Proceedings of International Conference on Mathematics
and Mathematics Education (ICMME 2018)
Turk. J. Math. Comput. Sci.
10(2018) 126–133

http://dergipark.gov.tr/tjmcs

MATDER

On Fuzzy Sub-H-Groups

Sibel Demiralp[a,*], Gülnur Haçat[a]

[a]Department of Mathematics, Faculty of Science and Letter, Kastamonu University, 37000, Kastamonu, Turkey.

Received: 15-08-2018 • Accepted: 13-11-2018

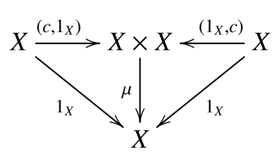
Abstract. In this paper we introduce fuzzy sub-H-group and give some examples. We show that there exist a natural transformation between $[Y, Z]$ and $[X, Z]$ where Y is a fuzzy sub-H-group of X. Also we prove that if Y is a fuzzy subspace of X, then ΩY is a fuzzy sub-H-group of ΩX.

2010 AMS Classification: 55P45, 54A40, 20N25.

Keywords: Fuzzy H-space, fuzzy sub-H-group, fuzzy H-homomorphism, fuzzy H-isomorphism.

1. Introduction

Zadeh introduced the concepts of fuzzy sets and fuzzy set operations in [13]. In 1968, Chang developed a theory of fuzzy topological spaces [1]. After that, basic concepts from homotopy theory were discussed in fuzzy settings. In this direction, Chong-you [3] introduced the concept of fuzzy paths. Also in [2], fuzzy homotopy concepts in fuzzy topological spaces were conceived. Then the fundamental group of a fuzzy topological space was developed in [7] . Later many topics of algebraic topology were extended to fuzzy topology. For example, the concept of fuzzy H-spaces and fuzzy H-groups have been introduced by Demiralp and Guner in [4]. An H-space is a pair (X, μ) where (X, p) is a pointed topological space, $\mu : X \times X \longrightarrow X$ is a continuous multiplication which makes the diagram



homotopy commutative, i.e. $\mu \circ (1_x, c) \simeq 1_x$ and $\mu \circ (c, 1_x) \simeq 1_x$, for the constant map $c(x) = p$. An H-group is an H-space whose multiplication is homotopy associative and has a homotopy inverse [5].

The most important example of an H-group is the loop spaces.

*Corresponding Author
Email addresses: sdemiralp@kastamonu.edu.tr (S. Demiralp), ghacat@ogr.kastamonu.edu.tr (G. Haçat)

2. Preliminaries

In this section we recall some basic notions concerning fuzzy set theory.

Definition 2.1. [6] Let X be a non empty set. A fuzzy set A in X is a function $A : X \to [0,1]$. 1_X and 0_X are the constant fuzzy sets taking values 1 and 0 , respectively. The collection of all fuzzy sets in X is denoted by I^X. The set

$$\operatorname{supp} A = \{x \in X \mid A(x) > 0\}$$

is called the support of fuzzy set A.

Definition 2.2. [9] A fuzzy point p_λ in a set X is a fuzzy set such that

$$p_\lambda(x) = \begin{cases} \lambda, & x = p \\ 0, & x \neq p \end{cases}$$

where $0 < \lambda \leqslant 1$.

Definition 2.3. [8] A fuzzy topology on a set X is a family $\tau \subseteq I^X$ which satisfies the following conditions:
(i) $0_X, 1_X \in \tau$.
(ii) $A, B \in X \Rightarrow A \wedge B \in \tau$.
(iii) $A_j \in \tau$ for all $j \in J$ (where J is an index set) $\Rightarrow \underset{j\in J}{\vee} A_j \in \tau$.
Then the pair (X, τ) is called fuzzy topological space. Every member of τ is called fuzzy open sets.

