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Some Algebraic and Topological Properties of New Lucas Difference Sequence Spaces

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Abstract. Karakaş and Karabudak [22], introduced the Lucas sequence spaces $X(E)$ and studied their some properties. The main purpose of this study is to introduce the Lucas difference sequence spaces $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$ by using the Lucas sequence. Also, we prove that the spaces $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$, are linearly isomorphic to spaces c_0 and c, respectively. Besides this, the $\alpha-,\beta-$ and $\gamma-$duals of this spaces have been computed, their bases have been constructed and some topological properties of these spaces have been studied. Finally, the classes of matrices $(c_0(\hat{L},\Delta):\mu)$ and $(c(\hat{L},\Delta):\mu)$ have been characterized, where μ is one of the sequence spaces ℓ_∞, c and c_0 and derives the other characterizations for the special cases of μ.

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1. Introduction

By a sequence space, we understand a linear subspace of the space $w=\mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely non-zero sequences, where $\mathbb{N}=\{0,1,2,...\}$. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and $p-$absolutely convergent series, respectively, where $1<p<\infty$.

We shall assume throughout unless stated otherwise that $p,q>1$ with $p^{-1}+q^{-1}=1$ and $0<r<1$, and use the convention that any term with negative subscript is equal to naught.

Let λ,μ be two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,k\in\mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A:\lambda\to\mu$ if for every sequence $x=(x_k)\in\lambda$ the sequence $Ax=\{(Ax)_n\}$, the $A-$transform of x is in μ where

$$(Ax)_n=\sum_k a_{nk}x_k \quad (n\in\mathbb{N}). \tag{1.1}$$

By (λ,μ), we denote the class of all matrices A such that $A:\lambda\to\mu$. Thus, $A\in(\lambda,\mu)$ if and only if the series on the right hand side of (1.1) converges for all $n\in\mathbb{N}$ and $x\in\lambda$, and we have $Ax=\{(Ax)_n\}_{n\in\mathbb{N}}\in\mu$ for all $x\in\lambda$. A sequence x is said to be $A-$summable to α if Ax converges to α which is called the $A-$limit of x.

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Let X be a sequence space and A be an infinite matrix. The set X_A is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.2}$$

is called the domain of A in X which is a sequence space.

Let Δ denote the matrix $\Delta = (\Delta_{nk})$ defined by

$$\Delta_{nk} = \begin{cases} (-1)^{n-k} & , \quad n-1 \leq k \leq n, \\ 0 & , \quad 0 \leq k < n-1 \text{ or } k > n \end{cases}$$

for all $k, n \in \mathbb{N}$.

In the literature, the matrix domain λ_Δ is called the difference sequence space whenever λ is a normed or paranormed sequence space. The idea of difference sequence spaces was introduced by Kızmaz [23]. In 1981, Kızmaz [23] defined the sequence spaces

$$X(\Delta) = \{x = (x_k) \in w : (x_k - x_{k+1}) \in X\}$$

for $X = \ell_\infty, c$ and c_0. The difference space bv_p, consisting of all sequences (x_k) such that $(x_k - x_{k-1})$ is in the sequence space ℓ_p, was studied in the case $0 < p < 1$ by Altay and Başar [2] and in the case $1 \leq p \leq \infty$ by Başar and Altay [12] and Çolak et al. [15]. The paranormed difference sequence space

$$\Delta\lambda(p) = \{x = (x_k) \in w : (x_k - x_{k+1}) \in \lambda(p)\}$$

was examined by Ahmad and Mursaleen [1] and Malkowsky [28] where $\lambda(p)$ is any of the paranormed spaces $\ell_\infty(p), c(p)$ and $c_0(p)$ defined by Simons [30] and Maddox [26].

