Proceedings of International Conference on Mathematics
and Mathematics Education (ICMME 2018)
Turk. J. Math. Comput. Sci.
10(2018) 207–214

http://dergipark.gov.tr/tjmcs

MATDER

Some Inequalities Related to $\eta-$Strongly Convex Functions

Seda Kılınç[a,*], Abdullah Akkurt[a], Hüseyin Yıldırım[a]

[a]*Department of Mathematics, Kahramanmaraş Sütcü İmam University,Kahramanmaraş,Turkey.*

Received: 08-08-2018 • Accepted: 08-11-2018

Abstract. The aim of this paper, is to establish some new inequalities of Hermite-Hadamard type by using $\eta-$strongly convex fuction. Moreover, we also consider their relevances for other related known results.

2010 AMS Classification: 03E72,90B50.

Keywords: Hermite-Hadamard inequality, $\eta-$convex fuction, Riemann-Liouville.

1. Introduction and Preliminares

The relationship between theory of convex functions and theory of inequalities has occured as a result of many researches investigation of these theories. A very intersting result in this regard is due to Hermite and Hadamard independently that is Hermite-Hadamard's inequality. This remarkable result of Hermite and Hadamard can be viewed as necessary and sufficient condition for a function to be convex. The $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval I of real numbers $a, b \in I$ and $a < b$, we have,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x) \leq \frac{f(a)+f(b)}{2}$$

Both inequalities hold in the reversed direction if f is concave.

The classical Hermite-Hadamard inequalities have attracted many researchers since 1893 [1–16]. Researchers investigated Hermite-Hadamard inequalities involving fractional integrals according to the associated fractional integral equalities and different types of convex functions.

Definition 1.1. A function $f : I \to \mathbb{R}$ is called convex with respect to $\eta-$convex, if

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y))$$

for all $x, y \in I$ and $t \in [0, 1]$.

*Corresponding Author
Email addresses: sedaaa_kilinc@hotmail.com (S. Kılınç), abdullahmat@gmail.com (A. Akkurt), hyildir@ksu.edu.tr (H. Yıldırım)

Definition 1.2. A function $f : I \to \mathbb{R}$ is called convex with respect to $\eta-$strongly convex $c > 0$, if

$$f(tx + (1-t)y) \le f(y) + t\eta(f(x), f(y)) - ct(1-t)(x-y)^2$$

or

$$f(tx + (1-t)y) \le f(y) + t\eta(f(x), f(y)) - ct(1-t)\eta^2(x, y)$$

for all $x, y \in I$ and $t \in [0,1]$.

Proposition 1.3. *If f:$[a,b] \to \mathbb{R}$ is $\eta-$strongly convex,then*

$$\max_{x\in[a,b]} f(x) \le \max\{f(b), f(b) + t\eta(f(a), f(b))\}.$$

Proof. For any $x \in [a,b]$, we have $x = ta + (1-t)b$ for some $t \in [0,1]$, which implies that

$$\begin{aligned} f(x) &= f(ta + (1-t)b) \le f(b) + t\eta(f(a), f(b)) - ct(1-t)\eta^2(a, b) \\ &\le \max\{f(b), f(b) + t\eta(f(a), f(b))\} \end{aligned}$$

since x is arbitrary, so

$$\max_{x\in[a,b]} f(x) \le \max\{f(b), f(b) + t\eta(f(a), f(b))\}$$

and the statement is proved. □

2. Main Results

In this section, we obtain our main results.

Theorem 2.1. *A function $f : I \to \mathbb{R}$ is $\eta-$strongly convex if and only if for any $x_1, x_2, x_3 \in I$, with $x_1 < x_2 < x_3$,*

$$\det\begin{bmatrix} 1 & x_1 & \eta(f(x_1), f(x_2)) \\ 1 & x_2 & f(x_2) - f(x_3) \\ 1 & x_3 & 0 \end{bmatrix} \ge 0$$

