of simple introductory examples and by comparing with Lagrangian duality. In Section 3, the conventional surrogate dual method is briefly given. Section 4 explains the Nonlinear p-norm surrogate constraint method which can be employed when the conventional surrogate dual method fails. In Section 5, some numerical examples are given. The paper is concluded with some remarks in Section 6.

2. Preliminaries

In the following theorem, it is proven that Lagrangian bounds are less than or equal to surrogate bounds. For well-documented detailed explanation, see [10]. So, the use of surrogate constraints in solving combinatorial optimization problems can provide a powerful alternative for calculating improved lower bounds for branch and bound procedures. Strong relaxations give more accurate approximations of original problems, but they should be easy to solve compared to original ones.

Theorem 2.1 (Strong Surrogate Duality)**.** *Surrogate lower bound is tighter than the Lagrangian lower bound.*

Proof. Suppose $\mu^T(g(x) - b) \leq 0$ *for any* $\mu \geq 0$ *and* $x \in X$, *then*

$$\begin{aligned}\theta_L(\mu) &= f(x^*) + \mu^T(g(x^*) - b)\\ &\leq f(x) + \mu^T(g(x) - b) \leq f(x), \forall x \in X \Longrightarrow\\ \theta_L(\mu) &\leq \min_x \left\{f(x) : \mu^T(g(x) - b) \leq 0, x \in X\right\} = \theta_S(\mu), \forall \mu \geq 0\end{aligned}$$

$$\Longrightarrow \max_{\mu\geq 0} \theta_L(\mu) \leq \max_{\mu \geq 0} \theta_S(\mu).$$

Definition 2.2. A difference between the optimal values of the primal and the dual objective functions is called *duality gap*.

Example 2.3. Consider the simple following example in [3]:

$$\begin{aligned}\min\quad & -x_1 - x_2\\ \text{s.t.}\quad & x_1 + 2x_2 - 3 \leq 0,\\ & x_1, x_2 = 0, 1, 2, \text{or } 3.\end{aligned}$$

The feasible region, optimal solution, and optimal objective value are simply obtained as follows:

$$S = \left\{(0,0)^T, (1,0)^T, (2,0)^T, (3,0)^T, (0,1)^T, (1,1)^T\right\},$$
$$x^* = (3,0)^T,$$
$$f(x^*) = -3.$$

Corresponding Lagrangian dual problem can be constructed as follows:

$$\begin{aligned}\theta_L(\mu) &= \min_{x_1,x_2}\{-x_1 - x_2 + \mu(x_1 + 2x_2 - 3) : x \in X\} \\ &= \min_{x_1,x_2}\{(\mu-1)x_1 + (2\mu-1)x_2 - 3\mu : x \in X\},\end{aligned}$$

where

$$x_1^* = \begin{cases} 3, & \text{if } \mu \le 1 \\ 0, & \text{otherwise} \end{cases}, \qquad x_2^* = \begin{cases} 3, & \text{if } \mu \le 1/2 \\ 0, & \text{otherwise} \end{cases}.$$

If we maximize the function

$$\theta_L(\mu) = \begin{cases} 6\mu - 6, & \text{if } 0 \le \mu \le 1/2 \\ -3, & \text{if } 1/2 \le \mu \le 1 \\ -3\mu, & \text{if } \mu \ge 1 \end{cases},$$

we find $\mu^* \in [1/2, 1]$, and $\theta_L(\mu^*) = -3$, that means no duality gap.

Example 2.4. Consider the simple following example in [36]:

$$\begin{aligned} \min \quad & -x_1 - 2x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 - 9 \le 0, \\ & x_1 + 4x_2 - 8 \le 0, \\ & x_1, x_2 \in \{0, 1, 2, 3, 4, 5\}. \end{aligned}$$

The feasible region, alternative optimal solutions, and optimal objective value are simply obtained as follows:

$$S = \left\{(0,0)^T, (1,0)^T, (2,0)^T, (3,0)^T, (0,1)^T, (1,1)^T, (2,1)^T, (0,2)^T\right\},$$
$$x^* = (0,2)^T, \text{or } (2,1)^T,$$
$$f(x^*) = -4.$$