Recently, Başar et al. [13] have defined the sequence spaces $bv(u, p)$ and $bv_\infty(u, p)$ by

$$bv(u, p) = \{x = (x_k) \in w : \sum_k |u_k(x_k - x_{k+1})|^{p_k} < \infty\}$$

and

$$bv_\infty(u, p) = \{x = (x_k) \in w : \sup_{k\in\mathbb{N}} |u_k(x_k - x_{k+1})|^{p_k} < \infty\}$$

where $u = (u_k)$ is an arbitrary fixed sequence and $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. These spaces are generalization of the space bv_p for $1 \leq p \leq \infty$. Quite recently, Kirişçi and Başar [24] have introduced and studied the generalized difference sequence spaces

$$\hat{X} = \{x = (x_k) \in w : B(r, s)x \in X\}$$

where X denotes any of the spaces ℓ_∞, c, c_0 and ℓ_p with $1 \leq p < \infty$, and $B(r, s)x = (sx_{k-1} + rx_k)$ with $r, s \in \mathbb{R} \setminus \{0\}$. Following Kirişçi and Başar [24], Sönmez [31] has examined the sequence space $X(B)$ as the set of all sequences whose $B(r, s, t)-$ trasforms are in the space $X \in \{\ell_\infty, c, c_0, \ell_p\}$, where $B(r, s, t)$ denotes the triple band matrix $B(r, s, t) = \{b_{nk}\{r, s, t\}\}$ defined by

$$b_{nk}\{r, s, t\} = \begin{cases} r & , \quad n = k \\ s & , \quad n = k+1 \\ t & , \quad n = k+2 \\ 0 & , \quad otherwise \end{cases}$$

for all $k, n \in \mathbb{N}$ and $r, s, t \in \mathbb{R} \setminus \{0\}$. Also, several authors studied matrix transformations on sequence spaces that are the matrix domain of the difference operator, or of the matrices of some classical methods of summability in different sequence spaces, for instance we refer to [3–11, 14, 16–18, 20, 21, 27, 29, 33–35] and references therein.

In this paper, we define the Lucas difference matrix $\hat{L}$ by using the Lucas sequence $\{L_{nk}\}_{n,k=1}^{\infty}$ and introduce new sequence spaces $\ell_p(\hat{L})$ and $\ell_\infty(\hat{L})$ related to the matrix domain of $\hat{F}$ in the sequence spaces ℓ_p and ℓ_∞, respectively, where $1 \leq p < \infty$. This study is organized as follows. In Section 2, we give some notations and basic concepts including the Lucas sequence and a BK-space. In Section 3, we define a new band matrix with Fibonacci numbers and introduce the sequence spaces $\ell_p(\hat{L})$ and $\ell_\infty(\hat{L})$. Also,we establish some inclusion relations concerning these spaces and construct the basis of the space $\ell_p(\hat{L})$ for $1 \leq p < \infty$. In Section 4, we determine the $\alpha-, \beta-, \gamma-$duals of the spaces $\ell_p(\hat{L})$ and $\ell_\infty(\hat{L})$.

2. The Lucas Difference Sequence Space $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$

Define the sequence $\{f_n\}_{n=0}^{\infty}$ of Fibonacci numbers and the sequence $\{L_n\}_{n=0}^{\infty}$ of Lucas numbers given by the linear recurrence relations

$$\begin{aligned} f_0 &= 0,\ f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2}, n \geq 2 \\ L_0 &= 2,\ L_1 = 1 \text{ and } L_n = L_{n-1} + L_{n-2}, n \geq 2. \end{aligned}$$

Fibonacci and Lucas numbers have many interesting properties and applications in arts, sciences and architecture. For example, the ratio sequences of Fibonacci and Lucas numbers converges to the golden ratio which is important in sciences and arts. Also, some basic properties of Fibonacci and Lucas numbers [25] are given as follows:

$$\begin{gathered} \sum_{k=1}^{n} f_k^2 = f_n f_{n+1} \text{ and } \sum_{k=1}^{n} f_k = f_{n+2} - 1,\ n \geq 1 \\ \sum_{k=1}^{n} L_{2k} = L_{2n+1} - 1 \text{ and } \sum_{k=1}^{n} L_k = L_{n+2} - 3. \end{gathered}$$

