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Smarandache Curves of Salkowski Curve According to Frenet Frame

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Abstract. In this paper, when the Frenet vectors of Salkowski curve are taken as the position vectors, the curvature and the torsion of Smarandache curves are calculated. These values are expressed depending upon the Salkowski curve.

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1. Introduction

Salkowski curves are, to the best of the author's knowledge, the first known family of curves with constant curvature but non-constant torsion with an explicit parametrization. They were defined in an earlier paper [4, 5]. A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [3]. Special Smarandache curves have been studied by some authors. Ahmad T.Ali studied some special Smarandache curves in the Euclidean space. He studied Frenet-Serret invariants of a special case [1]. Bektaş, Ö. and Yüce, S., studied some special Smarandache curves according to Darboux Frame in E^3 [2]. In this paper, special Smarandache curves belonging to Salkowski curve such as TN, NB, TB and TNB drawn by Frenet frame are defined and some related results are given.

2. Preliminaries

In differential geometry, special curves have an important role. One of these curves Smarandache curves. Smarandache curves was firstly defined by M. Turgut and S. Yılmaz in 2008 [3]. Let $\gamma = \gamma(t)$ be a regular curve with unit

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speed. Then the Frenet apparatus of the curve (γ) are [1]

$$\begin{cases} T(t) = \gamma'(t), & N(t) = \frac{T'(t)}{\|T'(t)\|}, \quad B(t) = T(t) \wedge N(t) \\ \kappa(t) = \|T'(t)\|, & \tau(t) = \frac{\det\left(\gamma'(t), \gamma''(t), \gamma'''(t)\right)}{\left(\|\gamma'(t) \wedge \gamma''(t)\|\right)^2} \\ T' = \kappa N, & N' = -\kappa T + \tau B, \quad B' = -\tau N \end{cases} \tag{2.1}$$

Another important curve is Salkowski curves. Firstly, definition of Salkowski curves is given by Salkowski [4] and finally definition of Salkowski curves is given by Monterde [5].

Definition 2.1. For any $m \in \mathbb{R}$ with $m \neq \mp\frac{1}{\sqrt{3}}$, 0, let us define the space curve

$$\begin{aligned} \gamma_m(t) \quad = \quad & \frac{1}{\sqrt{1+m^2}}\Big(-\frac{1-n}{4(1+2n)}\sin((1+2n)t) - \frac{1+n}{4(1-2n)}\sin((1-2n)t) - \frac{1}{2}\sin(t), \\ & \frac{1-n}{4(1+2n)}\cos((1+2n)t) + \frac{1+n}{4(1-2n)}\cos((1-2n)t) + \frac{1}{2}\cos(t), \frac{1}{4m}\cos(2nt)\Big) \quad \text{where} \quad n = \frac{m}{\sqrt{1+m^2}}. \end{aligned}$$

The geometric elements of the Salkowski curve γ_m are

$$\|\gamma_m(t)\| = \frac{cos(nt)}{\sqrt{1+m^2}} \quad \text{so the curve is regular in} \quad [-\frac{\pi}{2n}, \frac{\pi}{2n}], \quad \kappa = 1, \quad \tau = \tan(nt).$$

The Frenet apparatus are

$$\begin{aligned} T(t) &= -\Big(\cos(t)\cos(nt) + n\sin(t)\sin(nt), \sin(t)\cos(nt) - n\cos(t)\sin(nt), \frac{n}{m}\sin(nt)\Big), \\ N(t) &= n\Big(\frac{\sin(t)}{m}, -\frac{\cos(t)}{m}, -1\Big), \\ B(t) &= \Big(-\cos(t)\sin(nt) + n\sin(t)\cos(nt), -\sin(t)\sin(nt) - n\cos(t)\cos(nt), \frac{n}{m}\cos(nt)\Big). \end{aligned}$$

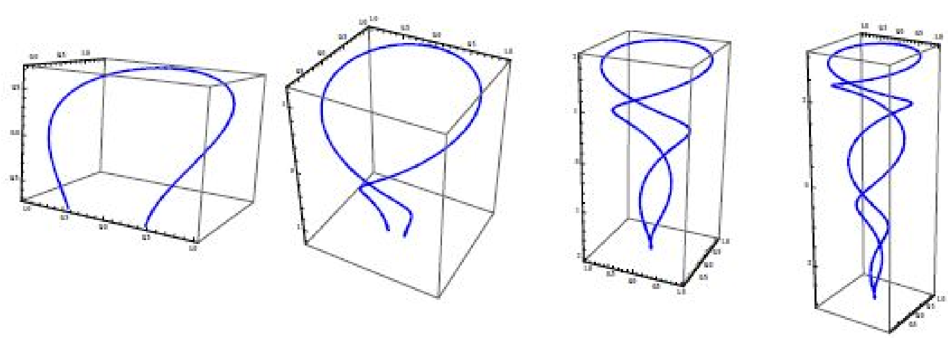

FIGURE 1. Salkowski Curve, $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$.

3. Smarandache Curves of Salkowski Curve According to Frenet Frame

In this section we shall investigate some curves such that they are obtained with binary and triple summations of the position vectors of Frenet vectors of a Salkowski curve.