and

$$f(x_1) \le f(x_3) + \eta(f(x_1), f(x_3))$$

Proof. Suppose that f is a $\eta-$strongly convex. Consider arbitrary, $c > 0$, $x_1, x_2, x_3 \in I$, with $x_1 < x_2 < x_3$.So there is a $t \in (0,1)$ such that $x_2 = tx_1 + (1-t)x_3$, namely $t = \frac{x_2 - x_3}{x_1 - x_3}$. From $\eta-$strongly convexity of f we have

$$\begin{aligned} f(x_2) = f(tx_1 + (1-t)x_3) &\le f(x_3) + t\eta(f(x_1), f(x_3)) \\ &-ct(1-t)\eta^2(x_1, x_3) \end{aligned}$$

or

$$\begin{aligned} f(x_2) = f(tx_1 + (1-t)x_3) &\le f(x_3) + \tfrac{x_2 - x_3}{x_1 - x_3}\eta(f(x_1), f(x_3)) \\ &-c\tfrac{(x_2 - x_3)(x_1 - x_2)}{(x_1 - x_3)^2}\eta^2(x_1, x_3) \end{aligned}$$

$$\begin{aligned} (x_3 - x_1)[f(x_3) - f(x_2)] + (x_3 - x_2)\eta(f(x_1), f(x_3)) \\ -c\tfrac{(x_2 - x_3)(x_1 - x_2)}{(x_1 - x_3)}\eta^2(x_1, x_3) \ge 0 \end{aligned}$$

which is equivalent to above determinat being nonnegative, Also for $t = 1$, $\frac{x_2 - x_3}{x_1 - x_3} = 1$,namely $x_1 = x_2$,

$$f(x_1) \le f(x_3) + \eta(f(x_1), f(x_3))$$

and also for $t = 0$

$$f(x_3) \le f(x_3)$$

For the inverse implications, consider $x, y \in I$, with $x < y$.Choosing any $t \in (0,1)$ we have $x < tx + (1-t)y < y$,and so

$$\det\begin{bmatrix} 1 & x & \eta(f(x), f(y)) \\ 1 & tx + (1-t)y & f(tx + (1-t)y) - f(y) + ct(1-t)\eta^2(x, y) \\ 1 & y & 0 \end{bmatrix} \ge 0$$

By expanding this determinat we reach to the inequality

$$f(tx + (1-t)y) \le f(y) + t\eta(f(x), f(y)) - ct(1-t)\eta^2(x, y)$$

for any $t \in (0,1)$ that gives $\eta-$strongly convex.

From assumption for $t = 1$,we have

$$f(x) \leq f(y) + \eta(f(x), f(y))$$

namely $\eta-$convex. □

Theorem 2.2. *For a function $f : I \to \mathbb{R}$ the following assertions are equivalent.*

a. f is $\eta-$strongly convex function;
b. For any $x, y, z \in I$ with $x < y < z$ we have

$$\frac{\eta(f(x),f(z))}{x-z} - c\frac{(x-y)}{(x-z)^2}\eta^2(x,z) \leq \frac{f(y)-f(z)}{y-z} \quad \text{and } f(x) \leq f(y) + \eta(f(x), f(y))$$
$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)) - ct(1-t)\eta^2(x,y)$$
$$\text{for } t = 1,\ f(x) \leq f(y) + \eta(f(x), f(y))$$

Proof. Supposed that f is $\eta-$strongly convex and $x, y, z \in I$ with $x < y < z$, then there is a $t \in (0,1)$, such that $y = tx + (1-t)z$.So we have $t = \frac{y-z}{x-z}$. Also

$$f(y) \leq f(z) + t\eta(f(x), f(z)) - ct(1-t)\eta^2(x,z)$$

or

$$f(y) - f(z) \leq \frac{y-z}{x-z}\eta(f(x), f(z)) - c\frac{(y-z)(x-y)}{(x-z)^2}\eta^2(x,z)$$

hence

$$\frac{\eta(f(x),f(z))}{x-z} - c\frac{(x-y)}{(x-z)^2}\eta^2(x,z) \leq \frac{f(y)-f(z)}{y-z}$$