Recently, Karakaş and Karabudak [22] established the regular matrix E by using Lucas numbers. Then, by introducing the sequence space $X(E)$ with the help of matrix E, they showed that the space $X(E)$ is a $BK-$ space, where $E = (L_{nk})$ is defined by

$$L_{nk} = \begin{cases} \frac{L_{k-1}^2}{L_n L_{n-1}} & , \quad 1 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$ and

$$X(E) = \{x = (x_k) \in w : y = Ex \in X\}.$$

In [19], they have defined the sequence spaces $c_0(\hat{F})$ and $c(\hat{F})$ as follows:

$$c_0(\hat{F}) = \{x \in w : \hat{F}x \in c_0\} \text{ and } c(\hat{F}) = \{x \in w : \hat{F}x \in c\}$$

where $\hat{F} = (\hat{f}_{nk})$ is the double band matrix defined by the sequence (f_n) of Fibonacci numbers as follows

$$f_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n} & , \quad k = n-1, \\ \frac{f_n}{f_{n+1}} & , \quad k = n, \\ 0 & , \quad 0 \leq k < n-1 \text{ or } k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Now, let us define the sets $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$ as the sets of all sequence spaces whose $\hat{L} = (L_{nk})$ transforms are in the well-know sequence spaces c_0 and c, respectively, namely,

$$\begin{aligned} c_0(\hat{L},\Delta) &= \left\{x \in w : \lim_{n\to\infty}\left(\frac{L_n L_{n-1} + 2}{L_{n-1}^2} x_n - \frac{L_{n-1}L_{n-2} + 2}{L_{n-1}^2} x_{n-1}\right) = 0\right\}, \\ c(\hat{L},\Delta) &= \left\{x \in w : \exists l \in \mathbb{C} \ni \lim_{n\to\infty}\left(\frac{L_n L_{n-1} + 2}{L_{n-1}^2} x_n - \frac{L_{n-1}L_{n-2} + 2}{L_{n-1}^2} x_{n-1}\right) = l\right\} \end{aligned}$$

where $\hat{L} = (\hat{L}_{nk})$ is the double band matrix defined by the sequence (L_n) of Lucas numbers as follows

$$L_{nk} = \begin{cases} -\frac{L_{n-1}L_{n-2}+2}{L_{n-1}^2} & , \quad k = n-1, \\ \frac{L_n L_{n-1}+2}{L_{n-1}^2} & , \quad k = n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k, n \in \mathbb{N}$. With the help of the notation of (1.2), the spaces $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$ can be rewritten as follows:

$$c_0(\hat{L},\Delta) = (c_0)_{\hat{L}} \text{ and } c(\hat{L},\Delta) = (c)_{\hat{L}}. \tag{2.1}$$

Define the sequence $y = (y_n)$, which will be frequently used, by the $\hat{L}-$transform of a sequence $x = (x_n)$, i.e.,

$$y_n = (\hat{L}x)_n = \frac{L_n L_{n-1} + 2}{L_{n-1}^2} x_n - \frac{L_{n-1}L_{n-2} + 2}{L_{n-1}^2} x_{n-1}, (n \in \mathbb{N}). \tag{2.2}$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *The sets $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$ are linear spaces with coordinatewise addition and scalar multiplication that are $BK-$spaces with norm*

$$\|x\|_{c_0(\hat{L},\Delta)} = \|\hat{L}x\|_\infty \text{ and } \|x\|_{c(\hat{L},\Delta)} = \|\hat{L}x\|_\infty.$$

Proof. The proof of the first part of the theorem is a routine verification, and so we omit it. Furthermore, since (2.1) holds, c_0 and c are $BK-$spaces with respect to its natural norm, and the matrix $\hat{L}$ is a triangle, Theorem 4.3.2 of Wilansky [36] implies that the spaces $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$ are $BK-$spaces with the given norms. This completes the proof. □

Theorem 2.2. *$c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$ are linearly isomorphic to the spaces c_0 and c, respectively, i.e., $c_0(\hat{L},\Delta) \cong c_0$ and $c(\hat{L},\Delta) \cong c$.*

Proof. Since the proof is similar for the space $c(\hat{L},\Delta)$, we consider only the space $c_0(\hat{L},\Delta)$. To prove this theorem, we should show the existence of a linear bijection between the spaces $c_0(\hat{L},\Delta)$ and c_0. Consider the transformation S from $c_0(\hat{L},\Delta)$ to c_0 by $y = Sx = \hat{L}x$. The linearity of S is clear. Further, it is obvious that $x = \theta$ whenever $Sx = \theta$ and hence S is injective where $\theta = (0,0,0,...)$.