Definition 3.1. Let $\gamma = \gamma(t)$ be a Salkowski curve in E^3 and $\{T, N, B\}$ be the Frenet frame. Then $\gamma_1(t)$-Smarandahce curve is by

$$\gamma_1(t) = \frac{1}{\sqrt{2}}(T(t) + N(t)).$$

According to this definition we can parametrize the $\gamma_1(t)$ - Smarandache curve as in that form

$$\gamma_1(t) = \frac{1}{\sqrt{2}}\Big(\cos(t)\cos(nt) + n\sin(t)\sin(nt) + \frac{n}{m}\sin(t), \sin(t)\cos(nt) - n\cos(t)\sin(nt) - \frac{n}{m}\cos(t), -\frac{n}{m}\sin(nt) - n\Big). \tag{3.1}$$

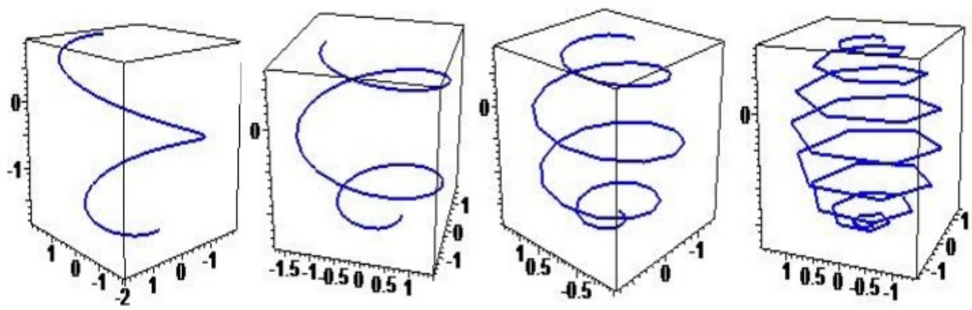

Figure 2. TN-Smarandache Curve, $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$.

Theorem 3.2. *Let $\gamma(t)$ be a Salkowski curve in $E^3$and $\{T, N, B\}$ be the Frenet Frame. Then the Frenet frame of the $\gamma_1(t)$-Smarandahce curve is given $\{T_{\gamma_1}, N_{\gamma_1}, B_{\gamma_1}\}$,*

$$\begin{aligned}
T_{\gamma_1}(t) &= \Big(\frac{n}{m}\sin(t)\cos(nt) + \cos(t), -\frac{n}{m}\cos(t)\cos(nt) + \sin(t), -n\cos(nt)\Big),\\
N_{\gamma_1}(t) &= \Big(\lambda_1\Big(\frac{n}{m}\cos(t)\cos(nt) - \sin(t)\Big) + n\sin(nt)\Big(\cos(t)\cos(nt) - \frac{n}{m}\sin(t)\Big),\\
&\quad \lambda_1\Big(\frac{n}{m}\sin(t)\cos(nt) + \cos(t)\Big) + n\sin(nt)\Big(\sin(t)\cos(nt) + \frac{n}{m}\cos(t)\Big), n^2\sin(nt)\Big),\\
B_{\gamma_1}(t) &= \Big(n\cos(t)\cos(nt) + n^2\sin(t)\sin(nt) + \frac{n^2}{m}\sin(t)\cos^2(nt),\\
&\quad n\sin(t)\cos(nt) - n^2\cos(t)\sin(nt) - \frac{n^2}{m}\cos(t)\cos^2(nt),\\
&\quad \frac{n^2}{m^2}\cos^2(nt)\frac{n^2}{m}\sin(nt) + 1\Big).
\end{aligned}$$

Proof. If we take the derivative in equation (3.1) we get

$$\gamma_1'(t) = \frac{1}{\sqrt{2}}\frac{n}{m}\left(a_1T + b_1N + c_1B\right). \tag{3.2}$$

Here the coefficients a_1, b_1 and c_1 are given

$$a_1 = \frac{n}{m}\sin(t)\cos(nt) + \cos(t), \quad b_1 = -\frac{n}{m}\cos(t)\cos(nt) + \sin(t), \quad c_1 = -n\cos(nt).$$

If we take the norm in the equation (3.2),

$$\|\gamma_1'(t)\| = \frac{n}{m}\sqrt{\lambda_1} \quad where \quad \lambda_1 = \cos^2(nt) + 1.$$

We obtained the tangent of $\gamma_1(t)$-Smarandahce curve as in

$$T_{\gamma_1}(t) = \frac{1}{\sqrt{\lambda_1}}\left(a_1 T + b_1 N + c_1 B\right).$$

The derivative in the (3.2) is

$$\gamma_1''(t) = \frac{1}{\sqrt{2}}\frac{n}{m}\left(a_2 T + b_2 N + c_2 B\right), \tag{3.3}$$

here the coefficients are given

$$\begin{aligned} a_2 &= -\frac{n}{m}\cos(t)\cos(nt) - \frac{n^2}{m}\sin(t)\sin(nt) - \sin(t), \\ b_2 &= \frac{n}{m}\sin(t)\cos(nt) + \frac{n^2}{m}\cos(t)\sin(nt) + \cos(t), \\ c_2 &= n^2\sin(nt). \end{aligned}$$

From equations (3.2) and (3.3) we have

$$\gamma_1'(t)\wedge\gamma_1''(t) = \frac{1}{2}\frac{n^2}{m^2}(a_3 T + b_3 N + c_3 B), \tag{3.4}$$

then the coefficients are given

$$\begin{aligned} a_3 &= n\cos(t)\cos(nt) + n^2\sin(t)\sin(nt) + \frac{n^2}{m}\sin(t)\cos^2(nt), \\ b_3 &= n\sin(t)\cos(nt) - n^2\cos(t)\sin(nt) - \frac{n^2}{m}\cos(t)\cos^2(nt), \\ c_3 &= \frac{n^2}{m^2}\cos^2(nt) + \frac{n^2}{m}\sin(nt) + 1. \end{aligned}$$