For the inverse implications, consider $x, y \in I$ with $x < y$. It is clear that for any $t \in (0,1)$, $x < tx + (1-t)y < y$. It follows that

$$\frac{\eta(f(x),f(y))}{x-y} - c\frac{(1-t)}{x-y}\eta^2(x,y) \leq \frac{f(tx+(1-t)y)-f(y)}{tx+(1-t)y-y}$$

that is equivalent to

$$\frac{\eta(f(x),f(y))}{x-y} - c\frac{(1-t)}{x-y}\eta^2(x,y) \leq \frac{f(tx+(1-t)y)-f(y)}{t(x-y)}$$

therefore

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)) - ct(1-t)\eta^2(x,y)$$

for any $x, y \in I$ with $x < y$ and $t \in (0,1)$ .so f is $\eta-$strongly convex. □

Theorem 2.3. *For a function $f : I \to \mathbb{R}$ the following assertions are equivalent.*

a. f is $\eta-$strongly convex function;
b. For any $x, y, z \in I$ with $x < y < z$ we have

$$\frac{\eta(f(x),f(z))}{x-z} - c\frac{(x-y)}{(x-z)^2}\eta^2(x,z) \leq \frac{f(y)-f(z)}{y-z} \quad \text{and } f(x) \leq f(y) + \eta(f(x), f(y))$$
$$f(tx + (1-t)z) \leq f(z) + t\eta(f(x), f(z)) - ct(1-t)\eta^2(x,z)$$
$$\text{for } t = 1,\ f(x) \leq f(z) + \eta(f(x), f(z))$$

Proof. with the same argument as theorem 2 proof is completed. □

Theorem 2.4. *Supposed that $f : I \to \mathbb{R}$ is a η-strongly convex function and η is bounded from above on $f(I) \times f(I)$. Then f satisfies a Lipschitz conditionon any closed interval $[a,b]$, contained in the interior I° of I. Hence, f is absolutely continuons on $[a,b]$ and continuons on I°.*

Proof. Let M_η be upper bound of η on $f(I)\times f(I)$.Consider closed interval $[a,b]$ in I° and choose $\varepsilon>0$ such that $[a-\varepsilon,b+\varepsilon]$ belong to I. Supposed that x,y are distinct points of $[a,b]$. Set $z=y+\frac{\varepsilon}{|y-x|}(y-x)$ and $t=\frac{|y-x|}{|y-x|+\varepsilon}$.So it is not hard to see that $z\in[a-\varepsilon,b+\varepsilon]$ and $y=tz+(1-t)x$.Then

$$\begin{aligned} f(y) &\le f(x)+t\eta(f(z),f(x))-ct(1-t)\eta^2(z,x)\\ &\le f(x)+tM_\eta-ct(1-t)\eta^2(z,x)\end{aligned}$$

this implies that

$$\begin{aligned} f(y)-f(x) &\le tM_\eta-ct(1-t)\eta^2(z,x)\\ &=\frac{|y-x|}{|y-x|+\varepsilon}M_\eta-c\frac{|y-x|.\varepsilon}{(|y-x|+\varepsilon)^2}\eta^2(z,x)\\ &\le\frac{|y-x|}{\varepsilon}M_\eta-c\frac{|y-x|.\varepsilon}{\varepsilon}\eta^2(z,x)\\ &=K|y-x|-c|y-x|.\eta^2(z,x)\\ &=|y-x|\left[K-c\eta^2(z,x)\right]\\ &=F|y-x|\end{aligned}$$

where $K=\frac{M_\eta}{\varepsilon},\ K-c\eta^2(z,x)=F$.

Also if we change the place of x,y in above argument we have $f(x)-f(y)\le F|y-x|$.Therefore $|f(y)-f(x)|\le F|y-x|$.