Let us take any $y \in c_0$ and define the sequence $x = \{x_k\}$ by

$$x_k = \frac{1}{L_kL_{k-1}+2}\sum_{j=1}^{k} L_{j-1}^2 y_j; \text{ for all } k \in \mathbb{N}.$$

Then, we have

$$\begin{aligned}
\lim_{k\to\infty}(\hat{L}x)_k &= \lim_{k\to\infty}\left(\frac{L_kL_{k-1}+2}{L_{k-1}^2}x_k - \frac{L_{k-1}L_{k-2}+2}{L_{k-1}^2}x_{k-1}\right)\\
&= \lim_{k\to\infty}\left(\frac{L_kL_{k-1}+2}{L_{k-1}^2}\frac{1}{L_kL_{k-1}+2}\sum_{j=1}^{k}L_{j-1}^2y_j - \frac{L_{k-1}L_{k-2}+2}{L_{k-1}^2}\frac{1}{L_{k-1}L_{k-2}+2}\sum_{j=1}^{k-1}L_{j-1}^2y_j\right)\\
&= \lim_{k\to\infty} y_k = 0
\end{aligned}$$

which says us that $x \in c_0(\hat{L},\Delta)$. Additionally, we have for every $x \in c_0(\hat{L},\Delta)$ that

$$\begin{aligned}
\|x\|_{c_0(\hat{L},\Delta)} &= \sup_{k\in\mathbb{N}}\left|\frac{L_kL_{k-1}+2}{L_{k-1}^2}x_k - \frac{L_{k-1}L_{k-2}+2}{L_{k-1}^2}x_{k-1}\right|\\
&= \sup_{k\in\mathbb{N}}|y_k| = \|y\|_\infty < \infty.
\end{aligned}$$

Consequently, we see from here that S is surjective. Hence, S is a linear bijection which therefore says us that the spaces $c_0(\hat{L},\Delta)$ and c_0 are linearly isomorphic, as desired. □

We wish to exhibit some inclusion relations concerning with the spaces $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$, in the present section. Here and after, by λ we denote any of the sets $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$ and μ denotes any of the spaces c_0 or c.

Theorem 2.3. *The inclusions $\mu \subset \lambda$ hold.*

Proof. Let $x = (x_k) \in \mu$. Then, since it is immediate that

$$\begin{aligned}
\|x\|_\lambda = \|\hat{L}x\|_\infty &= \sup_{k\in\mathbb{N}}\left|\sum_{j=k-1}^{k}\frac{(-1)^{k-j}L_jL_{j-1}+2}{L_{k-1}^2}x_j\right|\\
&\leq \|x\|_\infty \sup_{k\in\mathbb{N}}\sum_{j=k-1}^{k}\frac{(-1)^{k-j}L_jL_{j-1}+2}{L_{k-1}^2} = \|x\|_\infty.
\end{aligned}$$

The inclusion $\mu \subset \lambda$ holds. □

Since the isomorphism S, defined in Theorem 2.2, is surjective, the inverse image of the basis of the spaces c_0 and c is the basis of the new spaces $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$. Therefore, we have the following theorem without proof.