Generally, there exists a gap between the optimal value of the primal problem and the optimal value of even the tightest relaxation. Corresponding Lagrangian dual problem is as follows:

$$\max_{\mu \ge 0} \theta_L(\mu)$$

such that

$$\begin{aligned}\theta_L(\mu) &= \min_{x_1,x_2}\{-x_1 - 2x_2 + \mu_1(3x_1 + 2x_2 - 9) + \mu_2(x_1 + 4x_2 - 8) : x \in X\} \\ &= \begin{cases} 16\mu_1 + 17\mu_2 - 15, & \text{if } 2\mu_1 + 4\mu_2 \le 2,\ 3\mu_1 + \mu_2 \le 1 \\ 6\mu_1 - 3\mu_2 - 5, & \text{if } 2\mu_1 + 4\mu_2 > 2,\ 3\mu_1 + \mu_2 \le 1 \\ \mu_1 + 12\mu_2 - 10, & \text{if } 2\mu_1 + 4\mu_2 \le 2,\ 3\mu_1 + \mu_2 > 1 \\ -9\mu_1 - 8\mu_2, & \text{if } 2\mu_1 + 4\mu_2 > 2,\ 3\mu_1 + \mu_2 > 1 \end{cases},\end{aligned}$$

which is maximum for $\left(\mu_1^*, \mu_2^*\right)^T = (1/5, 2/5)^T$, and $\theta_L\left(\mu_1^*, \mu_2^*\right) = -5$, that is, there exists a duality gap.

3. Cutting Plane Procedure, Conventional Surrogate Dual Search

The algorithm we use can be summarized as follows. We use normalized surrogate multipliers, i.e., $\mu_1 + \ldots + \mu_m \le 1$, we take a weighted sum of some constraints with multipliers $\mu_i \ge 0$, and replace these constraints with a single surrogate constraint. We choose initial surrogate multipliers arbitrarily, initial lower bound as $-\infty$. In Step 1, we solve the relaxed problem, if major constraints are satisfied, then we stop. Otherwise, we update the lower bound (if we get a greater bound) and surrogate multipliers, then we go to Step 1, and repeat. For adjusting the multipliers, in a separate second stage, we use auxiliary linear programming problems using the levels of constraint violation. If any constraint is violated, then the associated multiplier will have increased at the next iteration. For more detail, see [18]

Let us show how to implement the algorithm to reduce the domain with cuts on Example 2.4, we start with the following surrogate relaxation:

$$\begin{array}{ll} \min & -x_1 - 2x_2 \\ \text{s.t.} & \mu_1 (3x_1 + 2x_2 - 9) + \mu_2 (x_1 + 4x_2 - 8) \leq 0, \\ & x_1, x_2 \in \{0, 1, 2, 3, 4, 5\}. \end{array}$$

The initial vector of multipliers is taken $\mu^{(0)} = (0.5, 0.5)^T$, so the optimal solution of the relaxed problem and the optimal objective value are obtained as $x^{(1)} = (1, 2)^T$, and $f(x^{(1)}) = -5$, respectively. This solution satisfies the first constraint $(3x_1 + 2x_2 - 9 = -2 < 0)$ but does not the second one $(x_1 + 4x_2 - 8 = 1 > 0)$. Auxiliary linear programming problem of the first iteration can be given as follows:

$$\begin{array}{ll} \max & \beta \\ \text{s.t.} & \beta \leq -2\mu_1 + \mu_2, \\ & \mu_1 + \mu_2 \leq 1, \\ & \mu_1, \mu_2 \geq 0, \end{array}$$

and the solution of this problem is $\mu^{(1)} = (0, 1)^T$. If we use these weights in the relaxed problem, we get the optimal solution as $x^{(2)} = (4, 1)^T$, and the optimal objective value as $f(x^{(2)}) = -6$. We don't update the lower bound since we don't yield a greater bound. Similarly, this solution satisfies the second constraint $(x_1 + 4x_2 - 8 = 0 \geq 0)$ but does not the first one $(3x_1 + 2x_2 - 9 = 5 > 0)$. In the next iteration, by adding the constraint $\beta \leq 5\mu_1$ to the first auxiliary problem, the multipliers are updated as $\mu^{(2)} = (0.125, 0.875)^T$ from the second auxiliary linear programming problem. By using these multipliers in the relaxed problem, we get $x^{(3)} = (3, 1)^T$, and $f(x^{(3)}) = -5$. While the second constraint is satisfied $(x_1 + 4x_2 - 8 = -1 < 0)$, the first one is not $(3x_1 + 2x_2 - 9 = 2 > 0)$. Again, by adding the constraint $\beta \leq 2\mu_1 - \mu_2$, we get third auxiliary linear programming problem which is optimal for $\mu^{(3)} = (0, 0)^T$. Thus, the best lower bound is -5. In some cases like Example 2.4, if there exists a duality gap, the conventional surrogate dual search method fails to find the optimal solutions of the primal problem.