It follows that if we choose $\delta<\frac{\varepsilon}{k}$, then f is absolutely continuons.Finally since $[a,b]$ is arbitrary on I°, then f is continuons on I°. □

Theorem 2.5. *Supposed that $f:[a,b]\to\mathbb{R}$ is a $\eta-$strongly convex function such that η is bounded from above on $f([a,b])\times f([a,b])$. Then*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)-\frac{1}{2}M_\eta+\frac{c}{3}\eta^2(a,b)\le\frac{1}{b-a}\int_a^b f(x)\,dx\\ &\le\frac{1}{2}\left[f(a)+f(b)\right]+\frac{1}{4}\{\eta f(a),f(b))+\eta(f(b),f(a))\}-\frac{c}{12}\left[\eta^2(a,b)+\eta^2(b,a)\right]\\ &\le\frac{1}{2}\left[f(a)+f(b)\right]+\frac{M_\eta}{2}-\frac{c}{12}\left[\eta^2(a,b)+\eta^2(b,a)\right]\end{aligned}$$

Proof. For the right side of inequality consider an arbitary point $x=ta+(1-t)b$ with $t\in[0,1]$. So $f(x)\le f(b)+t\eta(f(a),f(b))-ct(1-t)\eta^2(a,b)$ with $t=\frac{x-b}{a-b}$. It follows that

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx &\le\frac{1}{b-a}\int_a^b\left[f(b)+t\eta(f(a),f(b))-ct(1-t)\eta^2(a,b)\right]dx\\ &=\frac{1}{b-a}\left((b-a)f(b)+\frac{\eta(f(a),f(b))}{(b-a)}\frac{(b-a)^2}{2}-c\eta^2(a,b)\int_a^b\frac{(a-x)(x-b)}{(b-a)^2}dx\right)\\ &=f(b)+\frac{1}{2}\eta(f(a),f(b))-c\frac{(b-a)^3}{6(b-a)^3}\eta^2(a,b)\\ &=f(b)+\frac{1}{2}\eta(f(a),f(b))-\frac{c}{6}\eta^2(a,b)\end{aligned}$$

Therefore we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx &\le\min\left\{f(b)+\frac{1}{2}\eta(f(a),f(b))-\frac{c}{6}\eta^2(a,b),f(a)+\frac{1}{2}\eta(f(b),f(a))-\frac{c}{6}\eta^2(b,a)\right\}\\ &\le\frac{1}{2}\left[f(a)+f(b)\right]+\frac{1}{4}\{\eta f(a),f(b))+\eta(f(b),f(a))\}-\frac{c}{12}\left[\eta^2(a,b)+\eta^2(b,a)\right]\\ &\le\frac{1}{2}\left[f(a)+f(b)\right]+\frac{M_\eta}{2}-\frac{c}{12}\left[\eta^2(a,b)+\eta^2(b,a)\right]\end{aligned}$$

For the left side o inequality, $\eta-$strongly convex of f implies that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &=f\left(\frac{a+b}{4}-t\frac{(b-a)}{4}+\frac{a+b}{4}+t\frac{(b-a)}{4}\right)\\ &=f\left(\frac{1}{2}\left(\frac{a+b-t(b-a)}{2}\right)+\frac{1}{2}\left(\frac{a+b+t(b-a)}{2}\right)\right)\\ &\le f\left(\frac{a+b+t(b-a)}{2}\right)+\frac{1}{2}\eta\left(f\left(\frac{a+b-t(b-a)}{2}\right),f\left(\frac{a+b+t(b-a)}{2}\right)\right)\\ &\quad-\frac{c}{4}\eta^2\left(\frac{a+b-t(b-a)}{2},\frac{a+b+t(b-a)}{2}\right)\\ &\le f\left(\frac{a+b+t(b-a)}{2}\right)+\frac{1}{2}M_\eta-\frac{c}{4}\eta^2\left(\frac{a+b-t(b-a)}{2},\frac{a+b+t(b-a)}{2}\right)\end{aligned}$$

for all $t\in[0,1]$. So

$$f\left(\frac{a+b+t(b-a)}{2}\right)\ge f\left(\frac{a+b}{2}\right)-\frac{1}{2}M_\eta+\frac{c}{4}\eta^2\left(\frac{a+b-t(b-a)}{2},\frac{a+b+t(b-a)}{2}\right)$$

Also with the same argument we have

$$f\left(\frac{a+b-t(b-a)}{2}\right) \geq f\left(\frac{a+b}{2}\right) - \frac{1}{2}M_\eta + \frac{c}{4}\eta^2\left(\frac{a+b+t(b-a)}{2}, \frac{a+b-t(b-a)}{2}\right)$$