Definition 2.4. [1] Let (X, τ) be a fuzzy topological space and $X' \subset X$. Then

$$\tau' = \{A\,|_{X'} : A \in \tau\}$$

is a fuzzy topology on X' and (X', τ') is called the fuzzy subspace of (X, τ).

Definition 2.5. [4] Let X and Y be two sets, $f : X \to Y$ be a function and A be a fuzzy set in X, B be a fuzzy set in Y.
(1) the image of A under f is the fuzzy set $f(A)$ defined such that,

$$f(A)(y) = \begin{cases} \underset{x\in f^{-1}(y)}{\vee} A(x), & \text{if } f^{-1}(y) \neq \varnothing, \\ 0, & \text{otherwise} \end{cases}$$

for all $y \in Y$.
(2) the inverse image of B under f is the fuzzy set $f^{-1}(B)$ in X defined such that $f^{-1}(B)(x) = B(f(x))$, for all $x \in X$.

Definition 2.6. [6] Let (X, τ) and (Y, τ') be two fuzzy topological spaces.A function $f : (X, \tau) \to (Y, \tau')$ is fuzzy continuous if $f^{-1}(V) \in \tau$, for all $V \in \tau'$. The set of all fuzzy continuous functions from (X, τ) to (Y, τ') is denoted by $FC(X, Y)$.

Let $(A, \tau_A), (B, \tau_B)$ be fuzzy subspaces of X and Y, respectively, and $f \in FC(X, Y)$ such that $f(A) \subset B$. If for all $U \in \tau_B$, $f^{-1}(U) \cap A \in \tau_A$ then f is called relative fuzzy continuous.

Definition 2.7. [11] Let $(X, .)$ be a group, (X, τ) be a fuzzy topological space. If the function $(X, \tau) \times (X, \tau) \to (X, \tau)$, $(x, y) \to x.y^{-1}$ is relative fuzzy continuous, then (X, τ) is called a fuzzy topological group.

Definition 2.8. [4] Let (X, τ) be a fuzzy topological space and p_λ be a fuzzy point in X. The pair (X, p_λ) is called a pointed fuzzy topological space (PFTS) and p_λ is called the base point of (X, p_λ).

Definition 2.9. [3] Let (X, T) be a (classical) topological space. Then

$$\widetilde{T} = \left\{A \in I^X \mid SuppA \in T\right\}$$

is a fuzzy topology on X, called the fuzzy topology on X introduced by T and $(X, \widetilde{T})$ is called the fuzzy topological space introduced by (X, T) .

Let ε_I denote Euclidean subspace topology on I and $(I, \widetilde{\varepsilon_I})$ denote the fuzzy topological space introduced by the topological space (I, ε_I) .

Definition 2.10. [10] Let $(X,\tau),(Y,\tau')$ be fuzzy topological spaces and $f,\ g\in FC(X,Y)$. If there exist a fuzzy continuous function

$$F:(X,\tau)\times(I,\widetilde{\varepsilon_I})\rightarrow(Y,\tau')$$

such that $F(x,0)=f(x)$ and $F(x,1)=g(x)$, for all $x\in X$, then f and g are called fuzzy homotopic. The map F is called fuzzy homotopy from f to g and it is written $f\simeq g$. Also if for a fuzzy point p_λ of (X,τ), $F(p,t)=f(p)=g(p)$ then f and g are called fuzzy homotopic relative to p_λ. If $f=g$ then $f\simeq g$ with the fuzzy homotopy $F(x,t)=f(x)=g(x)$, for all $t\in I$.

Definition 2.11. [9] Let $f:(X,\tau)\rightarrow(Y,\tau^{'})$ be a fuzzy continuous function. If there is a fuzzy continuous function $f':(Y,\tau^{'})\rightarrow(X,\tau)$ satisfies the following conditions:
(i) $f\circ f'\simeq 1_Y$
(ii) $f'\circ f\simeq 1_X$
then, f is called a fuzzy homotopy equivalence. Further, fuzzy topological spaces are called fuzzy homotopic equivalent spaces and denoted by $X\simeq Y$.