Now, let us try to explain the relationship between Lagrange and surrogate multipliers, again on Example 2.4. We use Lagrangian/Surrogate relaxation [26]:

$$\begin{array}{ll} \min & -x_1 - 2x_2 + \lambda (\mu_1 (3x_1 + 2x_2 - 9) + \mu_2 (x_1 + 4x_2 - 8)) \\ \text{s.t.} & x_1, x_2 \in \{0, 1, 2, 3, 4, 5\}. \end{array}$$

We found Lagrangian multipliers as: $\lambda_1 = \lambda\mu_1 = \frac{1}{5}, \lambda_2 = \lambda\mu_2 = \frac{2}{5}$, and $\frac{\mu_1}{\mu_2} = \frac{\lambda_1}{\lambda_2} = \frac{1}{2}$, so $\mu_1 = \frac{1}{3}, \mu_2 = \frac{2}{3}$. Surrogate relaxation problem with these multipliers:

$$\begin{array}{ll} \min & -x_1 - 2x_2 \\ \text{s.t.} & x_1 + 2x_2 - 5 \leq 0, \\ & x_1, x_2 \in \{0, 1, 2, 3, 4, 5\}, \end{array}$$

where $x^* = (1, 2)^T$, or $(3, 1)^T$, or $(5, 0)^T$, $f(x^*) = -5$.

4. Nonlinear (p–norm) Surrogate Dual Method

This method reshapes the feasible region to reduce the duality gap. For more detail, see [16]. Suppose $p \in [1, \infty]$,

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & \left\{ \sum_{i=1}^{m} [\mu_i g_i(x)]^p \right\}^{1/p} \leq \left\{ \sum_{i=1}^{m} [\mu_i b_i]^p \right\}^{1/p}, \\ & x \in X, \end{array}$$

where μ satisfies the following:

$$\mu_1 b_1 = \ldots = \mu_m b_m,$$

$$\mu_1 + \ldots + \mu_m = 1.$$

Now, let us show how to apply the $p-$norm surrogate constraint method on Example 2.4. For $p = 5$:

$$\begin{array}{ll} \min & -x_1 - 2x_2 \\ \text{s.t.} & \left(\frac{3x_1+2x_2}{9}\right)^5 + \left(\frac{x_1+4x_2}{8}\right)^5 \leq 2, \\ & x_1, x_2 \in \{0, 1, 2, 3, 4, 5\}, \end{array}$$

where $x^* = (0, 2)^T$, $f(x^*) = -4$. Note that, because we choose the proper weights, they are dropped from the formulation by some rearrangements.

4.1. How can we choose the parameter p?

When the conventional method fails, enumerate all the solutions subject to $f(x)$=optimal dual objective value. Calculate g_i's for all those solutions. To eliminate these solutions from the feasible region, enforce the inequality

$$\sum_{i=1}^{m} \left(\frac{g_i}{b_i}\right)^p > m.$$

In Example 2.4, $-x_1 - 2x_2 = -5$ holds for $(1, 2)^T$, $(3, 1)^T$, and $(5, 0)^T$. The solutions $(3, 1)^T$ and $(5, 0)^T$ are eliminated for $p \geq 2$. The best value of p is 5, since

$$\min \left\{ p : \left(\frac{7}{9}\right)^p + \left(\frac{9}{8}\right)^p > 2 \right\} = 5.$$

5. Minimum-cost flow problems

We are interested in network problems, and we need their integer programming formulations. In minimum-cost flow problems, we try to construct the feasible flow from the source nodes to the sink nodes with minimum cost. Some integer programming problems, involving transportation problems, transshipment problems, assignment problems, shortest path problems (with or without time windows), and maximal flow problems can be seen those type of problems.

5.1. Application, Maximal flow problem. In maximal flow problems, we try to find the maximum flow of some commodity from the source nodes to the sink nodes, when a graph has upper capacity limitations on the arcs. The network in Figure 1 has two sources 0 and 1, and three sinks 5, 6, and 7. The number attached to each arc represents the own upper capacity. For intermediate nodes 2, 3 and 4, it is necessary to ensure that the total incoming flow equals the total outgoing flow.