Finally using the change of variable we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{b-a}\left[\int_a^{a+b} f(x)\,dx + \int_{a+b}^b f(x)\,dx\right]\\ &= \frac{1}{2}\int_0^1\left[f\left(\frac{a+b-t(b-a)}{2}\right) + f\left(\frac{a+b+t(b-a)}{2}\right)\right]dt\\ &\geq \frac{1}{2}\int_0^1\left[2f\left(\frac{a+b}{2}\right) - M_\eta + \frac{c}{2}\eta^2\left(\frac{a+b+t(b-a)}{2}, \frac{a+b-t(b-a)}{2}\right)\right]dt\\ &= f\left(\frac{a+b}{2}\right) - \frac{1}{2}M_\eta + \frac{c}{3}\eta^2(a,b).\end{aligned}$$

□

Definition 2.6. A function $g : [a,b] \to \mathbb{R}$ is said to be symmetric with respect to $\frac{a+b}{2}$ on $[a,b]$ if

$$g(x) = g(a+b-x), \text{ for any } a \leq x \leq b.$$

Theorem 2.7 (Hermite-Hadamard-Fejer right inequality)**.** *Supposed that $f : [a,b] \to \mathbb{R}$ is a $\eta-$strongly convex function such that η is bounded from above on $f([a,b]) \times f([a,b])$. Also supposed that $g : [a,b] \to \mathbb{R}^+$, is integrable and symmetric with respect to $\frac{a+b}{2}$.*

$$\begin{aligned}&\int_a^b f(x)g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\\ &+\frac{[\eta(f(a),f(b))+\eta(f(b),f(a))]}{2(b-a)}\int_a^b (b-x)g(x)\,dx\\ &-\frac{c}{2}\left[\eta^2(a,b)+\eta^2(b,a)\right]\int_a^b \frac{(b-x)(x-a)}{(b-a)^2}g(x)\,dx.\end{aligned}$$

Proof. From f is $\eta-$strongly convex function, using the change of variable and the fact that g is symmetric with respect to $\frac{a+b}{2}$, we get two inequalities.

First

$$\begin{aligned}&\int_a^b f(x)g(x)\,dx \leq (b-a)\int_0^1\left[f(b) + t\eta(f(a),f(b)) - ct(1-t)\eta^2(a,b)\right]\\ &\times g(ta+(1-t)b)\,dt\\ &= (b-a)\left[\int_0^1 f(b)g(ta+(1-t)b)\,dt + \eta(f(a),f(b))\int_0^1 tg(ta+(1-t)b)\,dt\right.\\ &\left.-c\eta^2(a,b)\int_0^1 t(1-t)g(ta+(1-t)b)\,dt\right].\end{aligned} \tag{2.1}$$

Second

$$\begin{aligned}&\int_a^b f(x)g(x)\,dx \leq (b-a)\int_0^1\left[f(a) + t\eta(f(b),f(a)) - ct(1-t)\eta^2(b,a)\right]\\ &\times g((1-t)a+tb)\,dt\\ &= (b-a)\left[\int_0^1 f(a)g((1-t)a+tb)\,dt + \eta(f(b),f(a))\int_0^1 tg((1-t)a+tb)\,dt\right.\\ &\left.-c\eta^2(b,a)\int_0^1 t(1-t)g((1-t)a+tb)\,dt\right].\end{aligned} \tag{2.2}$$

Finally if we add (2.1) and (2.2) we obtain

$$\begin{aligned}2\int_a^b f(x)g(x)\,dx &\leq (b-a)[f(a)+f(b)]\int_0^1 g((1-t)a+tb)\,dt\\ &+(b-a)[\eta(f(a),f(b))+\eta(f(b),f(a))]\int_0^1 tg((1-t)a+tb)\,dt\\ &-c\left[\eta^2(a,b)+\eta^2(b,a)\right]\int_0^1 t(1-t)g((1-t)a+tb)\,dt\end{aligned}$$