A map $f:(X,\tau)\longrightarrow(Y,\tau')$ is a fuzzy monomorphism if and only if when $f\circ g\simeq f\circ h$, then $g\simeq h$.
The fuzzy homotopy relation "$\simeq$" is an equivalence relation. Thus the set $FC(X,Y)$ is partitioned into equivalence classes, calling fuzzy homotopy classes. The fuzzy homotopy class of a function f is denoted by $[f]$. The set of all fuzzy homotopy classes of the fuzzy continuous functions from (X,p_λ) and (Y,q_η) is denoted by $\left[(X,p_\lambda),(Y,q_\eta)\right]$. .

Let (X,p_λ) and (Y,q_η) be pointed fuzzy topological spaces. If $f:(X,p_\lambda)\rightarrow(Y,q_\eta)$ is a fuzzy continuous function then it is assumed that all subsets contain the basepoint, f preserves the base point, i.e. $f(p)=q$ and that all fuzzy homotopies are relative to the base point.

Definition 2.12. [5] Let (X,τ) be a fuzzy topological space. If $\alpha:(I,\widetilde{\varepsilon_I})\longrightarrow(X,\tau)$ is a fuzzy continuous function and the fuzzy set E is connected in $(I,\widetilde{\varepsilon_I})$ with $E(0)>0$ and $E(1)>0$, then the fuzzy set $\alpha(E)$ in (X,τ) is called a fuzzy path in (X,τ). The fuzzy points $(\alpha(0))_{E(0)}=\alpha\left(0_{E(0)}\right)$ and $(\alpha(1))_{E(1)}=\alpha\left(1_{E(1)}\right)$ are called the initial point and the terminal point of the fuzzy path $\alpha(E)$, respectively.

Definition 2.13. [3] Let A be a fuzzy set in a fuzzy topological space (X,τ). If for any two fuzzy points $a_\lambda,b_\eta\in A$, there is a fuzzy path contained in A with initial point a_λ and terminal point b_η, then A is said to be fuzzy path connected in (X,τ).

Definition 2.14. [3] A fuzzy path $\alpha(A)$ which the initial point and the terminal point are p_λ, is called a fuzzy loop in (X,p_λ) based at p_λ. The set of all fuzzy loops in (X,p_λ) based at p_λ is called fuzzy loop space. This space is a fuzzy topological space having the fuzzy compact-open topology. It is denoted by $\Omega(X,p_\lambda)$.

3. Fuzzy H-Groups

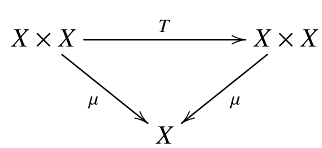
In this section we recall the concept of fuzzy H-space and fuzzy H-group.

Definition 3.1. [4] Let (X,p_λ) be a pointed fuzzy topological space, $\mu:X\times X\rightarrow X$ is a fuzzy continuous multiplication and $c:X\rightarrow X,\ c:x\rightarrow p$ is a constant function. If $\mu\circ(c,1_X)\simeq 1_X\simeq\mu\circ(1_X,c)$ then (X,p_λ) is called a fuzzy H-space and c is called homotopy identity of (X,p_λ). Here, $(c,1_X)(x)=(c(x),1_X(x))=(p,x)$ for all $x\in X$.

Definition 3.2. [4] Let the PFTS (X,p_λ) be a fuzzy H-space with the fuzzy continuous multiplication μ. If there exist a function

$$T:X\times X\rightarrow X\times X,\ T(x,y)=(y,x)$$

which makes the diagram



homotopy commutative, i.e. $\mu \circ T \simeq \mu$, then μ is called fuzzy homotopy abelian and (X, p_λ) is called an abelian fuzzy H-space. If $\mu \circ (\mu \times 1_X) \simeq \mu \circ (1_X \times \mu)$ then μ is called fuzzy homotopy associative. If there exist a fuzzy continuous function $\phi : X \longrightarrow X$ which makes the diagram

$$\begin{array}{ccccc} X & \xrightarrow{(\phi,1_X)} & X \times X & \xleftarrow{(1_X,\phi)} & X \\ & {\scriptstyle c}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle c} & \\ & & X & & \end{array}$$

homotopy commutative, i.e. $\mu \circ (\phi, 1_X) \simeq c \simeq \mu \circ (1_X, \phi)$, then ϕ is called fuzzy homotopy inverse of μ.

