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Some Inequalities for Ricci Solitons

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Abstract. We deal with a submanifold of a Ricci soliton $(\bar{M}, \bar{g}, V, \lambda)$ and obtain that under what conditions such a submanifold is Ricci soliton. Moreover, we establish some inequalities for Ricci solitons to obtain the relationships between the intrinsic or extrinsic invariants.

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1. Introduction

The notion of Ricci soliton is appeared after Hamilton introduced the Ricci flow in 1982 and the self similar solutions of such a flow is Ricci soliton. According to the definition of Hamilton, a Riemannian manifold (M, g) is called a Ricci soliton if it admits a smooth vector field V on M such that

$$\frac{1}{2}\mathcal{L}_V g + Ric + \lambda g = 0, \tag{1.1}$$

where $\mathcal{L}_V g$ is a Lie-derivative of the metric tensor g with respect to V, Ric is the Ricci tensor of (M, g), λ is a constant and X, Y are arbitrary vector fields on M. Hence the Ricci soliton denotes (M, g, V, λ).

A smooth vector field V is called a potential field of the Ricci soliton. A Ricci soliton (M, g, V, λ) is said to be shrinking, steady or expanding according to $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. Moreover, if the potential field V is zero or Killing, then the Ricci soliton becomes trivial, in which case the metric is Einstein. If the potential field V is the gradient of some smooth function f on M, then the Ricci soliton is called gradient and denotes by (M, g, f, λ). Obviously, a gradient Ricci soliton (M, g, f, λ) is said to be trivial, if its potential function f is a constant.

Ricci solitons have become popular after Grigori Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904, also because Ricci solitons model the formation of singularities in the Ricci flow, and they correspond to self-similar solutions, i.e., they are stationary points of the Ricci flow in the space of metrics modulo diffeomorphisms and scalings (for details, see [13]). In 2011, the authors dealt with immersions of a Ricci soliton (M, g, V, λ) into a Riemannian manifold $\overline{M}$ and showed that a shrinking Ricci soliton immersed into a space form with constant mean curvature must be Gaussian soliton [1]. Then, B. Y. Chen and S. Deshmukh in [9] classified the Ricci solitons with concurrent potential fields and they derived a necessary and sufficient condition for a submanifold to be

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a Ricci soliton in a Riemannian manifold equipped with a concurrent vector field in. Also, they completely classified Ricci solitons on Euclidean hypersurfaces arisen from the position vector field of hypersurfaces (see [11]). Therefore, one can find a lot of papers on Ricci solitons in the literature (for details, we refer to [8, 10, 15].

Recently, some mathematicians are interested in for studying the concept of Ricci soliton for different kinds of manifolds, such as contact, paracontact, Sasakian, and etc. For example, in [2], Bejan and Crasmareanu investigated the Eisenhart problem of finding parallel tensors treated already in the framework of quasi-constant curvature manifolds for symmetric case for giving some characterizations in terms of Ricci solitons. Also, they studied the class of parallel symmetric tensor fields of (0,2)-type and possible Lorentz Ricci solitons (see [3]). On the other hand, in [5], the authors dealt with $\eta-$Ricci solitons in the case when its potential vector field is the characteristic vector field ξ and when it is torse-forming. They obtained some results for $\eta-$Ricci solitons on $\varepsilon-$almost paracontact metric manifolds (for other papers, we refer to [4, 6, 16].

The present paper is organized as follows: In Section 2, we recall some basic notions which are needed. In Section 3, we deal with a submanifold M of a Ricci soliton $(\overline{M}, \bar{g}, V, \lambda)$ and calculate the scalar curvature of such a submanifold M. Moreover, we establish an inequality for Ricci solitons to obtain the relationships between the intrinsic or extrinsic invariants, such as the scalar curvature, sectional curvature or mean curvature.

2. Preliminaries

2.1. Basic Formulas and Definitions for Submanifolds

We recall now some notions from [9]:

Let $(\overline{M}, \bar{g})$ be an $m-$dimensional Riemannian manifold and $\phi : M \to \overline{M}$ be an isometric immersion from an $n-$dimensional Riemannian manifold (M, g) into $(\overline{M}, \bar{g})$. The Levi-Civita connections of ambient manifold $\overline{M}$ and the submanifold M will be denote by ∇ and $\overline{\nabla}$, respectively.

For any $X, Y \in \Gamma(TM)$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.1}$$

where $\nabla_X Y$ and $h(X, Y)$ are the tangential and the normal parts of $\overline{\nabla}_X Y$ and the equation (2.1) is called the Gauss formula and h is called the second fundamental form of M.