If we take the norm in equation (3.4), it becomes

$$\|\gamma_1'(t)\wedge\gamma_1''(t)\| = \frac{1}{2}\frac{n^3}{m^3}\sqrt{\lambda_1^2 + 2m\mu_1} \quad \textit{where} \quad \mu_1 = \lambda_1\sin(nt) + m.$$

From the equaiton (2.1) binormal vector of $\gamma_1(t)$-Smarandahce curve is given as

$$B_{\gamma_1}(t) = \frac{1}{\frac{n}{m}\sqrt{\lambda_1^2 + 2m\mu_1}}(a_4 T + b_4 N + c_4 B)$$

with the coefficients as follows

$$\begin{aligned} a_4 &= n\cos(t)\cos(nt) + n^2\sin(t)\sin(nt) + \frac{n^2}{m}\sin(t)\cos^2(nt), \\ b_4 &= n\sin(t)\cos(nt) - n^2\cos(t)\sin(nt) - \frac{n^2}{m}\cos(t)\cos^2(nt), \\ c_4 &= \frac{n^2}{m^2}\cos^2(nt)\frac{n^2}{m}\sin(nt) + 1. \end{aligned}$$

From (2.1) principal normal vector of $\gamma_1(t)$-Smarandahce curve can be written as

$$N_{\gamma_1}(t) = \frac{1}{\frac{n}{m}\sqrt{\lambda_1^3 + 2\lambda_1\mu_1}}\left(a_5 T + b_5 N + c_5 B\right)$$

and the coefficients are

$$\begin{aligned} a_5 &= \lambda_1\Big(\frac{n}{m}\cos(t)\cos(nt) - \sin(t)\Big) + n\sin(nt)\Big(\cos(t)\cos(nt) - \frac{n}{m}\sin(t)\Big), \\ b_5 &= \lambda_1\Big(\frac{n}{m}\sin(t)\cos(nt) + \cos(t)\Big) + n\sin(nt)\Big(\sin(t)\cos(nt) + \frac{n}{m}\cos(t)\Big), \\ c_5 &= n^2\sin(nt). \end{aligned}$$

Theorem 3.3. *Let $\gamma(t)$ be a Salkowski curve in E^3. Then the curvature and torsion according to γ_1-Smarandache curve are, respectively,*

$$\kappa_{\gamma_1}(t) = \sqrt{\frac{2\lambda_1^2 + 4m\mu_1}{\lambda_1^3}}, \quad \tau_{\gamma_1}(t) = \frac{\rho_1 \sqrt{2}\cos(nt)}{\lambda_1^2 + 2m\mu_1}.$$

Proof. From the expressions (2.1), curvature of the $\gamma_1(t)$-Smarandahce curve can be written

$$\kappa_{\gamma_1}(t) = \sqrt{\frac{2\lambda_1^2 + 4m\mu_1}{\lambda_1^3}}.$$

If we take the derivative in equation (3.3), it becomes

$$\gamma_1'''(t) = \frac{1}{\sqrt{2}}\frac{n}{m}\left(a_6 T + b_6 N + c_6 B\right). \tag{3.5}$$

Here the coefficients a_6, b_6 and c_6 are

$$\begin{aligned} a_6 &= -\Big(\frac{n}{m} + \frac{n^3}{m}\Big)\sin(t)\cos(nt) - 2\frac{n^2}{m}\cos(t)\sin(nt) - \cos(t), \\ b_6 &= \Big(\frac{n}{m} + \frac{n^3}{m}\Big)\cos(t)\cos(nt) - 2\frac{n^2}{m}\sin(t)\sin(nt) - \sin(t), \\ c_6 &= n^3\cos(nt). \end{aligned}$$

From equations (3.2), (3.3) and (3.5) torsion of the $\gamma_1(t)$-Smarandahce curve is

$$\tau_{\gamma_1}(t) = \frac{\rho_1 \sqrt{2}\cos(nt)}{\lambda_1^2 + 2m\mu_1}, \quad \textit{where} \quad \rho_1 = -3m^2\sin(nt) - m\lambda_1.$$

Definition 3.4. Let $\gamma = \gamma(t)$ be a Salkowski curve in E^3 and $\{T, N, B\}$ be the Frenet frame. Then $\gamma_2(t)$-Smarandahce curve is by

$$\gamma_2(t) = \frac{1}{\sqrt{2}}(N(t) + B(t)).$$

According to this definition we can parametrize the $\gamma_2(t)$ - Smarandache curve as in that form

$$\gamma_2(t) = \frac{1}{\sqrt{2}}\Big(-\cos(t)\sin(nt) + n\sin(t)\cos(nt) + \frac{n}{m}\sin(t), -\sin(t)\sin(nt) - n\cos(t)\cos(nt) - \frac{n}{m}\cos(t), \frac{n}{m}\cos(nt) - n\Big). \tag{3.6}$$