Definition 3.3. [4] A fuzzy H-group is a fuzzy H-space which has a fuzzy homotopy associative multiplication and a fuzzy homotopy inverse.

Example 3.4. Let X be a fuzzy topological space. Then $\Omega(X, p_\lambda)$ is a fuzzy H-group with the base point $w_0(A)$ which is the equal p_λ at any point, fuzzy continuous multiplication $m : \Omega(X, p_\lambda) \times \Omega(X, p_\lambda) \longrightarrow \Omega(X, p_\lambda)$ defined such that, for any $\alpha(E), \beta(D) \in \Omega(X, p_\lambda)$

$$m(\alpha(E), \beta(D))(t) = \begin{cases} \alpha\left((2t)_{E(2t)}\right) & , 0 \le t \le \frac{1}{2} \\ \beta\left((2t-1)_{D(2t-1)}\right) & , \frac{1}{2} \le t \le 1. \end{cases}$$

Example 3.5. Let $(X, \cdot)$ be a group with the identity element e and $(X, e_\lambda, \cdot)$ be a fuzzy topological group. Then (X, e_λ) is a fuzzy H-group with the multiplication " $\cdot$ ".

4. Main Results

In this section we define fuzzy H-isomorphism and give some examples. Then we define fuzzy sub-H-group and give some properties.

Throughout this section we assume that X is a fuzzy H-group with the continuous multiplication μ constant map c and homotopy inverse ϕ.

Definition 4.1. Let X and Y be fuzzy H-groups. A fuzzy continuous map $f : X \longrightarrow Y$ is called a fuzzy H-homomorphism whenever $f \circ \mu \simeq \eta \circ (f \times f)$ where η is the multiplication of Y. Also, f is called a fuzzy H-isomorphism if there exists a fuzzy H-homomorphism $g : Y \longrightarrow X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. In this case, X and Y are called fuzzy H-isomorphic.

Example 4.2. Let Y be a fuzzy topological space, $p_\lambda, q_\delta \in Y$ be fuzzy points and $\alpha(B)$ be a fuzzy path with the initial point p_λ and the terminal point q_δ. Let define a map

$$\alpha^+ : \Omega(Y, p_\lambda) \longrightarrow \Omega(Y, q_\delta)$$

such that $\alpha^+(\beta(D)) = m(\alpha^{-1}(B), m(\beta(D), \alpha(B)))$. Then it is clear that α^+ is a fuzzy H-homomorphism. Also

$$\alpha^+ \circ \left(\alpha^{-1}\right)^+ \simeq 1_{\Omega X}$$
$$\left(\alpha^{-1}\right)^+ \circ \alpha^+ \simeq 1_{\Omega X}.$$

Therefore α^+ is a fuzzy H-isomorphism.

Theorem 4.3. *Let (X, p_λ) and (Y, q_η) be fuzzy topological spaces and $f \in FC(X, Y)$. Then $f_+ : \Omega(X, p_\lambda) \longrightarrow \Omega(Y, q_\eta)$ defined by $f_+(\alpha(B)) = (f \circ \alpha)(B)$ is a fuzzy H-homomorphism. Also if f is a fuzzy homotopy equivalence, then f_+ is a fuzzy H-isomorphism.*