So the change of variable $x = ta + (1-t)b$ implies that

$$\begin{aligned}&\int_a^b f(x)g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\\ &+\frac{[\eta(f(a),f(b))+\eta(f(b),f(a))]}{2(b-a)}\int_a^b (b-x)g(x)\,dx\\ &-\frac{c}{2}\left[\eta^2(a,b)+\eta^2(b,a)\right]\int_a^b \frac{(b-x)(x-a)}{(b-a)^2}g(x)\,dx.\end{aligned}$$

□

Theorem 2.8 (Hermite-Hadamard-Fejer left inequality)**.** *Supposed that $f:[a,b]\to\mathbb{R}$ is a $\eta-$strongly convex function such that η is bounded from above on $f([a,b])\times f([a,b])$. Also supposed that $g:[a,b]\to\mathbb{R}^{+}$, is integrable and symmetric with respect to $\frac{a+b}{2}$.*

$$\begin{array}{c} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx-\frac{1}{2}\int_a^b \eta\left(f(a+b-x),f(x)\right)g(x)\,dx \\ +\frac{c}{4}\int_a^b \eta^2(a+b-x,x)\,g(x)\,dx\le\int_a^b f(x)\,g(x)\,dx. \end{array}$$

Proof. From $\eta-$strongly convex of f we have

$$\begin{array}{c} f\left(\frac{a+b}{2}\right)=f\left(\frac{ta-ta+a+b-tb+tb}{2}\right)=f\left(\frac{ta+(1-t)b}{2}+\frac{tb+(1-t)a}{2}\right) \\ \le f\left(tb+(1-t)a\right)+\frac{1}{2}\eta\left(f\left(ta+(1-t)b\right),f\left(tb+(1-t)a\right)\right) \\ -\frac{c}{4}\eta^2\left(ta+(1-t)b,tb+(1-t)a\right) \end{array}$$

Also with the change of variable $x=tb+(1-t)a$, $t=\frac{x-a}{b-a}$ we have

$$\begin{array}{l} f\left(\frac{a+b}{2}\right)\int_0^1 g\left(tb+(1-t)a\right)(b-a)\,dt \\ \le\int_0^1 f\left(tb+(1-t)a\right)g\left(tb+(1-t)a\right)(b-a)\,dt \\ +\frac{1}{2}\int_0^1 \eta\left(f\left(ta+(1-t)b\right),f\left(tb+(1-t)a\right)\right)g\left(tb+(1-t)a\right)(b-a)\,dt \\ -\frac{c}{4}\int_0^1 \eta^2\left(ta+(1-t)b,tb+(1-t)a\right)g\left(tb+(1-t)a\right)(b-a)\,dt \\ =\int_a^b f(x)\,g(x)\,dx+\frac{1}{2}\int_a^b \eta\left(f(a+b-x),f(x)\right)g(x)\,dx \\ -\frac{c}{4}\int_a^b \eta^2(a+b-x,x)\,g(x)\,dx \end{array}$$

□

Let $f:I\to\mathbb{R}$ be a $\eta-$strongly convex function. For $x_1,x_2\in I$, and $\alpha_1,\alpha_2\in[0,1]$. Define $T_i=\sum_{j=1}^{i}\alpha_j$ and choose α_i such that $T_n=1$. So

$$\begin{array}{l} f\left(\sum_{i=1}^{n}\alpha_i\,x_i\right)=f\left(\sum_{i=1}^{n}\alpha_i\,x_i\frac{T_n}{T_n}\right) \\ =f\left(\sum_{i=1}^{n-1}\alpha_i\,x_i\frac{T_{n-1}}{T_{n-1}}+\alpha_n x_n\right) \\ \le f(x_n)+T_{n-1}\eta\left(f\left(\sum_{i=1}^{n-1}\alpha_i\,\frac{x_i}{T_{n-1}}\right),f(x_n)\right) \\ \quad -c\prod_{i=1}^{n}\alpha_i\,\eta^2\left(\sum_{i=1}^{n-1}\alpha_i\,\frac{x_i}{T_{n-1}},x_n\right). \end{array}$$

