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Local T_1 Preordered Spaces

Mehmet Baran

Department of Mathematics, Faculty of Sciences, Erciyes University, 38039, TURKEY.

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Abstract. The aim of this paper is to characterize local T_1 preordered spaces as well as to investigate some invariance properties of them.

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1. Introduction

There is a connection between topology and order. A topological space defines an preordered (reflexive and transitive) relation and given a preordered relation on a set one can get a topology (see [14, 19]). Domain theory which can be considered as a branch of order theory studies special kinds of partially ordered sets, namely, directed complete partial orders of a domain, i.e., of a non-empty subset of the order in which each two elements have some upper bound that is an element of this subset has a least upper bound. The primary motivation for the study of domains, which was initiated by Dana Scott in the late 1960s, was the search for a denotational semantics of the lambda calculus, especially for functional programming languages in computer [15, 16, 18, 23–26].

In 1991, Baran [2], introduced a local T_1 object in a topological category which was used to define the notion of strongly closed subobject of an object in a topological category [3] and it is shown, in [7–9], and [11] that they form appropriate closure operators in the sense of Dikranjan and Giuli [13] in the category convergence spaces [14, 21] limit spaces [14, 21], and semi uniform convergence spaces [22]. The other use of a local T_1 property is to define the notion of local completely regular and local normal objects [6] in set-based topological categories.

In this paper, we characterize local T_1 preorderd spaces and investigate some invariance properties of them.

2. Preliminaries

The category **Prord** of preordered sets has as objects the pairs (B, R), where B is a set and R is reflexive and transitive relation on B and has as morphisms $(B, R) \to (B_1, R_1)$ those functions $f : B \to B_1$ such that if aRb, then $f(a)R_1 f(b)$ for all $a, b \in B$.

Recall, [1, 21], that a functor $U : \mathcal{E} \to \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if U is concrete (i.e., faithful and amnestic (i.e., if $U(f) = id$ and f is an isomorphism, then $f = id$)), has small (i.e., sets) fibers, and for which every U-source has an initial lift or, equivalently, for which each U-sink has a final lift. Note that a topological functor $U : \mathcal{E} \to \mathcal{B}$ is said to be normalized if constant objects, i.e., subterminals, have a unique structure.

Email address: baran@erciyes.edu.tr

Note also that U has a left adjoint called the discrete functor D. Recall, in [1,21] that an object $X \in \mathcal{E}$ is discrete if and only if every map $U(X) \to U(Y)$ lift to map $X \to Y$ for each object $Y \in \mathcal{E}$.
Note that **Prord** is a topological category over **Set**, the category of sets and functions [20] and [21].

2.1. A source $\{f_i : (B, R) \to (B_i, R_i), i \in I\}$ is initial in **Prord** if and only if for all $a, b \in B$, aRb if and only if $f_i a R_i f_i b$ for all $i \in I$ [20] and [21].

2.2. An epimorphism $f : (B, R) \to (B_1, R_1)$ is final in **Prord** if and only if for all $a, b \in B_1$, aR_1b if and only if there exists a sequence $a_i \in B_1, i = 1, 2, ...n$ with $a = a_1R_1a_2R_1a_3R_1...R_1a_n = b$ such that for each $k = 1, 2, ...n - 1$, there is a pair $c_k, c_{k+1} \in B$ such that $f(c_k) = a_k, f(c_{k+1}) = a_{k+1}$ and c_kRc_{k+1} [20].

2.3. The discrete structure R on B in **Prord** is given by aRb if and only if $a = b$, for $a, b \in B$.

3. Local T_1 Preordered Spaces

In this section, we characterize T_1 preordered spaces at a point p and give some invariance properties of them.

Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p [2], i.e., two disjoint copies of B identified at p, or in other words, the pushout of $p : 1 \to B$ along itself (where 1 is the terminal object in **Set**, the category of sets). More precisely, if i_1 and $i_2 : B \to B \vee_p B$ denote the inclusion of B as the first and second factor, respectively, then $i_1 p = i_2 p$ is the pushout diagram. A point x in $B \vee_p B$ will be denoted by $x_1 (x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = p_2$.

The skewed $p-$axis map, $S_p : B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$ and the fold map at p, $\triangledown_p : B \vee_p B \to B$ is given by $\triangledown_p(x_i) = x$ for $i = 1, 2$ [2].

Definition 3.1. Let (X, τ) be a topological space and $p \in X$. For each point x distinct from p, there exists a neighborhood of p missing x and there exists a neighborhood of x missing p, then (X, τ) is said to be T_1 at p [2,5].

Theorem 3.2. *Let (X, τ) be a topological space and $p \in X$. Then (X, τ) is T_1 at p if and only if the initial topology induced by $\{S_p : X \vee_p X \to (X^2, \tau_*)$ and $\nabla_p : X \vee_p X \to (X, P(X))\}$ is discrete, where τ_* is the product topology on X^2.*

Proof. The proof is given in [5]. □

Let $\mathcal{U} : \mathcal{E} \to \textbf{\textit{Set}}$ be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$ and p be a point in B.

Definition 3.3. If the initial lift of the $\mathcal{U}$-source $\{S_p : B \bigvee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \bigvee_p B \to \mathcal{UD}(B) = B\}$ is discrete, then X is called T_1 at p, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.

