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Comparison Theorems for One Sturm-Liouville Problem With Nonlocal Boundary Conditions

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Abstract. In this study we present a new approach for investigation of some Sturm-Liouville systems with nonlocal boundary conditions. In the theory of boundary value problems for two-order differential equations the basic concepts and methods have been formulated studying the problems of classical mathematical physics. However, many modern problems, which arise as the mathematical modeling of some systems and processes in the fields of physics, such as the vibration of strings, the interaction of atomic particles motivate to formulate and investigate the new ones, for example, a class of Sturm-Liouville problems with nonlocal boundary conditions. Such conditions arise when we cannot measure data directly at the boundary. In this case, the problem is formulated, where the value of the solution and its derivative is linked to interior points of the considered interval. Sturm-Liouville problems together with transmission conditions at some interior points is very important for solving many problems of mathematical physics. In this study we present a new approach for investigation of boundary value problems consisting of the two interval Sturm-Liouville equations. This kind of boundary value transmission problems are connected with various physical transfer problems (for example, heat and mass transfer problems). We define a new Hilbert space and linear differential operator in it such a way that the considered nonlocal problem can be interpreted as an spectral problem. We investigate the main spectral properties of the problem under consideration. Particularly we present a new criteria for Sturm-Comparison theorems. Our main result generalizes the classical comparison theorem for regular Sturm-Liouville problems.

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1. Introduction

There is a permanent interest in obtaining new sufficient conditions for the oscillation or nonoscillation of solutions of various types of second order equations.The relationship between the oscillatory and other fundamental properties of the solutions of Sturm-Liouville type differential equations are of central importance in the theory of boundary value problems. The oscillation theory for the solutions of differential equations is one of the traditional trends in the qualitative theory of differential equations. Its essence is to establish conditions for the existence of oscillating (nonoscillating) solutions, to study the laws of distribution of the zeros, the maxima and minima of the solution, to obtain estimates of the distance between the consecutive zeros and of the number of zeros in a given interval, as well as to describe the relationship between the oscillatory and other fundamental properties of the solutions of various classes of differential equations. Therefore, the comparison and oscillation theorems of Sturm have remained a topic of considerable interest.

Transmission problems appear frequently in various fields of physics and technics. For example, in electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (see, [13] and the references listed therein). Another completely different field is that of "hydraulic fracturing" (see, [7]) used in order to increase the flow of oil from a reservoir into a producing oil well. In recent years, Sturm-Liouville problems with transmission conditions have been an important research topic in mathematical physics [1–3, 5, 6, 8–12]. We give a method for proving the comparison and oscillation theorem of the discontinuous Sturm-liouville problem (2.1) − (2.2).

2. Comparison Theorems

In this study we investigated one discontinuous eigenvalue problem which consists of Sturm-Liouville equation,

$$\tau y := -y''(x) + q(x)y(x) = 0 \tag{2.1}$$

to hold on two disjoint intervals (a, c) and (c, b) , where discontinuity in y and y' at the interior singular point $x = c$ are prescribed by transmission conditions

$$y(c-) = y(c+), \quad y'(c-) = y'(c+), \tag{2.2}$$

where the potential $q(x)$ is real-valued, continuous on $[a, c) \cup (c, b]$ and has a finite limits $q(c\mp) = \lim_{x\to c\mp} q(x)$.

Theorem 2.1. *(Sturm Comparison Theorem) Suppose $y = y_1(x)$ is a real solution of the equation*

$$L_1 y := -y'' + g(x)y = 0 \tag{2.3}$$

on two disjoint intervals $[a, c)$ and $(c, b]$ satisfying the transmission conditions

$$y(c-) = y(c+), \quad y'(c-) = y'(c+) \tag{2.4}$$

and $y = y_2(x)$ is a real solution of the equation

$$L_2 y := -y'' + h(x)y = 0 \tag{2.5}$$

on $[a, c)\cup \in (c, b]$ with the same transmission conditions (2.4). Fortormore, suppose that $g(x) > h(x)$ on whole $[a, c)\cup \in (c, b]$. If $(y'z - yz')|_{c+0} > 0$ then there is at least one zero of y_2 between any two successive zeros y_1.