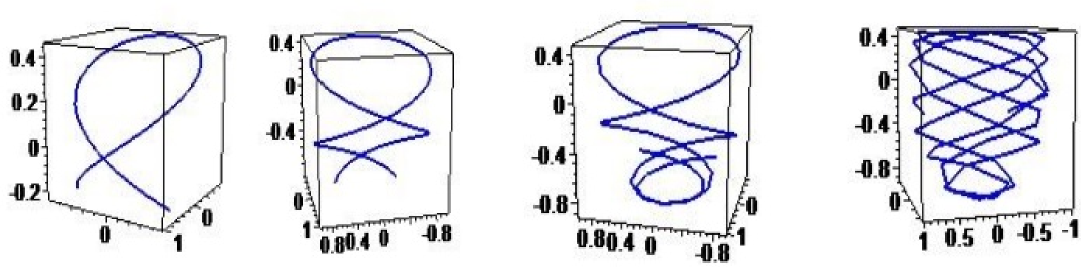

Figure 3. NB-Smarandache Curve, $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$.

Theorem 3.5. *Let $\gamma(t)$ be a Salkowski curve in $E^3$and $\{T, N, B\}$ be the Frenet Frame. Then the Frenet frame of the $\gamma_2(t)$-Smarandahce curve is given $\{T_{\gamma_2}, N_{\gamma_2}, B_{\gamma_2}\}$ (as figure 2),*

$$\begin{aligned}
T_{\gamma_2}(t) &= \Big(\frac{n}{m}\sin(t)\sin(nt)+\cos(t), -\frac{n}{m}\cos(t)\sin(nt)+\sin(t), -n\sin(nt)\Big),\\
N_{\gamma_2}(t) &= \Big(\lambda_2\Big(\frac{n}{m}\cos(t)\sin(nt)-\sin(t)\Big)+n\cos(nt)\Big(\frac{n}{m}\sin(t)-\cos(t)\sin(nt)\Big),\\
&\quad \lambda_2\Big(\frac{n}{m}\sin(t)\sin(nt)+\cos(t)\Big)-n\cos(nt)\Big(\sin(t)\sin(nt)+\frac{n}{m}\cos(t)\Big),\\
&\quad -n^2\cos(nt)\Big),\\
B_{\gamma_2}(t) &= \Big(n\cos(t)\sin(nt)-n^2\sin(t)\cos(nt)+\frac{n^2}{m}\sin(t)\sin^2(nt),\\
&\quad n\sin(t)\sin(nt)+n^2\cos(t)\cos(nt)-\frac{n^2}{m}\cos(t)\sin^2(nt),\\
&\quad \frac{n^2}{m^2}\sin^2(nt)-\frac{n^2}{m}\cos(nt)+1\Big).
\end{aligned}$$

Proof. If we take the derivative in equation (3.6) we get

$$\gamma_2'(t) = \frac{1}{\sqrt{2}}\frac{n}{m}\left(a_7T+b_7N+c_7B\right). \tag{3.7}$$

Here the coefficients a_7, b_7 and c_7 are given

$$a_7 = \frac{n}{m}\sin(t)\sin(nt)+\cos(t), \quad b_7 = -\frac{n}{m}\cos(t)\sin(nt)+\sin(t), \quad c_7 = -n\sin(nt).$$

If we take the norm in the equation (3.7),

$$\|\gamma_2'(t)\| = \frac{n}{m}\sqrt{\lambda_2} \quad where \quad \lambda_2 = \sin^2(nt)+1.$$

We obtained the tangent of $\gamma_2(t)$-Smarandahce curve as in

$$T_{\gamma_2}(t) = \frac{1}{\sqrt{\lambda_2}}\left(a_7T+b_7N+c_7B\right).$$

The derivative in the (3.7) is

$$\gamma_2''(t) = \frac{1}{\sqrt{2}}\frac{n}{m}\left(a_8T+b_8N+c_8B\right), \tag{3.8}$$

here the coefficients are given

$$\begin{aligned}
a_8 &= \frac{n}{m}\cos(t)\sin(nt)+\frac{n^2}{m}\sin(t)\cos(nt)-\sin(t),\\
b_8 &= \frac{n}{m}\sin(t)\sin(nt)-\frac{n^2}{m}\cos(t)\cos(nt)+\cos(t),\\
c_8 &= -n^2\cos(nt).
\end{aligned}$$

From equations (3.7) and (3.8) we have

$$\gamma_2'(t)\wedge\gamma_2''(t) = \frac{1}{2}\frac{n^2}{m^2}(a_9T+b_9N+c_9B), \tag{3.9}$$

then the coefficients are given

$$
\begin{aligned}
a_9 &= n\cos(t)\sin(nt) - n^2\sin(t)\cos(nt) + \frac{n^2}{m}\sin(t)\sin^2(nt),\\
b_9 &= n\sin(t)\sin(nt) + n^2\cos(t)\cos(nt) - \frac{n^2}{m}\cos(t)\sin^2(nt),\\
c_9 &= \frac{n^2}{m^2}\sin^2(nt) - \frac{n^2}{m}\cos(nt) + 1.
\end{aligned}
$$

If we take the norm in equation (3.9), it becomes

$$\|\gamma_2'(t)\wedge\gamma_2''(t)\| = \frac{1}{2}\frac{n^3}{m^3}\sqrt{\lambda_2^2 - 2m\mu_2} \quad \textit{where} \quad \mu_2 = \lambda_2\cos(nt) - m.$$