Similarly, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$,

$$\overline{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{2.2}$$

where $-A_V X$ and $\nabla^\perp_X V$ are the tangential and normal parts of $\overline{\nabla}_X V$, respectively. The equation (2.2) is called the Weingarten formula. Here A_V and $\nabla^\perp$ denote a shape operator or the fundamental tensor of Weingarten with respect to the normal section V and the normal connection of M in the ambient space $\overline{M}$.

For any $V \in \Gamma(TM^\perp)$, the shape operator and the second fundamental form are related by

$$\overline{g}(h(X, Y), V) = g(A_V X, Y).$$

The mean curvature vector field H of M in $\overline{M}$ is given by

$$H = (\frac{1}{n}) \sum_{i=1}^{n} h(e_i, e_i)$$

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis of submanifold M. Moreover, the submanifold M is totally umbilical if and only if

$$h(X, Y) = g(X, Y)H$$

where H denotes the mean curvature vector field of M, for any $X, Y \in \Gamma(TM)$.

The equations of Gauss and Codazzi are given by the following

$$\begin{aligned} g(R(X, Y)Z, W) &= \overline{g}(\overline{R}(X, Y)Z, W) + \overline{g}(h(X, W), h(Y, Z)) \\ &\quad - \overline{g}(h(X, Z), h(Y, W)) \\ (\overline{R}(X, Y)Z)^\perp &= (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \end{aligned} \tag{2.3}$$

for any vectors $X, Y, Z, W \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$. Here $(\overline{R}(X, Y)Z)^{\perp}$ is the normal component of $\overline{R}(X, Y)Z$ and $\overline{\nabla}h$ is defined by

$$(\overline{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Let P be a 2-plane section spanned by orthonormal vectors X and Y. From (2.3), one has

$$K(P) = \overline{K}(P) - \|h(X, Y)\|^2 + g(h(X, X), h(Y, Y)).$$

Let $\{e_1, e_2, ..., e_n\}$ be any orthonormal basis for T_pM, at $p \in M$. The Ricci tensor Ric is defined by

$$Ric(X, Y) = \sum_{j=1}^{n} R(e_j, X, Y, e_j)$$

where R is the Riemannian curvature tensor of M, for any $X, Y \in T_pM$.

Now, let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r belongs to an orthonormal basis $\{e_{n+1}, ..., e_m\}$ of the normal space $T_p^{\perp}M$. We put

$$h^r_{ij} = g(h(e_i, e_j), e_r) \quad \text{and} \quad \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

Let $(\overline{M}, \bar{g})$ be an $m-$dimensional Riemannian manifold and $\phi : M \to \overline{M}$ be an isometric immersion from an $n-$dimensional Riemannian manifold (M, g) into $(\overline{M}, \bar{g})$. Then, the Ricci tensor Ric can be written as

$$\overline{Ric}(X, Y) = \overline{Ric}\mid_{T_pM} (X, Y) + \overline{Ric}\mid_{T_p^{\perp}M} (X, Y)$$

for any $X, Y \in T_pM$.
Throughout this paper, we suppose that the normal part of Ricci tensor $\overline{Ric}\mid_{T_p^{\perp}M} (X, Y)$ vanishes.

Let $\overline{M}$ be a Riemannian manifold and $\phi : M \to \overline{M}$ be an isometric immersion such that K_{ij} and $\overline{K}_{ij}$ denote the sectional curvatures of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\overline{M}$, respectively. Thus, K_{ij} and $\overline{K}_{ij}$ are called the "intrinsic" and "extrinsic" sectional curvatures of the Span$\{e_i, e_j\}$ at $p \in M$, respectively. From (2.3), we get

$$K_{ij} = \overline{K}_{ij} + \sum_{r=n+1}^{m} (h^r_{ii} h^r_{jj} - (h^r_{ij})^2). \tag{2.4}$$

In the view of (2.4), it follows that

$$2\tau(p) = 2\overline{\tau}(T_pM) + n^2\|H\|^2 - \|h\|^2, \tag{2.5}$$

where

$$\overline{\tau}(T_pM) = \sum_{1 \le i < j \le n} \overline{K}_{ij}$$

denotes the scalar curvature of the $n-$plane section T_pM in the ambient manifold $\overline{M}$. Thus, $\tau(p)$ and $\overline{\tau}(T_pM)$ are called the "intrinsic" and "extrinsic" scalar curvatures of the submanifold M at $p \in M$, respectively.