Proof. Let x_1 and x_2 with $x_1 < x_2$ be consecutive zeros of $y_1 = y_1(x)$. Suppose, it possible,that $y_2 = y_2(x)$ does not have a zero in the interval (x_1, x_2). Lagrange's identity (**see, [4]**) gives

$$y_2 L_1 y_1 - y_1 L_2 y_2 = \frac{d}{dx}\{y_1 y_2' - y_2 y_1'\} + \{g(x) - h(x)\}y_1 y_2 = 0$$

Hence

$$y_2(x)L_1 y_1(x) - y_1(x)L_2 y_2(x) = \frac{d}{dx}\{\frac{dy_2(x)}{dx}y_1(x) - y_2(x)\frac{dy_1(x)}{dx}\} + \{g(x) - h(x)\}y_1(x)y_2(x) = 0$$

Integrating on both sides of the last equation from x_1 to x_2, we get

$$p(x)(y_1' y_2 - y_2' y_1)|_{x_1}^{x_2} = \int_{x_1}^{x_2} \{g(x) - h(x)\}y_1 y_2 dx \tag{2.6}$$

Case 1. **1a)** Consider the case $x_1 \in [a,c)$ and $x_2 \in (c,b]$. Integrating on both sides of the equation (2.6) over (x_1, c) we get

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} (y_1' y_2 - y_2' y_1)|_{x_1}^{c-\epsilon} = \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \int_{x_1}^{c-\epsilon} \{g(x) - h(x)\} y_1(x) y_2(x) dx.$$

Since $y_1(x_1) = 0$ we have

$$[\lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} W(y_1, y_2; c-\epsilon) - y_1'(x_1) y_2(x_1)] = \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \int_{x_1}^{c-\epsilon} \{g(x) - h(x)\} y_1(x) y_2(x) dx. \tag{2.7}$$

Similarly we have

$$[y_1'(x_2) y_2(x_2) - \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} W(y_1, y_2; c+\epsilon)] = \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \int_{c+\epsilon}^{x_2} \{g(x) - h(x)\} y_1(x) y_2(x) dx \tag{2.8}$$

Withoutloss of generality we shall assume that $y_1(x) > 0$ and $y_2(x) > 0$ in the interval $(x_1, c) \cup (c, x_2)$. Multiplying (2.7) throughout by (2.8) and summing we have

$$y_1'(x_2) y_2(x_2) - y_1'(x_1) y_2(x_1) + \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \{W(y_1, y_2; c-\epsilon_1) - W(y_1, y_2; c+\epsilon)\}$$

$$= \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \int_{x_1}^{c-\epsilon} \{g(x) - h(x)\} y_1 y_2 dx + \int_{c+\epsilon}^{x_2} \{g(x) - h(x)\} y_1 y_2 dx\}$$

From the transmission conditions it follows that

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} W(y_1, y_2; c-\epsilon) - \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} W(y_1, y_2; c+\epsilon) = 0$$

Substituting this in the previous equality gives

$$\begin{aligned} y_1'(x_2) y_2(x_2) - y_1'(x_1) y_2(x_1) &= \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \{\int_{x_1}^{c-\epsilon} \{g(x) - h(x)\} y_1 y_2 dx \\ &+ \int_{c+\epsilon}^{x_2} \{g(x) - h(x)\} y_1 y_2 dx\} \end{aligned} \tag{2.9}$$

Since $y_1(x) > 0$ and $y_2(x) > 0$ in $(x_1, c) \cup (c, x_2)$, the integral on right-hand side of the last equality is positive. On the other hand, since $y_1(x_1) = 0$ and $y_1(x) > 0$ in the right neighborhood of the point x_1 we have $y_1'(x_1) = \lim_{h \to 0} \frac{y_1(x_1+h)}{h} \geq 0$. But $y_1(x)$ cannot vanish at the point $x = x_1$ because then it would follow from well-known Cauchy-Picard's Theorem for the solution $y_1'(x) = 0$ for all $x \in [a,c)$, which is impossible). Consequently, $y_1'(x_1) > 0$. By the similar way we can show that $y_2'(x_2) < 0$. The left-hand side of the inequality

$$y_1'(x_2) y_2(x_2) - y_1'(x_1) y_2(x_1) \leq 0,$$

which presents as with a contradiction; right-hand side of the equality (2.9) > 0, but left-hand side of the same equality ≤ 0. The proof is complete for the case.