From the equaiton (2.1) binormal vector of $\gamma_2(t)$-Smarandahce curve is given as

$$B_{\gamma_2}(t) = \frac{1}{\frac{n}{m}\sqrt{\lambda_2^2 - 2m\mu_2}}(a_{10}T + b_{10}N + c_{10}B)$$

with the coefficients as follows

$$
\begin{aligned}
a_{10} &= n\cos(t)\sin(nt) - n^2\sin(t)\cos(nt) + \frac{n^2}{m}\sin(t)\sin^2(nt),\\
b_{10} &= n\sin(t)\sin(nt) + n^2\cos(t)\cos(nt) - \frac{n^2}{m}\cos(t)\sin^2(nt),\\
c_{10} &= \frac{n^2}{m^2}\sin^2(nt) - \frac{n^2}{m}\cos(nt) + 1.
\end{aligned}
$$

From (2.1) principal normal vector of $\gamma_2(t)$-Smarandahce curve can be written as

$$N_{\gamma_2}(t) = \frac{1}{\frac{n}{m}\sqrt{\lambda_2^3 - 2m\lambda_2\mu_2}}(a_{11}T + b_{11}N + c_{11}B)$$

and the coefficients are

$$
\begin{aligned}
a_{11} &= \lambda_2\Big(\frac{n}{m}\cos(t)\sin(nt) - \sin(t)\Big) + n\cos(nt)\Big(\frac{n}{m}\sin(t) - \cos(t)\sin(nt)\Big),\\
b_{11} &= \lambda_2\Big(\frac{n}{m}\sin(t)\sin(nt) + \cos(t)\Big) - n\cos(nt)\Big(\sin(t)\sin(nt) + \frac{n}{m}\cos(t)\Big),\\
c_{11} &= -n^2\cos(nt).
\end{aligned}
$$

Theorem 3.6. *Let $\gamma(t)$ be a Salkowski curve in E^3. Then the curvature and torsion according to γ_2-Smarandache curve are, respectively,*

$$\kappa_{\gamma_2}(t) = \sqrt{\frac{2\lambda_2^3 - 4m\mu_2}{\lambda_2^3}}, \quad \tau_{\gamma_2}(t) = \frac{\rho_2\sqrt{2}\sin(nt)}{\lambda_2^2 - 2m\mu_2}.$$

Proof. From the expressions (2.1), curvature of the $\gamma_2(t)$-Smarandahce curve can be written

$$\kappa_{\gamma_2}(t) = \sqrt{\frac{2\lambda_2^3 - 4m\mu_2}{\lambda_2^3}}.$$

If we take the derivative in equation (3.8), it becomes

$$\gamma_2'''(t) = \frac{1}{\sqrt{2}}\frac{n}{m}(a_{12}T + b_{12}N + c_{12}B). \tag{3.10}$$

Here the coefficients a_{12}, b_{12} and c_{12} are

$$\begin{aligned} a_{12} &= -\Big(\frac{n}{m}+\frac{n^3}{m}\Big)\sin(t)\sin(nt)+2\frac{n^2}{m}\cos(t)\cos(nt)-\cos(t),\\ b_{12} &= \Big(\frac{n}{m}+\frac{n^3}{m}\Big)\cos(t)\sin(nt)+2\frac{n^2}{m}\sin(t)\cos(nt)-\sin(t),\\ c_{12} &= n^3\sin(nt). \end{aligned}$$

From equations (3.7), (3.8) and (3.10) torsion of the $\gamma_2(t)$-Smarandahce curve is

$$\tau_{\gamma_2}(t)=\frac{\rho_2\sqrt{2}\sin(nt)}{\lambda_2^2-2m\mu_2},\quad \textit{where}\quad \rho_2=3m^2\cos(nt)-m\lambda_2.$$

Definition 3.7. Let $\gamma=\gamma(t)$ be a Salkowski curve in E^3 and $\{T,N,B\}$ be the Frenet frame. Then $\gamma_3(t)$-Smarandahce curve is by

$$\gamma_3(t) = \frac{1}{\sqrt{2}}(T(t)+B(t)).$$

According to this definition we can parametrize the $\gamma_3(t)$ - Smarandache curve as in that form

$$\begin{aligned} \gamma_3(t) = \frac{1}{\sqrt{2}}\Big(&-\cos(t)\cos(nt)-\cos(t)\sin(nt)-n\sin(t)\sin(nt)+n\sin(t)\cos(nt),\\ &-\sin(t)\cos(nt)-\sin(t)\sin(nt)+n\cos(t)\sin(nt)-n\cos(t)\cos(nt)\frac{n}{m}\cos(nt)-\frac{n}{m}\sin(nt)\Big). \end{aligned} \tag{3.11}$$

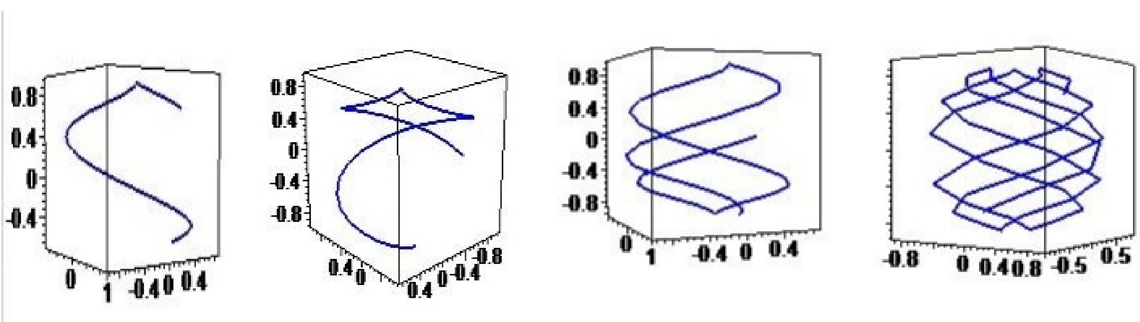

Figure 4. TB-Smarandache Curve, $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$.