Theorem 2.4. *Define the sequence $b^{(k)} = \{b_n^{(k)}\}_{n\in\mathbb{N}}$ of elements of the spaces $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)} = \begin{cases} \dfrac{L_{k-1}^2}{L_nL_{n-1}+2} & , \quad k \geq n, \\ 0 & , \quad 0 \leq k < n. \end{cases}$$

Let $\lambda_k = (\hat{L}x)_k$ for all $k \in \mathbb{N}$. Then the following assertions are true:

(i): *The sequence $\{b^{(k)}\}_{k\in\mathbb{N}}$ is a basis for the space $c_0(\hat{L},\Delta)$ and any $x \in c_0(\hat{L},\Delta)$ has a unique representation of the form*

$$x = \sum_k \lambda_k b^{(k)}.$$

(ii): *The set $\{e, b^{(k)}\}_{k\in\mathbb{N}}$ is a basis for the space $c(\hat{L},\Delta)$ and any $x \in c(\hat{L},\Delta)$ has a unique representation of the form*

$$x = le + \sum_k [\lambda_k - l]\, b^{(k)},$$

where $l = \lim_{k\to\infty}(\hat{L}x)_k$.

Remark 2.5. It is well known that every Banach space X with a Schauder basis is separable.

From Theorem 2.4 and Remark 2.5, we can give the following corollary:

Corollary 2.6. *The spaces $\ell_p(\hat{L})$ is separable.*

3. Duals of The New Sequence Spaces

In this section, we state and prove the theorems determining the $\alpha-,\beta-$ and $\gamma-$ duals of the sequence space $c_0(\hat{L},\Delta)$. Since the case $p = 1$ can be proved by analogy, we omit the proof of that case and consider only the case $1 < p \leq \infty$.

The set $S(\lambda,\mu)$ defined by

$$S(\lambda,\mu) = \{z = (z_k) \in w : xz = (x_kz_k) \in \mu \text{ for all } x = (x_k) \in \lambda\} \tag{3.1}$$

is called the multiplier space of the sequence spaces λ and μ. One can easily observe for a sequence space ν with $\lambda \supset \nu \supset \mu$ that the inclusions

$$S(\lambda,\mu) \subset S(\nu,\mu) \text{ and } S(\lambda,\mu) \subset S(\lambda,\nu)$$

hold. With the notation of (3.1), the alpha-, beta- and gamma-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined by

$$\lambda^\alpha = S(\lambda,\ell_1), \lambda^\beta = S(\lambda, cs) \text{ and } \lambda^\gamma = S(\lambda, bs).$$

For to give the alpha-, beta- and gamma-duals of the space $c_0(\hat{L},\Delta)$ of non-absolute type, we need the following Lemma;

Lemma 3.1. *[32] $A \in (c_0 : \ell_1) = (c : \ell_1)$ if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} \right| < \infty$$

where $\mathcal{F}$ denotes the collection of all finite subsets of $\mathbb{N}$.

Lemma 3.2. *[32]*

(i) *$A \in (c_0 : c)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty \tag{3.2}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n\to\infty} a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}. \tag{3.3}$$

(ii) *$A \in (c : c)$ if and only if (3.2) and (3.3) hold and*

$$\exists \alpha \in \mathbb{C} \ni \lim_{n\to\infty} \sum_k a_{nk} = \alpha. \tag{3.4}$$

Lemma 3.3. *[32] $A \in (c_0 : \ell_\infty) = (c : \ell_\infty)$ if and only if (3.3) holds.*

Theorem 3.4. *Define the set d_1 as follows:*

$$d_1 = \left\{ a = (a_k) \in w : \sup_{K \in \mathcal{F}} \sum_{n=0}^{\infty} \left| \sum_{k \in K} \frac{L_{k-1}^2}{L_n L_{n-1} + 2} a_n \right| < \infty \right\}$$

Then, $\left\{c_0(\hat{L}, \Delta)\right\}^\alpha = \left\{c(\hat{L}, \Delta)\right\}^\alpha = d_1$.