Proof. Let $\alpha(B), \beta(C) \in \Omega(X, p_\lambda)$, then

$$\begin{aligned} m \circ (f_+ \times f_+)(\alpha(B), \beta(C))(t) &= m(f_+(\alpha(B)), f_+(\beta(C))(t) \\ &= m((f \circ \alpha)(B), (f \circ \beta)(C))(t) \\ &= \begin{cases} (f \circ \alpha)\left((2t)_{B(2t)}\right) & , 0 \le t \le \frac{1}{2} \\ (f \circ \beta)\left((2t-1)_{C(2t-1)}\right) & , \frac{1}{2} \le t \le 1 \end{cases} \\ &= \begin{cases} f_+\left(\alpha(2t)_{B(2t)}\right) & , 0 \le t \le \frac{1}{2} \\ f_+\left(\beta(2t-1)_{C(2t-1)}\right) & , \frac{1}{2} \le t \le 1 \end{cases} \\ &= f_+ \circ m(\alpha(B), \beta(C)). \end{aligned}$$

So f_+ is a fuzzy H-homomorphism.

Let $g : (Y, q_\eta) \longrightarrow (X, p_\lambda)$ be the fuzzy homotopy equivalence of f. Then

$$g_+ : \Omega(Y, q_\eta) \longrightarrow \Omega(X, p_\lambda), g_+(\gamma(D)) = (g \circ \gamma)(D)$$

is a fuzzy H-homomorphism.

$$\begin{aligned} (f_+ \circ g_+)(\gamma(D)) &= f_+((g \circ \gamma)(D)) \\ &= ((f \circ g) \circ \gamma)(D) \\ &\simeq (1_Y \circ \gamma)(D) = \gamma(D) \end{aligned}$$

$$\begin{aligned} (g_+ \circ f_+)(\alpha(B)) &= g_+((f \circ \alpha)(B)) \\ &= ((g \circ f) \circ \alpha)(B) \\ &\simeq (1_X \circ \alpha)(B) = \alpha(B) \end{aligned}$$

Therefore f_+ is a fuzzy H-isomorphism. □

Definition 4.4. [4] The category whose objects are pointed fuzzy topological spaces and the set of morphisms is

$$\hom\left((X, p_\lambda), (Y, q_\eta)\right) = \left[(X, p_\lambda), (Y, q_\eta)\right]$$

is called the homotopy category of the pointed fuzzy topological spaces.

Theorem 4.5. *[12] For any category C and object Y of C, there is a contravariant functor Π^Y(or covariant functor Π_Y) from C to the category of sets and functions which associates to an object X (or Z) of C the set $\Pi^Y(X) = hom(X, Y)$ (or $\Pi_Y(Z) = hom(Y, Z)$) and to a morphism $f : X \longrightarrow X'$ (or $h : Z \longrightarrow Z'$) the function*

$$\Pi^Y(f) = f^* : \hom(X', Y) \longrightarrow \hom(X, Y)$$

(or $h_* : hom(Y, Z) \longrightarrow hom(Y, Z')$) defined by $f^*(g') = g' \circ f$, for $g' : X' \longrightarrow Y$ (or $h_*(g) = h \circ g$ for $g : Y \longrightarrow Z$).

Theorem 4.6. *[4] Let a pointed fuzzy topological space (X, p_λ) be a fuzzy H-group. Then Π^X is a contravariant functor from the homotopy category of the fuzzy pointed topological spaces to the category of groups and homomorphisms.*

Definition 4.7. [12] Let C, D be two categories and $F, G : C \longrightarrow D$ two functors from C to D. A natural transformation T from F to G is a function which

i) to each $X \in C$ assigns a morphism $T(X) \in \hom_D(F(X), G(X))$, i.e. $T(X) : F(X) \longrightarrow G(X)$;

ii) for each morphism $f \in \hom_C(X, Y)$ satisfies

$$T(Y) \circ F(f) = G(f) \circ T(X).$$

Theorem 4.8. *Let X and Y be two fuzzy H-groups and $g : X \longrightarrow Y$ be a map. Then g_* is a natural transformation from Π^X to Π^Y.*