Theorem 3.8. *Let $\gamma(t)$ be a Salkowski curve in $E^3$and $\{T,N,B\}$ be the Frenet Frame. Then the Frenet frame of the $\gamma_3(t)$-Smarandahce curve is given $\{T_{\gamma_3},N_{\gamma_3},B_{\gamma_3}\}$ (as figure 2),*

$$\begin{aligned} T_{\gamma_3}(t) &= \Big(\frac{n}{m}\sin(t),-\frac{n}{m}\cos(t),-n\Big),\\ N_{\gamma_3}(t) &= (\cos(t),-\sin(t),0),\\ B_{\gamma_3}(t) &= \Big(n\sin(t),-n\cos(t),\frac{n}{m}\Big). \end{aligned}$$

Proof. If we take the derivative in equation (3.11) we get

$$\gamma_3'(t)=\frac{1}{\sqrt{2}}\frac{n}{m}(\cos(nt)+\sin(nt))\,(a_{13}T+b_{13}N+c_{13}B)\,. \tag{3.12}$$

Here the coefficients a_{13}, b_{13} and c_{13} are given

$$a_{13}=\frac{n}{m}\sin(t),\quad b_{13}=-\frac{n}{m}\cos(t),\quad c_{13}=-n.$$

If we take the norm in the equation (3.12),

$$\|\gamma_3'(t)\| = \frac{1}{\sqrt{2}}\frac{n}{m}(\cos(nt) + \sin(nt)) \quad where \quad \lambda_3 = \cos(nt) + \sin(nt).$$

We obtained the tangent of $\gamma_3(t)$-Smarandahce curve as in

$$T_{\gamma_3}(t) = (a_{13}T + b_{13}N + c_{13}B).$$

The derivative in the (3.12) is

$$\gamma_3''(t) = \frac{1}{\sqrt{2}}\frac{n}{m}(a_{14}T + b_{14}N + c_{14}B), \tag{3.13}$$

here the coefficients are given

$$a_{14} = \frac{n}{m}\cos(t)\lambda_3 + \frac{n^2}{m}\sin(t)\varphi_1, \quad b_{14} = \frac{n}{m}\sin(t)\lambda_3 - \frac{n^2}{m}\cos(t)\varphi_1, \quad c_{14} = -n^2\varphi_1$$

where $\varphi_1 = \cos(nt) - \sin(nt)$. From equations (3.12) and (3.13) we have

$$\gamma_3'(t) \wedge \gamma_3''(t) = \frac{1}{2}\frac{n^3}{m^3}(a_{15}T + b_{15}N + c_{15}B), \tag{3.14}$$

then the coefficients are given

$$a_{15} = n\sin(t), \quad b_{15} = n\cos(nt), \quad c_{15} = \frac{n}{m}.$$

If we take the norm in equation (3.14), it becomes

$$\|\gamma_3'(t) \wedge \gamma_3''(t)\| = \frac{1}{2}\frac{n^3}{m^3}\lambda_3$$

From the equaiton (2.1) binormal vector of $\gamma_3(t)$-Smarandahce curve is given as

$$B_{\gamma_3}(t) = (a_{16}T + b_{16}N + c_{16}B)$$

with the coefficients as follows

$$a_{16} = n\sin(t), \quad b_{16} = -n\cos(t), \quad c_{16} = \frac{n}{m}.$$

From (2.1) principal normal vector of $\gamma_3(t)$-Smarandahce curve can be written as

$$N_{\gamma_3}(t) = (a_{17}T + b_{17}N + c_{17}B)$$

and the coefficients are

$$a_{17} = \cos(t), \quad b_{17} = -\sin(t), \quad c_{17} = 0.$$

Theorem 3.9. *Let $\gamma(t)$ be a Salkowski curve in E^3. Then the curvature and torsion according to γ_3-Smarandache curve are, respectively,*

$$\kappa_{\gamma_3}(t) = \frac{\sqrt{2}}{\lambda_3}, \quad \tau_{\gamma_3}(t) = \frac{m\sqrt{2}}{\lambda_3}.$$

Proof. From the expressions (2.1), curvature of the $\gamma_3(t)$-Smarandahce curve can be written

$$\kappa_{\gamma_3}(t) = \frac{\sqrt{2}}{\lambda_3}.$$

If we take the derivative in equation (3.13), it becomes

$$\gamma_3'''(t) = \frac{1}{\sqrt{2}}\frac{n}{m}(a_{18}T + b_{18}N + c_{18}B). \tag{3.15}$$

Here the coefficients a_{18}, b_{18} and c_{18} are

$$\begin{aligned}
a_{18} &= -\Big(\frac{n}{m} + \frac{n^3}{m}\Big)\sin(t)\lambda_3 + 2\frac{n^2}{m}\cos(t)\varphi_1, \\
b_{18} &= \Big(\frac{n}{m} + \frac{n^3}{m}\Big)\cos(t)\lambda_3 + 2\frac{n^2}{m}\sin(t)\varphi_1, \\
c_{18} &= n^3\lambda_3.
\end{aligned}$$