Proof. Let $a = (a_k) \in w$. Then, we can easily derive with (2.2) that

$$a_n x_n = \sum_{k=1}^{n} \frac{L_{k-1}^2}{L_n L_{n-1} + 2} a_n y_k = (By)_n, \quad (n \in \mathbb{N}); \tag{3.5}$$

where $B = (b_{nk})$ is defined by the formula

$$b_{nk} = \begin{cases} \frac{L_{k-1}^2}{L_n L_{n-1}+2} a_n & , \quad (1 \leq k \leq n) \\ 0 & , \quad (k > n) \end{cases}, (n, k \in \mathbb{N}).$$

It follows from (3.5) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in c_0(\hat{L}, \Delta)$ if and only if $By \in \ell_1$ whenever $y \in c_0$, i.e., $a = (a_n)$ is in the $\alpha-$dual of the space $c_0(\hat{L}, \Delta)$ if and only if $B \in (c_0 : \ell_1)$. This gives the result by Lemma 3.1 that $\left\{c_0(\hat{L}, \Delta)\right\}^\alpha = d_1$. Similarly, we get $\left\{c(\hat{L}, \Delta)\right\}^\alpha = d_1$. □

Theorem 3.5. *Define the sets d_2, d_3 and d_4 as follows:*

$$\begin{aligned}
d_2 &= \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} \frac{L_{j-1}^2}{L_k L_{k-1} + 2} a_j \right| < \infty \right\}, \\
d_3 &= \left\{ a = (a_k) \in w : \exists \alpha_k \in \mathbb{C} \ni \lim_{n \to \infty} \left(\sum_{j=k}^{n} \frac{L_{j-1}^2}{L_k L_{k-1} + 2} a_j \right) = \alpha_k \textit{ for each } k \in \mathbb{N} \right\}, \\
d_4 &= \left\{ a = (a_k) \in w : \exists \alpha \in \mathbb{C} \ni \lim_{n \to \infty} \sum_{k=0}^{\infty} \left(\sum_{j=k}^{n} \frac{L_{j-1}^2}{L_k L_{k-1} + 2} a_j \right) = \alpha \right\}.
\end{aligned}$$

Then, $\{c_0(\hat{L}, \Delta)\}^\beta = d_2 \cap d_3$ and $\{c(\hat{L}, \Delta)\}^\beta = d_2 \cap d_3 \cap d_4$.

Proof. Consider the equation

$$\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\sum_{j=1}^{k} \frac{L_{j-1}^2}{L_k L_{k-1} + 2} y_j \right] a_k \\
&= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \frac{L_{j-1}^2}{L_k L_{k-1} + 2} a_j \right] y_k = (Dy)_n
\end{aligned} \tag{3.6}$$

where $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \sum_{j=k}^{n} \frac{L_{j-1}^2}{L_k L_{k-1}+2} a_j & , \quad (0 \leq k \leq n) \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$.

Thus, we decude by with (3.6) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c_0(\hat{L}, \Delta)$ if and only if $Dy \in c$ whenever $y = (y_k) \in c_0$. Therefore $a = (a_n) \in \{c_0(\hat{L}, \Delta)\}^\beta$ if and only if $D \in (c_0 : c)$. Then, we derive from (3.2) and (3.3) that

$$\begin{aligned}
&\sup_{n \in \mathbb{N}} \sum_{k} |d_{nk}| < \infty \\
&\exists \alpha_k \in \mathbb{C} \ni \lim_{n \to \infty} d_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}
\end{aligned}$$

which shows that $\{c_0(\hat{L}, \Delta)\}^\beta = d_2 \cap d_3$. It is clear that one can also prove $\{c(\hat{L}, \Delta)\}^\beta = d_2 \cap d_3 \cap d_4$. So, we leave the detailed proof to the reader. □

Theorem 3.6. $\{c_0(\hat{L},\Delta)\}^\gamma = \{c(\hat{L},\Delta)\}^\gamma = d_2$.

Proof. This is obtained in the similar way used in the proof of Theorem 3.5 by using Lemma 3.3 instead of Lemma 3.2. □

4. Matrix Transformations Related to The New Sequence Spaces

In this section, we characterize the matrix transformations from the spaces $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$ into any given sequence space μ and from the sequence space μ into the space $c_0(\hat{L},\Delta)$ and $c(\hat{L},\Delta)$, where μ is any of the spaces ℓ_∞, c and c_0.

Since $c_0(\hat{L},\Delta)\cong c_0$ (or $c(\hat{L},\Delta)\cong c$), we can say: The equivalence "$x\in c_0(\hat{L},\Delta)$ (or $x\in c(\hat{L},\Delta)$), if and only if $y\in c_0$ (or $y\in c$) " holds.