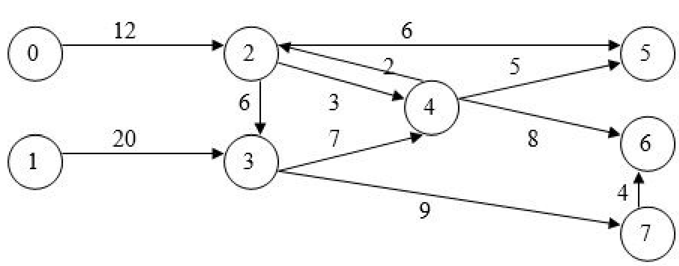

Figure 1. Maximum flow problem [39]

Linear programming formulation of the problem in Figure 1 is:

$$\begin{array}{ll} \max & x_{02} + x_{13} \\ \text{subject to} & x_{02} + x_{42} = x_{23} + x_{24} + x_{25}, \\ & x_{13} + x_{23} = x_{34} + x_{37}, \\ & x_{24} + x_{34} = x_{42} + x_{45} + x_{46}, \\ & \left.\begin{array}{l} x_{02} \leq 12, x_{13} \leq 20, x_{23} \leq 6, x_{24} \leq 3, x_{25} \leq 6, x_{34} \leq 7, \\ x_{37} \leq 9, x_{42} \leq 2, x_{45} \leq 5, x_{46} \leq 8, x_{76} \leq 4, \end{array}\right\} \begin{array}{c} \to \text{capacity} \\ \text{limitations} \\ \text{on arcs} \end{array} \\ & x_{02}, x_{13}, x_{23}, x_{24}, x_{25}, x_{34}, x_{37}, x_{42}, x_{45}, x_{46}, x_{76} \geq 0, \text{ integer} \end{array}$$

We consider four most restrictive capacities on arcs from nodes 2 to 4, 4 to 2, 4 to 5, and 7 to 6 to construct the surrogate constraint as:

$$\mu_1(x_{24} - 3) + \mu_2(x_{42} - 2) + \mu_3(x_{45} - 5) + \mu_4(x_{76} - 4) \leq 0.$$

The weights obtained at each step of the algorithm, the corresponding optimal solutions and the optimal objective function values can be given as follows:

$$\begin{aligned} \mu^{(0)} &= (0.25, 0.25, 0.25, 0.25)^T, x^{(1)} = (12, 16, 0, 6, 6, 7, 9, 0, 8, 5, 0)^T, f(x^{(1)}) = 28, \\ \mu^{(1)} &= (0, 0, 1, 0)^T, x^{(2)} = (12, 16, 0, 6, 6, 7, 9, 0, 5, 8, 0)^T, f(x^{(2)}) = 28, \\ \mu^{(2)} &= (1, 0, 0, 0)^T, x^{(3)} = (9, 16, 0, 3, 6, 0, 7, 9, 10, 0, 0)^T, f(x^{(3)}) = 25, \\ \mu^{(3)} &= (0.625, 0, 0.375, 0)^T, x^{(4)} = (10, 16, 0, 4, 6, 0, 7, 9, 3, 8, 0)^T, f(x^{(4)}) = 26, \\ \mu^{(4)} &= (0.875, 0, 0.125, 0)^T, x^{(5)} = (9, 16, 0, 3, 6, 0, 7, 9, 5, 5, 0)^T, f(x^{(5)}) = 25 \longrightarrow \text{Optimal Solution.} \end{aligned}$$

5.2. **Application, Shortest path problem with distance window only on the destination node, [17].** Consider the problem of finding the paths with distance window [28, 30], where the number next to each arc represents the arc length.

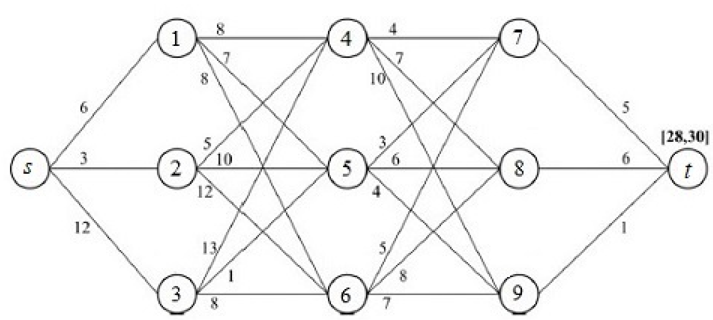

FIGURE 2. Distance confined shortest path problem

To formulate the problem which is given in Figure 2, we use an auxiliary variable as total distance:

$$\begin{aligned} z &= 6x_{s1} + 3x_{s2} + 12x_{s3} + 8x_{14} + 7x_{15} + 8x_{16} + 5x_{24} + 10x_{25} + 12x_{26} + 13x_{34} + x_{35} + 8x_{36} \\ & \quad + 4x_{47} + 7x_{48} + 10x_{49} + 3x_{57} + 6x_{58} + 4x_{59} + 5x_{67} + 8x_{68} + 7x_{69} + 5x_{7t} + 8x_{8t} + x_{9t}, \\ \text{Variables} &= 0 \text{ or } 1. \end{aligned}$$