From equations (3.12), (3.13) and (3.15) torsion of the $\gamma_3(t)$-Smarandahce curve is

$$\tau_{\gamma_3}(t) = \frac{m\sqrt{2}}{\lambda_3}.$$

Definition 3.10. Let $\gamma = \gamma(t)$ be a Salkowski curve in E^3 and $\{T, N, B\}$ be the Frenet frame. Then $\gamma_4(t)$-Smarandahce curve is by

$$\gamma_4(t) = \frac{1}{\sqrt{3}}(T(t) + N(t) + B(t)).$$

According to this definition we can parametrize the $\gamma_4(t)$ - Smarandache curve as in that form

$$\begin{aligned}\gamma_4(t) = \frac{1}{\sqrt{3}}\Big(& -\cos(t)\cos(nt) - \cos(t)\sin(nt) - n\sin(t)\sin(nt) + n\sin(t)\cos(nt) \\ & + \frac{n}{m}\sin(t), -\sin(t)\cos(nt) - \sin(t)\sin(nt) + n\cos(t)\sin(nt) - n\cos(t)\cos(nt) \\ & - \frac{n}{m}\cos(t), -\frac{n}{m}\sin(nt) + \frac{n}{m}\cos(nt) - n. \end{aligned} \tag{3.16}$$

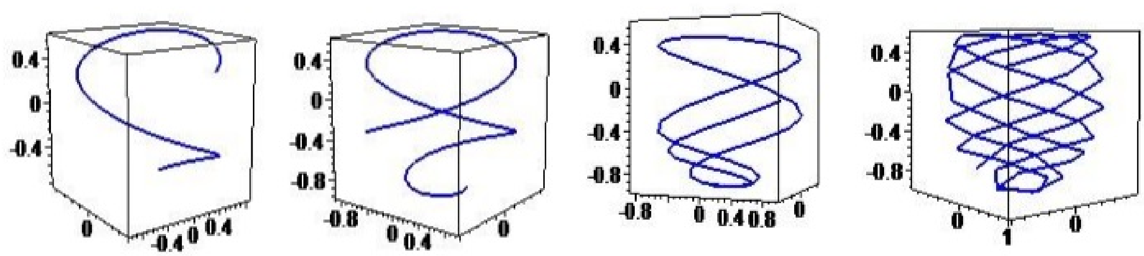

FIGURE 5. TNB-Smarandache Curve, $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$.

Theorem 3.11. *Let $\gamma(t)$ be a Salkowski curve in $E^3$and $\{T, N, B\}$ be the Frenet Frame. Then the Frenet frame of the $\gamma_4(t)$-Smarandahce curve is given $\{T_{\gamma_4}, N_{\gamma_4}, B_{\gamma_4}\}$ (as figure 2),*

$$\begin{aligned} T_{\gamma_4}(t) &= \Big(\frac{n}{m}\sin(t)\lambda_3 + \cos(t), -\frac{n}{m}\cos(t)\lambda_3 + \sin(t), -n\lambda_3\Big), \\ N_{\gamma_4}(t) &= \Big(\lambda_4\Big(\frac{n}{m}\cos(t)\lambda_3 - \sin(t)\Big) + n\varphi_1\Big(\frac{n}{m}\sin(t) - \cos(t)\lambda_3\Big), \\ &\quad \lambda_4\Big(\frac{n}{m}\sin(t)\lambda_3 + \cos(t)\Big) - n\varphi_1\Big(\frac{n}{m}\cos(t) + \sin(t)\lambda_3\Big), -n^2\varphi_1\Big) \\ B_{\gamma_4}(t) &= \Big(n\cos(t)\lambda_3 - n^2\sin(t)\varphi_1 + \frac{n^2}{m}\sin(t)\lambda_3^2, n\sin(t)\lambda_3 + n^2\cos(t)\varphi_1 \\ &\quad -\frac{n^2}{m}\cos(t)\lambda_3^2, \frac{n^2}{m^2}\lambda_3^2 - \frac{n^2}{m}\varphi_1 + 1\Big). \end{aligned}$$

Proof. If we take the derivative in equation (3.16) we get

$$\gamma_4'(t) = \frac{1}{\sqrt{3}}\frac{n}{m}\left(a_{19}T + b_{19}N + c_{19}B\right). \tag{3.17}$$

Here the coefficients a_{19}, b_{19} and c_{19} are given

$$a_{19} = \frac{n}{m}\sin(t)\lambda_3 + \cos(t), \quad b_{19} = -\frac{n}{m}\cos(t)\lambda_3 + \sin(t), \quad c_{19} = -n\lambda_3.$$

If we take the norm in the equation (3.17),

$$\|\gamma_4'(t)\| = \frac{1}{\sqrt{3}}\frac{n}{m}\sqrt{\lambda_4} \quad where \quad \lambda_4 = \lambda_3^2 + 1.$$

We obtained the tangent of $\gamma_4(t)$-Smarandahce curve as in

$$T_{\gamma_4}(t) = \frac{1}{\sqrt{\lambda_3}}\frac{n}{m}\left(a_{19}T + b_{19}N + c_{19}B\right).$$