In what follows, for brevity, we write

$$\tilde{a}_{nk} := \sum_{j=k}^{\infty} \frac{L_{j-1}^2}{L_kL_{k-1}+2}a_{nj}$$

for all $k,n\in\mathbb{N}$.

Theorem 4.1. *Suppose that the entries of the infinite matrices $A=(a_{nk})$ and $E=(e_{nk})$ are connected with the relation*

$$e_{nk} := \tilde{a}_{nk} \tag{4.1}$$

for all $k,n\in\mathbb{N}$. Let μ be any given sequence space. Then,

- *$A\in(c_0(\hat{L},\Delta):\mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}}\in c_0(\hat{L},\Delta)^\beta$ for all $n\in\mathbb{N}$ and $E\in(c_0:\mu)$.*
- *$A\in(c(\hat{L},\Delta):\mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}}\in c(\hat{L},\Delta)^\beta$ for all $n\in\mathbb{N}$ and $E\in(c:\mu)$.*

Proof. We prove only first part of the theorem. Suppose that (4.1) holds between $A=(a_{nk})$ and $E=(e_{nk})$, and take into account that the space $c_0(\hat{L},\Delta)$ and c_0 are linearly isomorphic.

Let $A\in(c_0(\hat{L},\Delta):\mu)$ and take any $y=(y_k)\in c_0$. Then $E\hat{L}$ exists and $\{a_{nk}\}_{k\in\mathbb{N}}\in d_2\cap d_3$ which yields that $\{e_{nk}\}_{k\in\mathbb{N}}\in c_0$ for each $n\in\mathbb{N}$. Hence, Ey exists and thus

$$\sum_k e_{nk}y_k = \sum_k a_{nk}x_k$$

for all $n\in\mathbb{N}$.

We have that $Ey=Ax$ which leads us to the consequence $E\in(c_0:\mu)$.

Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}}\in\{c_0(\hat{L},\Delta)\}^\beta$ for each $n\in\mathbb{N}$ and $E\in(c_0:\mu)$, and take any $x=(x_k)\in c_0(\hat{L},\Delta)$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{\infty} a_{nk}x_k = \sum_{k=0}^{\infty}\left[\sum_{j=k}^{n}\frac{L_{j-1}^2}{L_kL_{k-1}+2}a_{kj}\right]y_k$$

for all $n\in\mathbb{N}$, that $Ey=Ax$ and this shows that $A\in(c_0(\hat{L},\Delta):\mu)$. This completes the proof. □

Theorem 4.2. *Suppose that the elements of the infinite matrices $A=(a_{nk})$ and $B=(b_{nk})$ are connected with the relation*

$$b_{nk} := \sum_{j=n-1}^{n}\frac{(-1)^{n-j}L_jL_{j-1}+2}{L_{n-1}^2}a_{jk} \text{ for all } k,n\in\mathbb{N}. \tag{4.2}$$

Let μ be any given sequence space. Then, $A\in(\mu:c(\hat{L},\Delta))$ if and only if $B\in(\mu:c)$.

Proof. Let $z=(z_k)\in\mu$ and consider the following equality.

$$\sum_{k=0}^{m} b_{nk}z_k = \sum_{j=n-1}^{n}\frac{(-1)^{n-j}L_jL_{j-1}+2}{L_{n-1}^2}\left(\sum_{k=0}^{m}a_{jk}z_k\right) \text{ for all } m,n\in\mathbb{N}$$

which yields as $m\to\infty$ that $(Bz)_n=\{\hat{L}(Az)\}_n$ for all $n\in\mathbb{N}$. Therefore, one can observe from here that $Az\in c_0(\hat{L},\Delta)$ whenever $z\in\mu$ if and only if $Bz\in c_0$ whenever $z\in\mu$. This completes the proof. □

$$\lim_{n\to\infty}\sum_k |a_{nk}| = \sum_k \left|\lim_{n\to\infty} a_{nk}\right|, \tag{4.3}$$

$$\lim_{n\to\infty}\sum_k |a_{nk}| = 0, \tag{4.4}$$

$$\lim_{n\to\infty}\sum_k a_{nk} = 0. \tag{4.5}$$

The folowing results are due to Stieglitz and Tietz [32]:

Lemma 4.3. *Let $A = (a_{nk})$ be an infinite matrix. Then*

(i) *$A = (a_{nk}) \in (\ell_\infty : c_0)$ if and only if (4.4) holds.*
(ii) *$A = (a_{nk}) \in (\ell_\infty : c)$ if and only if (3.2) and (4.3) hold.*
(iii) *$A = (a_{nk}) \in (c : \ell_\infty)$ if and only if (3.2) holds.*
(iv) *$A = (a_{nk}) \in (c : c_0)$ if and only if (3.2), (4.5) hold and (3.3) holds with $\alpha_k = 0$.*
(v) *$A = (a_{nk}) \in (c_0 : c_0)$ if and only if (3.2) holds and (3.3) holds with $\alpha_k = 0$.*

Now, we can give the following results:

Corollary 4.4. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i) *$A \in (c_0(\hat{L}, \Delta) : \ell_\infty)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{c_0(\hat{L}, \Delta)\}^\beta$ for all $n \in \mathbb{N}$ and (3.2) holds with $\tilde{a}_{nk}$ instead of a_{nk}.*
(ii) *$A \in (c_0(\hat{L}, \Delta) : c)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{c_0(\hat{L}, \Delta)\}^\beta$ for all $n \in \mathbb{N}$ and (3.2) and (3.3) hold with $\tilde{a}_{nk}$ instead of a_{nk}.*
(iii) *$A \in (c_0(\hat{L}, \Delta) : c_0)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{c_0(\hat{L}, \Delta)\}^\beta$ for all $n \in \mathbb{N}$ and (3.2) and (4.3) hold with $\alpha_k = 0$ as $\tilde{a}_{nk}$ instead of a_{nk}.*

Corollary 4.5. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i) *$A \in (c(\hat{L}, \Delta) : \ell_\infty)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{c(\hat{L}, \Delta)\}^\beta$ for all $n \in \mathbb{N}$ and (3.2) holds with $\tilde{a}_{nk}$ instead of a_{nk}.*
(ii) *$A \in (c(\hat{L}, \Delta) : c)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{c(\hat{L}, \Delta)\}^\beta$ for all $n \in \mathbb{N}$ and (3.2)-(3.4) hold with $\tilde{a}_{nk}$ instead of a_{nk}.*
(iii) *$A \in (c(\hat{L}, \Delta) : c_0)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{c(\hat{L}, \Delta)\}^\beta$ for all $n \in \mathbb{N}$ and (3.2), (4.5) hold and (3.3) holds with $\alpha_k = 0$ as $\tilde{a}_{nk}$ instead of a_{nk}.*

Corollary 4.6. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i) *$A = (a_{nk}) \in (c_0 : c_0(\hat{L}, \Delta))$ if and only if (3.2) holds and (3.3) holds with $\alpha_k = 0$ as b_{nk} instead of a_{nk}.*
(ii) *$A = (a_{nk}) \in (c : c_0(\hat{L}, \Delta))$ if and only if (3.2), (4.5) hold and (3.3) holds with $\alpha_k = 0$ as b_{nk} instead of a_{nk}.*
(iii) *$A = (a_{nk}) \in (\ell_\infty : c_0(\hat{L}, \Delta))$ if and only if (4.4) holds with b_{nk} instead of a_{nk}.*
(iv) *$A = (a_{nk}) \in (c_0 : c(\hat{L}, \Delta))$ if and only if (3.2) and (3.3) hold with b_{nk} instead of a_{nk}.*
(v) *$A = (a_{nk}) \in (c : c(\hat{L}, \Delta))$ if and only if 3.2), (3.3) and (3.4) hold with b_{nk} instead of a_{nk}.*
(vi) *$A = (a_{nk}) \in (\ell_\infty : c(\hat{L}, \Delta))$ if and only if (3.2) and (4.3) hold with b_{nk} instead of a_{nk}.*

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