On the other hand, the divergence of any vector field X on $\Gamma(TM)$ is shown by $div(X)$ and defined by

$$div(X) = \sum_{i=1}^{n} g(\nabla_{e_i} X, e_i), \tag{2.6}$$

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis of $\Gamma(TM)$. For details, we refer to [7].

3. The Submanifolds of Ricci Solitons

The Ricci solitons are natural extension of Einstein manifolds and self-similar solutions to their Ricci flow equation which is proved by Hamilton in [12]. In this section, we give the scalar curvature of submanifold M of Ricci soliton $(\overline{M}, \bar{g}, \lambda, V)$ to obtain a relationship between the intrinsic and extrinsic invariants. Then, we give an inequality for the Ricci soliton $(\overline{M}, \bar{g}, V, \lambda)$ to characterize such a submanifold M.

We begin to this section with the following lemma:

Lemma 3.1. *Let $(\overline{M}, \bar{g}, V, \lambda)$ be a Ricci soliton and $\phi : M \to \overline{M}$ be an isometric immersion. Then, we have*

$$2\tau(p) + div(V) - n^2\|H\|^2 + \lambda n + \|h\|^2 = 0, \tag{3.1}$$

where the potential vector field $V \in \Gamma(TM)$ of the Ricci soliton, τ, $\|H\|$ and h denote the scalar curvature, mean curvature and second fundamental form of M, respectively.

Proof. Since $\overline{M}$ is a Ricci soliton, one has

$$\frac{1}{2}\sum_{i=1}^{n}\{g(\nabla_{e_i}V, e_i) + g(\nabla_{e_i}V, e_i)\} + \sum_{i=1}^{n}\overline{Ric}(e_i, e_i) + \sum_{i=1}^{n}\lambda g(e_i, e_i) \quad = \quad 0, \tag{3.2}$$

where $\{e_1, ..., e_n\}$ denotes an orthonormal basis of T_pM. Then, using (2.5) and (2.6) in the equality (3.2), one has

$$div(V) + 2\bar{\tau}\mid_{T_pM} + \lambda\sum_{i=1}^{n} g(e_i, e_i) \quad = \quad 0. \tag{3.3}$$

From (2.3) and (3.3), we have

$$div(V) + 2\tau(p) - \sum_{i,j=1}^{n}\{g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_i, e_j)\} + \lambda n = 0.$$

Then, the proof is completed. □

We here recall that the following lemma from [14]:

Lemma 3.2. *If $a_1, a_2, ..., a_n$ are $(n > 1)$ real numbers, then*

$$\frac{1}{n}\{\sum_{i=1}^{n} a_i\}^2 \quad \leq \quad \sum_{i=1}^{n} a_i^2,$$

with equality holding if and only if $a_1 = a_2 = ... = a_n$.

Using Lemma 3.2, we can improve the equality (3.1) as follows:

Theorem 3.3. *Let $(\overline{M}, \bar{g}, V, \lambda)$ be a Ricci soliton and $\phi : M \to \overline{M}$ be an isometric immersion. Then, we have*

$$2\tau(p) \quad \leq \quad -div(V) - \lambda n + n(n-1)\|H\|^2 \tag{3.4}$$

for $V \in \Gamma(TM)$. If the equality of (3.4) holds, then M is totally umbilical.

Proof. From (3.1), we get

$$\begin{aligned} 2\tau(p) \quad &= -div(V) + n^2\|H\|^2 - \lambda n - \|h\|^2 \\ &= -div(V) + n^2\|H\|^2 - \lambda n - \sum_{i=1}^{n}\sum_{s=n+1}^{m}(h_{ii}^s)^2 - \sum_{i,j\neq 1}^{n}\sum_{s=n+1}^{m}(h_{ij}^s)^2 \\ &\leq -div(V) + n^2\|H\|^2 - \lambda n - \frac{1}{n}(n^2\|H\|^2) - \sum_{i,j\neq 1}^{n}\sum_{s=n+1}^{m}(h_{ij}^s)^2 \\ &\leq -div(V) - \lambda n + n(n-1)\|H\|^2 - \sum_{i,j\neq 1}^{n}\sum_{s=n+1}^{m}(h_{ij}^s)^2 \end{aligned}$$

is found. Then,

$$2\tau(p) \quad \leq \quad -div(V) - \lambda n + n(n-1)\|H\|^2$$

is obtained which gives us (3.4). If the equality of (3.4) is satisfied, then M is totally umbilical. □

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