Theorem 3.4. *A preordered space (B, R) is T_1 at p if and only if for $x \in B$, if xRp or pRx, then $x = p$.*

Proof. Suppose (B, R) is T_1 at p. If $x \in B$ and xRp, then

$$\pi_1 S_p(x, p) R \pi_1 S_p(p, x) = xRp,$$

$$\pi_2 S_p(x, p) R \pi_2 S_p(p, x) = xRx$$

and

$$\triangledown_p(x, p) = x = \triangledown_p(p, x)$$

where $\pi_i : B^2 \to B$, $i = 1, 2$, are the projection maps. Since (B, R) is T_1 at p, it follows from 2.1, 2.3, and Definition 3.3 that $(x, p) = (p, x)$, *i.e*, $x = p$. Similarly, if pRx, then

$$\pi_1 S_p(p, x) R \pi_1 S_p(x, p) = pRx,$$

$$\pi_2 S_p(x, p) R \pi_2 S_p(p, x) = xRx.$$

Since (B, R) is T_1 at p, it follows that $(x, p) = (p, x)$, *i.e*, $x = p$.

Conversely, suppose that for $x \in B$, if xRp or pRx, then $x = p$. We show that (B, R) is T_1 at p. By 2.1, 2.3, and Definition 3.3, we need to show that for each pair u and v in the wedge $B \vee_p B$, $\pi_1 S_p(u) R \pi_1 S_p(v), \pi_2 S_p(u) R \pi_2 S_p(v)$, and $\triangledown_p(u) = \triangledown_p(v)$ if and only if $u = v$. If $u = v$, then $\pi_1 S_p(u) R \pi_1 S_p(v)$, $\pi_2 S_p(u) R \pi_2 S_p(v)$, and $\triangledown_p(u) = \triangledown_p(v)$ since R is reflexive. Suppose that $\pi_1 S_p(u) R \pi_1 S_p(v), \pi_2 S_p(u) R \pi_2 S_p(v)$, and $\triangledown_p(u) = \triangledown_p(v)$. It follows that u and v have the form (x, p) or (p, x) for some $x \in B$. If $u = (x, p)$ and $v = (p, x)$, then

$$\pi_1 S_p(u) R \pi_1 S_p(v) = xRp,$$

$$\pi_2 S_p(u) R \pi_2 S_p(v) = xRx$$

and

$$\nabla_p(u) = x = \nabla_p(v).$$

By the assumption, we have $x = p$, i.e., $u = v$.
If $u = (p, x)$ and $v = (x, p)$, then

$$\pi_1 S_p(u) R \pi_1 S_p(v) = pRx,$$
$$\pi_2 S_p(u) R \pi_2 S_p(v) = xRx$$

and

$$\nabla_p(u) = x = \nabla_p(v).$$

By the assumption, we have $x = p$, and consequently, $u = v$. Hence, (B, R) is T_1 at p. □

Theorem 3.5. *If (B, R) preordered space is T_1 at p and $M \subset B$ with $p \in M$, then M is T_1 at p.*

Proof. Let R_M be the initial structure on M induced by the inclusion map $i : M \subset B$ and for $x \in M$, xR_Mp or pR_Mx. If xR_Mp, then by 2.1, $i(x)Ri(p) = xRp$ and by Theorem 3.4, $x = p$ since (B, R) is T_1 at p. If pR_Mx, then by 2.1, $i(p)Ri(x) = pRx$ and consequently, $x = p$ since (B, R) is T_1 at p. Hence, (M, R_M) is T_1 at p. □

Theorem 3.6. *For all $i \in I$ and $p_i \in B_i$, (B_i, R_i) T_1 at p_i If and only if $(B = \prod_{i \in I}, R)$ is T_1 at p, where R is the product structure on B and $p = (p_1, p_2, ...)$.*

Proof. Suppose that $(B = \prod_{i \in I}, R)$ is T_1 at p. Since each (B_i, R_i) is isomorphic to a subspace of $(B = \prod_{i \in I}, R)$, it follows from Theorem 3.5 that (B_i, R_i) T_1 at p_i for all $i \in I$ and $p_i \in B_i$.

Suppose that (B_i, R_i) T_1 at p_i for all $i \in I$, $p_i \in B_i$ and for $x \in B$ xRp. By 2.1, $\pi_i(x) R_i \pi_i(p) = x_i R_i p_i$ for all $i \in I$. Since (B_i, R_i) T_1 at p_i, by Theorem 3.4, $x_i = p_i$ and consequently, $x = p$. If pRx, then by 2.1, $\pi_i(p) R_i \pi_i(x) = p_i R_i x_i$ for all $i \in I$. Since (B_i, R_i) T_1 at p_i, by Theorem 3.4, $x_i = p_i$ and consequently, $x = p$. Hence, by Theorem 3.4, (B, R) is T_1 at p. □

Theorem 3.7. *If (B_i, R_i) T_1 at p_i for all $i \in I$ and $p_i \in B_i$, then $(B = \coprod_{i \in I}, R)$ is T_1 at (i, p), where R is the coproduct structure on B and $(i, p) \in B$.*

Proof. Suppose that (B_i, R_i) T_1 at p_i for all $i \in I$, $p_i \in B_i$ and for $(j, x) \in B$, $(j, x)R(i, p)$. Note that by 2.2, for $(i, x), (j, y) \in B$, $(i, x)R(j, y)$ if and only if $i = j$ and xR_iy with $x, y \in B_i$, where (i, x) means $x_i \in B_i$. Since $(j, x)R(i, p)$, it follows that $i = j$ and xR_ip. (B_i, R_i) T_1 at p_i implies that by Theorem 3.4, $x_i = p_i$ and consequently, $(j, x) = (i, x) = (i, p)$. Similarly, if $(i, p)R(j, x)$ for $(j, x) \in B$, then by the same argument $(j, x) = (i, x) = (i, p)$. Hence, (B, R) is T_1 at p. □

Remark 3.8. In a topological category, T_1 at p and T_0 at p objects may be equivalent, see [10, 17] and all objects may be T_1 at p, for example, it is shown, in [6], that all prebornogical spaces are T_1 at p. Moreover, T_1 at p objects could be only discrete objects, see [12].

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