Again, we have to make sure that the equality of the total incoming and outgoing flows for all nodes s, 1 to 9, and t with the following shortest path constraints:

$$\begin{aligned}
x_{s1} + x_{s2} + x_{s3} &= 1,\\
x_{s1} &= x_{14} + x_{15} + x_{16},\\
x_{s2} &= x_{24} + x_{25} + x_{26},\\
x_{s3} &= x_{34} + x_{35} + x_{36},\\
x_{14} + x_{24} + x_{34} &= x_{47} + x_{48} + x_{49},\\
x_{15} + x_{25} + x_{35} &= x_{57} + x_{58} + x_{59},\\
x_{16} + x_{26} + x_{36} &= x_{67} + x_{68} + x_{69},\\
x_{47} + x_{57} + x_{67} &= x_{7t},\\
x_{48} + x_{58} + x_{68} &= x_{8t},\\
x_{49} + x_{59} + x_{69} &= x_{9t},\\
x_{7t} + x_{8t} + x_{9t} &= 1.
\end{aligned}$$

We first find the path with length 28 by using $p-$norm surrogate constraint method aggregating distance window constraints $(28.1 - z \leq 0.1)$ and $(z \leq 30)$ as a surrogate constraint:

$$\mu_1^2 (28.1 - z)^2 + \mu_2^2 z^2 \leq 2\left(\frac{30}{301}\right)^2,$$

where $\mu_1 = \frac{300}{301}, \mu_2 = \frac{1}{301}$. The optimal paths with length 28 are $(s \to 3 \to 6 \to 9 \to t)$ and $(s \to 1 \to 6 \to 8 \to t)$.

Second, the path with length 29 with considering the major constraints as $(29 - z \leq 1)$ and $(z \leq 30)$ and surrogate constraint as:

$$\mu_1^2 (29 - z)^2 + \mu_2^2 z^2 \leq 2\left(\frac{30}{31}\right)^2,$$

where $\mu_1 = \frac{30}{31}, \mu_2 = \frac{1}{31}$. The optimal path with length 29 is $(s \to 2 \to 6 \to 8 \to t)$. The path with length 30 can be found as similar way.

5.2.1. *How can we use the conventional method?* We aggregate only 3 equality constraints in our surrogate constraint as following:

$$\begin{aligned}
&\mu_1 (x_{s1} - x_{14} - x_{15} - x_{16}) + \mu_2 (-x_{s1} + x_{14} + x_{15} + x_{16}) + \mu_3 (x_{14} + x_{24} + x_{34} - x_{47} - x_{48} - x_{49})\\
&+\mu_4 (-x_{14} - x_{24} - x_{34} + x_{47} + x_{48} + x_{49}) + \mu_5 (x_{47} + x_{57} + x_{67} - x_{7t}) + \mu_6 (-x_{47} - x_{57} - x_{67} + x_{7t}) \leq 0.
\end{aligned}$$

An appropriate initial vector of multipliers leads to good lower bounds. To generate better starting solution, we should regard all constraints as equally important, or at least as important as the others, thus we take the initial weight vector as $\mu^0 = (0.2, 0.1, 0.15, 0.25, 0.2, 0.1)$, (taking into consideration that we cannot choose $\mu_1 = \mu_2, \mu_3 = \mu_4, \mu_5 = \mu_6$), and the distance window constraint as $z = 28$ (or $z = 29$), then we succeed in getting the optimal solution at the first iteration.

6. Concluding Remarks

The surrogate dual, in general, provides a tighter bound than the Lagrangian dual, i.e., gives a better lower approximation, yields smaller duality gaps, and reduces the size of constraint sets, and can also be used for nonlinear integer programming problems. Any optimal solution to the surrogate problem that is feasible for the primal is automatically optimal for the primal and no additional conditions are required, such as Lagrangian complementary slackness conditions $(\mu^T(g(x) - b) = 0)$. In order to solve by using the conventional method, easier relaxations can be created by careful partitioning of the major constraints into those which to be relaxed and to be enforced. For surrogate relaxations, branch-and-bound procedures have been proved computationally more efficient and convenient than dynamic programming approaches. But, the search of optimal surrogate multipliers can be more difficult than optimal Lagrange multipliers. With surrogate constraint modeling for transportation problems, we have to deal with nonlinear programming instead of linear programming, integer variables instead of continuous variables, too little-reduced problem size because of branching in integer programming. Consequently, the methods are not efficient for transportation problems.

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