Proof. For any map $f : Z \longrightarrow Z'$

$$\begin{aligned} (g_*(Z) \circ f^*)([h]) &= g_*(Z)([h \circ f]) = [g \circ h \circ f] \\ (f^* \circ g_*(Z))([h]) &= f^*([g \circ h]) = [g \circ h \circ f]. \end{aligned}$$

So g_* is a natural transformation. □

Theorem 4.9. *[4] Let Y be a pointed fuzzy topological space. Let the operation "$\circledast$" on $[(Y, q_\delta), (X, p_\lambda)]$ be identified such that,*

$$[g] \circledast [h] = [\mu \circ (g, h)]$$

for all $[g], [h] \in [(Y, q_\delta), (X, p_\lambda)]$. Then $([(Y, q_\delta), (X, p_\lambda)], \circledast)$ is a group with the unit element $[c]$, where $c : X \longrightarrow X$, $x \longrightarrow p$ is the constant function.

Theorem 4.10. *Let Y be a fuzzy H-group and $f : X \longrightarrow Y$ be a map. Then f_* is a natural transformation from Π^X to Π^Y in the category of groups and homomorphisms if and only if f is a fuzzy H-homomorphism.*

Proof. It is known that f_* is a natural transformation. To show that f_* is a homomorphism, let Z be any pointed fuzzy topological spaces and $g, h : Z \longrightarrow X$ be any functions. Then,

$$\begin{aligned} f_*(Z)([g] \circledast [h]) &= f_*(Z)([\mu \circ (g, h)]) \\ &= [f \circ \mu \circ (g, h)] \end{aligned}$$

$$\begin{aligned} f_*(Z)([g]) \circledast f_*(Z)([h]) &= [f \circ g] \circledast [f \circ h] \\ &= [\eta \circ (f \circ g, f \circ h)] \\ &= [\eta \circ (f \times f) \circ (g, h)]. \end{aligned}$$

Because f is a fuzzy H-homomorphism, $f \circ \mu \simeq \eta \circ (f \times f) \Rightarrow [f \circ \mu] = [\eta \circ (f \times f)]$. Consequently f_* is a homomorphism. □

Definition 4.11. Let Y be a pointed fuzzy subspace of X. If Y is itself an fuzzy H-group with the same base point as X, continuous multiplication $\mu \mid_{Y \times Y} = \eta$, homotopy inverse $\phi \mid_{Y \times Y} = \phi'$ and constant function $c \mid_{Y \times Y} = c'$ such that the inclusion map $i : Y \longrightarrow X$ is a fuzzy H-homomorphism, then Y is called a fuzzy sub-H-group of X.

Example 4.12. Let X be a fuzzy H-group. Then X itself and the one point space $\{p_\lambda\}$ are fuzzy sub-H-groups of X.

Example 4.13. Let $(G, e_\lambda, \cdot)$ be a fuzzy topological group and H be a fuzzy subgroup of G. Then H is a fuzzy sub-H-group of G.

Corollary 4.14. *If Y is a fuzzy sub-H-group of X, then there exists a fuzzy continuous multiplication $\eta : Y \times Y \to Y$ such that $i \circ \eta \simeq \mu \circ (i \times i)$.*

Theorem 4.15. *Let (Y, p_λ) be a PFTS and (Y', p_λ) be a pointed fuzzy subspace of Y. Then the fuzzy loop space $\Omega(Y', p_\lambda)$ is a fuzzy sub-H-group of the fuzzy loop space $\Omega(Y, p_\lambda)$.*

Proof. Let $i : \Omega(Y', p_\lambda) \longrightarrow \Omega(Y, p_\lambda)$ be the inclusion map. Then it is clear that $i \circ m \simeq m \circ (i \times i)$. □

Theorem 4.16. *Let Y be a fuzzy sub-H-group of X. Then for the fuzzy constant map $c' : Y \to Y$, $i \circ c' = c \circ i$.*