The derivative in the (3.17) is

$$\gamma_4''(t) = \frac{1}{\sqrt{3}}\frac{n}{m}\left(a_{20}T + b_{20}N + c_{20}B\right), \tag{3.18}$$

here the coefficients are given

$$\begin{aligned} a_{20} &= \frac{n}{m}\cos(t)\lambda_3 + \frac{n^2}{m}\sin(t)\varphi_1 - \sin(t), \\ b_{20} &= \frac{n}{m}\sin(t)\lambda_3 - \frac{n^2}{m}\cos(t)\varphi_1 + \cos(t), \\ c_{20} &= -n^2\varphi_1. \end{aligned}$$

From equations (3.17) and (3.18) we have

$$\gamma_4'(t) \wedge \gamma_4''(t) = \frac{1}{3}\frac{n^2}{m^2}(a_{21}T + b_{21}N + c_{21}B), \tag{3.19}$$

then the coefficients are given

$$\begin{aligned} a_{21} &= n\cos(t)\lambda_3 - n^2\sin(t)\varphi_1 + \frac{n^2}{m}\sin(t)\lambda_3^2, \\ b_{21} &= n\sin(t)\lambda_3 + n^2\cos(t)\varphi_1 - \frac{n^2}{m}\cos(t)\lambda_3^2, \\ c_{21} &= \frac{n^2}{m^2}\lambda_3^2 - \frac{n^2}{m}\varphi_1 + 1. \end{aligned}$$

If we take the norm in equation (3.19), it becomes

$$\|\gamma_4'(t) \wedge \gamma_4''(t)\| = \frac{1}{3}\frac{n^3}{m^3}\sqrt{\lambda_4^2 - m\mu_3} \quad where \quad \mu_3 = 2\varphi_1\lambda_4 - 3m.$$

From the equaiton (2.1) binormal vector of $\gamma_4(t)$-Smarandahce curve is given as

$$B_{\gamma_4}(t) = \frac{1}{\frac{n}{m}\sqrt{\lambda_4^2 - m\mu_3}}(a_{22}T + b_{22}N + c_{22}B)$$

with the coefficients as follows

$$\begin{aligned} a_{22} &= n\cos(t)\lambda_3 - n^2\sin(t)\varphi_1 + \frac{n^2}{m}\sin(t)\lambda_3^2, \\ b_{22} &= n\sin(t)\lambda_3 + n^2\cos(t)\varphi_1 - \frac{n^2}{m}\cos(t)\lambda_3^2, \\ c_{22} &= \frac{n^2}{m^2}\lambda_3^2 - \frac{n^2}{m}\varphi_1 + 1. \end{aligned}$$

From (2.1) principal normal vector of $\gamma_4(t)$-Smarandahce curve can be written as

$$N_{\gamma_4}(t) = \frac{1}{\sqrt{\lambda_4^3 - m\lambda_4\mu_3}}\left(a_{23}T + b_{23}N + c_{23}B\right)$$

and the coefficients are

$$\begin{aligned}
a_{23} &= \lambda_4\Big(\frac{n}{m}\cos(t)\lambda_3 - \sin(t)\Big) + n\varphi_1\Big(\frac{n}{m}\sin(t) - \cos(t)\lambda_3\Big),\\
b_{23} &= \lambda_4\Big(\frac{n}{m}\sin(t)\lambda_3 + \cos(t)\Big) - n\varphi_1\Big(\frac{n}{m}\cos(t) + \sin(t)\lambda_3\Big),\\
c_{23} &= -n^2\varphi_1.
\end{aligned}$$

Theorem 3.12. *Let $\gamma(t)$ be a Salkowski curve in E^3. Then the curvature and torsion according to γ_4-Smarandache curve are, respectively,*

$$\kappa_{\gamma_4}(t) = \sqrt{\frac{3\lambda_4^2 - 3m\mu_3}{\lambda_4^3}}, \quad \tau_{\gamma_4}(t) = \frac{m\sqrt{3}\lambda_3\rho_3}{\lambda_4^2 - m\mu_3}.$$

Proof. From the expressions (2.1), curvature of the $\gamma_4(t)$-Smarandahce curve can be written

$$\kappa_{\gamma_4}(t) = \sqrt{\frac{3\lambda_4^2 - 3m\mu_3}{\lambda_4^3}}.$$

If we take the derivative in equation (3.18), it becomes

$$\gamma_4'''(t) = \frac{1}{\sqrt{3}}\frac{n}{m}\left(a_{24}T + b_{24}N + c_{24}B\right). \tag{3.20}$$

Here the coefficients a_{24}, b_{24} and c_{24} are

$$\begin{aligned}
a_{24} &= -\Big(\frac{n}{m} + \frac{n^3}{m}\Big)\sin(t)\lambda_3 + 2\frac{n^2}{m}\cos(t)\varphi_1 - \cos(t),\\
b_{24} &= \Big(\frac{n}{m} + \frac{n^3}{m}\Big)\cos(t)\lambda_3 + 2\frac{n^2}{m}\sin(t)\varphi_1 - \sin(t),\\
c_{24} &= n^3\lambda_3.
\end{aligned}$$

From equations (3.17), (3.18) and (3.20) torsion of the $\gamma_4(t)$-Smarandahce curve is

$$\tau_{\gamma_4}(t) = \frac{m\sqrt{3}\lambda_3\rho_3}{\lambda_4^2 - m\mu_3}, \quad \textit{where} \quad \rho_3 = 3m\varphi_1 - \lambda_4.$$

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