Theorem 2.9. *Consider functions $f:I\to\mathbb{R}$ and $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, such that η is nondecreasing and nonnegative sublinear on first variable. also define*

$$\eta_f\left(x_i,x_{i+1},\cdots,x_n\right)=\eta\left(\eta_f\left(x_i,x_{i+1},\cdots,x_{n-1}\right),f(x_n)\right)$$

and $\eta_f(x)=f(x)$ for all $x\in I$. Then f is $\eta-$ strongly convex if f for any $n\ge 2$,

$$f\left(\sum_{i=1}^{n}\alpha_i\,x_i\right)\le\sum_{i=1}^{n}T_i\eta_f\left(x_i,x_{i+1,\cdots},x_n\right)-c\prod_{i=1}^{n}T_i\alpha_i\,\eta^2\left(\sum_{i=1}^{n-1}\alpha_i\,x_i,x_n\right).$$

$T_j=\sum_{j=1}^{i}\alpha_j$ for $i=1,2,\ldots n$ such that $T_n=1$.

Proof. Suppose that f is $\eta-$strongly convex. Since η is nondecreasing and nonnegative sublinear on first variable then for (3) it follows that

$$\begin{aligned}
& f\left(\sum_{i=1}^{n}\alpha_i\ x_i\right)=f\left(\sum_{i=1}^{n}\alpha_i\ x_i\frac{T_n}{T_n}\right)\\
& = f\left(\sum_{i=1}^{n-1}\alpha_i\ x_i\frac{T_{n-1}}{T_{n-1}}+\alpha_n x_n\right)\\
& \le f(x_n)+T_{n-1}\eta\left(f\left(\sum_{i=1}^{n-1}\alpha_i\ \frac{x_i}{T_{n-1}}\right),f(x_n)\right)\\
& -c\prod_{i=1}^{n}\alpha_i\ \eta^2\left(\sum_{i=1}^{n-1}\alpha_i\ \frac{x_i}{T_{n-1}},x_n\right)\\
& \le f(x_n)+T_{n-1}\eta\left(f\left(\frac{T_{n-2}}{T_{n-1}}\sum_{i=1}^{n-2}\alpha_i\ \frac{x_i}{T_{n-2}}\right)+\frac{\alpha_{n-1}}{T_{n-1}}x_{n-1},f(x_n)\right)\\
& -T_{n-1}c\prod_{i=1}^{n-1}\alpha_i\ \eta^2\left(\frac{T_{n-2}}{T_{n-1}}\sum_{i=1}^{n-2}\alpha_i\ \frac{x_i}{T_{n-2}}+\frac{\alpha_{n-1}}{T_{n-1}}x_{n-1},x_n\right)\\
& \le f(x_n)+T_{n-1}\eta\left(\left[f\left(x_{n-1}\right)+\frac{T_{n-2}}{T_{n-1}}\eta\left(f\left(\sum_{i=1}^{n-1}\alpha_i\ \frac{x_i}{T_{n-2}}\right),f\left(x_{n-1}\right)\right),f(x_n)\right]\right)\\
& -T_{n-1}c\prod_{i=1}^{n-1}\alpha_i\ \eta^2\left(\frac{T_{n-2}}{T_{n-1}}\sum_{i=1}^{n-2}\alpha_i\ \frac{x_i}{T_{n-2}}+\frac{\alpha_{n-1}}{T_{n-1}}x_{n-1},x_n\right)\\
& -T_{n-2}c\prod_{i=1}^{n-2}\alpha_i\ \eta^2\left(\frac{T_{n-3}}{T_{n-2}}\sum_{i=1}^{n-3}\alpha_i\ \frac{x_i}{T_{n-3}}+\frac{\alpha_{n-2}}{T_{n-2}}x_{n-2},x_{n-1},x_n\right)\\
& \le\cdots\le f(x_n)+T_{n-1}\eta\left(f(x_{n-1}),f(x_n)\right)+T_{n-2}\eta\left(\eta\left(f(x_{n-2}),f(x_{n-1})\right),f(x_n)\right)\\
& +\cdots+T_1\eta\left(\eta\left(\cdots\eta\left(f(x_1),f(x_2),f(x_3)\cdots\right)\right),f(x_n)\right)\\
& -\left(T_{n-1}+T_{n-2}+\cdots+T_1\right)c\prod_{i=1}^{n}\alpha_i\ \eta^2\left(\sum_{i=1}^{n-1}\alpha_i\ x_i,x_n\right)\\
& = f(x_n)+T_{n-1}\eta_f\left(x_{n-1},x_n\right)+T_{n-2}\eta_f\left(x_{n-2},x_{n-1},x_n\right)+\cdots+T_1\eta_f\left(x_1,x_2,\cdots,x_{n-1},x_n\right)\\
& -c\sum_{i=1}^{n-1}T_i\left(\prod_{i=1}^{n}\alpha_i\ \eta^2\left[\left(\eta\left(\cdots\eta\left(\left(x_1,x_2\right),x_3\right)\right),\ldots,x_n\right)\right]\right)\\
& = \sum_{i=1}^{n}T_i\eta_f\left(x_i,x_{i+1,\cdots,}x_n\right)-c\prod_{i=1}^{n}T_i\alpha_i\ \eta^2\left(\sum_{i=1}^{n-1}\alpha_i\ x_i,x_n\right).
\end{aligned}$$