Proof. Let $q_\eta \in Y$, then,

$$\begin{aligned} (c \circ i)(q_\eta) &= c(i(q_\eta)) = c(q_\eta) = p_\lambda \\ (i \circ c')(q_\eta) &= i(c'(q_\eta)) = i(p_\lambda) = p_\lambda. \end{aligned}$$

Therefore $i \circ c' = c \circ i$. □

Theorem 4.17. *Let Y be a fuzzy sub-H-group of X. Then there exists a fuzzy continuous function $\varphi : Y \longrightarrow Y$ such that $i \circ \varphi \simeq \phi \circ i$, where ϕ is fuzzy homotopy inverse of X.*

Proof. Let Z be any pointed fuzzy topological space and $f : Z \longrightarrow Y$ be any function. Then $i_*(Z)$ is a homomorphism from the group $\Pi^Y(Z)$ to the group $\Pi^X(Z)$. Since $[c']$ is the unit element of $\Pi^Y(Z)$, then

$$[f] \circledast [\varphi \circ f] = [\eta \circ (f, \varphi \circ f)] = [\eta \circ (1_Y, \varphi) \circ f] = [c' \circ f] = [c']$$

Therefore $[f]^{-1} = [\varphi \circ f]$. Similarly $[i \circ f]^{-1} = [\phi \circ i \circ f]$. So

$$i_*(Z)\left([f]^{-1}\right) = (i_*(Z)([f]))^{-1} = [i \circ f]^{-1} = [\phi \circ i \circ f]$$

and

$$i_*(Z)\left([f]^{-1}\right) = i_*(Z)([\varphi \circ f]) = [i \circ \varphi \circ f].$$

Therefore $[\phi \circ i \circ f] = [i \circ \varphi \circ f]$. If we take Z as Y and f as 1_Y, then $[i \circ \varphi] = [\phi \circ i]$. □

Theorem 4.18. *Let Y be a pointed fuzzy subspace of X. If*
i) there exists a fuzzy continuous multiplication $\eta : Y \times Y \to Y$ such that $i \circ \eta \simeq \mu \circ (i \times i)$,
ii) for the fuzzy constant map $c' : Y \to Y$, $i \circ c' = c \circ i$,
iii) there exists a fuzzy continuous map $\phi' : Y \longrightarrow Y$ such that $i \circ \phi' \simeq \phi \circ i$,
iv) the inclusion map $i : Y \longrightarrow X$ is a fuzzy monomorphism,
then Y is a fuzzy sub-H-group of X.

Proof. From i) and ii)

$$\begin{aligned} i \circ \eta \circ (1_Y, c') &\simeq \mu \circ (i \times i) \circ (1_Y, c') \\ &= \mu \circ (i \circ 1_Y, i \circ c') \\ &= \mu \circ (1_X \circ i, c \circ i) \\ &= \mu \circ (1_X, c) \circ i \\ &\simeq 1_X \circ i = i \circ 1_Y. \end{aligned}$$

Therefore $i \circ \eta \circ (1_Y, c') \simeq i \circ 1_Y$. Since i is a fuzzy monomorphism, $\eta \circ (1_Y, c') \simeq 1_Y$. By the same way, $\eta \circ (c', 1_Y) \simeq 1_Y$. Thus c' is a fuzzy homotopy identity for η.
From i)

$$\begin{aligned} i \circ \eta \circ (\eta \times 1_Y) &\simeq \mu \circ (i \times i) \circ (\eta \times 1_Y) \\ &= \mu \circ [(i \circ \eta) \times (i \circ 1_Y)] \\ &\simeq \mu \circ [(\mu \circ (i \times i)) \times (1_X \circ i)] \\ &= \mu \circ (\mu \times 1_X) \circ (i \times i \times i) \\ &\simeq \mu \circ (1_X \times \mu) \circ (i \times i \times i) \\ &= \mu \circ [(1_X \circ i) \times (\mu \circ (i \times i))] \\ &\simeq \mu \circ [(i \circ 1_Y) \times (i \circ \eta)] \\ &= \mu \circ (i \times i) \circ (1_Y \times \eta) \\ &\simeq i \circ \eta \circ (1_Y \times \eta). \end{aligned}$$