1b) Consider the case $(y_1'(c+0)y_2(c+0) - y_1(c+0)y_2'(c+0)) > 0$. In this case from (2.7) and (2.8) we have

$$-p_1^2 y_1'(x_1)y_2(x_1) + p_2^2 y_1'(x_2)y_2(x_2) + (\delta\gamma - 1) \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} (y_1' y_2 - y_2' y_1)|_{c+0}$$

$$= \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \int_{x_1}^{c-\epsilon} \{g(x) - h(x)\} y_1 y_2 dx + \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \int_{c+\epsilon}^{x_2} \{g(x) - h(x)\} y_1 y_2 dx \tag{2.10}$$

By virtue of the transmission conditions

$$sgn(y_1(x)y_2(x))_{x\in(x_1,c)} = sgn(y_1(x)y_2(x))_{x\in(c,x_2)}$$

Therefore, without loss of generality we can assume that $y_1(x)y_2(x) > 0$ in whole $[a, c) \cup \in (c, b]$. Then right-hand side of the equality (2.10) is greater than zero but the left-hand side of this equality is less than or equals zero.

Case 2. Consider the case $(x_1, x_2) \subset [a, c)$.

i)Let $y_1(x) > 0$ and $y_2(x) > 0$ in the interval (x_1, x_2). These conditions ensure that the integral on the right in (2.6) is positive. However, on the left, we have $y_1(x_1) = y_1(x_2) = 0$ with $y_1'(x_1) > 0$ and $y_1'(x_2) < 0$. The left-hand side therefore becomes

$$(y_1'(x_2)y_2(x_2) - y_1'(x_1)y_2(x_1)) < 0$$

which presents us with a contradiction: right-hand side > 0 and left-hand side < 0 . Since the conditions describing $y_1(x)$ are given, we conclude that $y_2(x)$ must change sign between $x = x_1$ and $x = x_2$. Thus $y_2(x) = 0$ (at least once) between the zeros of $y_1(x)$.

ii)Assume that both $y_1(x) < 0$ and $y_2(x) < 0$ are negative in in the interval (x_1, x_2). This could be represented in a similar way to the case i).

iii)Let $y_1(x) < 0$ and $y_2(x) > 0$ in the interval (x_1, x_2) These conditions ensure that the integral on the right in (2.6) is negative. However, on the left, we have $y_1(x_1) = y_1(x_2) = 0$ with $y_1'(x_1) < 0$ and $y_1'(x_2) > 0$. The left-hand side therefore becomes

$$(y_1'(x_2)y_2(x_2) - y_1'(x_1)y_2(x_1)) > 0$$

which presents us with a contradiction: right-hand side < 0 and left-hand side > 0 . Since the conditions describing $y_1(x)$ are given, we conclude that $y_2(x)$ must change sign between $x = x_1$ and $x = x_2$. Thus $y_2(x) = 0$ (at least once) between the zeros of $y_1(x)$.

iv)Let $y_1(x) > 0$ and $y_2(x) < 0$ in the interval (x_1, x_2). This is similar to the previous case. Let $(x_1, x_2) \subset (c, b]$. This case easily proved similarly to Case 2. □

Theorem 2.2. *Let* $\phi(x,\lambda) = \begin{cases} \phi_1(x,\lambda), & x \in [a,c) \\ \phi_2(x,\lambda), & x \in (c,b] \end{cases}$ *be the solution of equation (2.3), satisfying the initial conditions*

$$\phi_1(a,\lambda) = \sin\alpha, \quad \phi_1'(a,\lambda) = -\cos\alpha$$

and transmission conditions

$$\phi_2(c+,\lambda) = \phi_1(c-,\lambda), \phi_2'(c+,\lambda) = \phi_1'(c-,\lambda)$$

and let $\varphi(x,\lambda)$ *be the solution of the equation (2.5) with the same initial-transmission conditions. Furthermore, suppose that* $h(x) > g(x)$ *for* $\forall x[a, c) \cup \in (c, b]$*. Then if* $\phi(x)$ *has m zeros in* $[a, c) \cup (c, b]$ $\varphi(x)$ *has not fewer than m zeros in the same interval and* $k - th$ *zero of* $\varphi(x)$ *is less than the* $k - th$ *zero of* $\phi(x)$*.*

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