Therefore, since i is a fuzzy monomorphism, $\eta \circ (\eta \times 1_Y) \simeq \eta \circ (1_Y \times \eta)$. So η is fuzzy homotopy associative.
From iii)

$$\begin{aligned} i \circ c' &= c \circ i \\ &\simeq \mu \circ (1_X, \phi) \circ i \\ &= \mu \circ (1_X \circ i, \phi \circ i) \\ &\simeq \mu \circ (i \circ 1_Y, i \circ \phi') \\ &= \mu \circ (i \times i) \circ (1_Y, \phi') \\ &\simeq i \circ \eta \circ (1_Y, \phi'). \end{aligned}$$

Since i is a fuzzy monomorphism $c' \simeq \eta \circ (1_Y, \phi')$. By the same way, $c' \simeq \eta \circ (\phi', 1_Y)$. Thus, ϕ' is a homotopy inverse for η. Therefore Y is a fuzzy H-group.
Let q_δ be the base point of Y. From ii)

$$(i \circ c')(q_\delta) = i(c'(q_\delta)) = i(q_\delta) = q_\delta$$
$$(c' \circ i)(q_\delta) = c(i(q_\delta)) = c(q_\delta) = p_\lambda.$$

So $q_\delta = p_\lambda$. Also from i), i is a fuzzy H-homomorphism. Therefore Y is a fuzzy sub-H-group of X. □

References

[1] Chang, C. L., *Fuzzy topological spaces*, Journal of mathematical Analysis and Applications, **24**(1968), 182–190. 1, 2.4
[2] Chang, Y. Z., Wang Jin, L., *Fuzzy homotopy type invariance of fundamental group of fuzzy topological space*, Fuzzy Math, **4**(1984), 53–56. 1
[3] Chong-you, Z., *Fuzzy path and fuzzy connectedness*, Fuzzy sets and Systems, **14.3**(1984), 273–280. 1, 2.9, 2.13, 2.14
[4] Demiralp, S., Güner, E., *Some characterizations of Hopf group on fuzzy topological spaces*, Iranian Journal of Fuzzy Systems, **11.(6)**(2014), 111–121. 1, 2.5, 2.8, 3.1, 3.2, 3.3, 4.4, 4.6, 4.9

[5] Dugundgi, J., *Topology*, Allyn ans Boston, Boston, 1996. 1, 2.12
[6] Foster, D.H., *Fuzzy Topological Groups*, J.Math.Anal.Appl., **67**(1979), 549–564. 2.1, 2.6
[7] Freyd, P., *Abelian Categories*, Harper & Row, New York, 1964. 1
[8] Gümüs, S., Yildiz, C., *The Fuzzy Sheaf of the Groups Formed by Fuzzy Topological Space*, Int.Jorn. of Pure and Appl. Math., **37(1)**(2007), 101–112. 2.3
[9] Güner, E.,*Fuzzy contractibility*, Fac. Sci. Univ. Ank. Series A, **1**(2007), 11-16. 2.2, 2.11
[10] Güner, E., Balci, S., *On the fuzzy sheaf of the fundamental groups*, Taiwanese Journal of Mathematics, (2007), 1057-1062. 2.10
[11] Rosenfeld, A., *Fuzzy groups*, J. Math. Anal. Appl., **35**(1971), 512–517. 2.7
[12] Spainer, E.H., *Algebraic Topology*, Springer,Verlag, 1994. 4.5, 4.7
[13] Zadeh, L. A., *Fuzzy Sets*, Information and control, **8**(1965), 3. 1