For the inverse implication consider $n=2$ in (3). we omit the details. □

Theorem 2.10. *Suppose that* $f:[a,b]\to\mathbb{R}$ *is a differentiable* $\eta-$*strongly convex function on* (a,b) *and that* η *is measurable on* $f([a,b])\times f([a,b])$. *Then we have*

$$\begin{aligned}
& f'(y)\left(\frac{a+b}{2}-y\right)+c\left[\left(\frac{b^2+ab+a^2}{3}\right)-(a+b)y+y^2\right]\\
& \le\int_a^b\eta\left(f(x),f(y)\right)dx.
\end{aligned}$$

Proof. From the definition of $\eta-$strongly convex functions we have;

$$\frac{f(tx+(1-t)y)-f(y)}{t}+c(1-t)(x-y)^2\le\eta\left(f(x),f(y)\right)$$

for $t\in(0,1]$. Taking the limit $t\to0^+$, we get

$$f'(y)(x-y)+c(x-y)^2\le\eta\left(f(x),f(y)\right)$$

for any $x\in[a,b]$ and any $y\in(a,b)$.

Since is measurable on $f([a,b])\times f([a,b])$, then the integral

$$\begin{aligned}
& \int_a^b\eta\left(f(x),f(y)\right)dx\ge\int_a^b f'(y)(x-y)\,dx+\int_a^b c(x-y)^2\,dx\\
& =(b-a)f'(y)\left(\frac{a+b}{2}-y\right)+c\left[\left(\frac{b^3-a^3}{3}\right)-\left(b^2-a^2\right)y+(b-a)y^2\right]
\end{aligned}$$

□

References

[1] Aleman, A., *On some generalizations of convex sets and convex functions*, Anal. Numer.Theor. Approx., **14**(1985), 1–6. 1

[2] Bector, C.R., Singh, C., *B-Vex functions*, J. Optim. Theory. Appl., **71(2)**(1991), 237–253. 1

[3] De, B., *. . . netti, Sulla strati. . . cazioni convesse*, Ann. Math. Pura. Appl., **30**(1949), 173–183. 1

[4] Dragomir, S.S., *Inequalities of Hermite-Hadamard type for λ -convex functions on linear spaces*, Preprint RGMIA Res. Rep. Coll. **17**(2014), Art. 13, pp.18. [Online http://rgmia.org/papers/v17/v17a13.pdf]. 1

[5] Fejer, L., *Uberdie fourierreihen, II*, Math. Naturwise. Anz Ungar. Akad. Wiss., **24**(1906), 369–390. 1

[6] Hanson, M.A., *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl., **80**(1981), 545–550. 1

[7] Hyers, D.H., Ulam, S.M., *Approximately convex functions*, Proc. Amer. Math. Soc., **3**(1952), 821–828. 1

[8] Hsu, I., Kuller, R.G., *Convexity of vector-valued functions*, Proc. Amer. Math. Soc., **46**(1974), 363–366. 1

[9] Jensen, J.L.W.V., *On konvexe funktioner og uligheder mellem middlvaerdier*, Nyt. Tidsskr. Math. B., **16**(1905), 49-69. 1

[10] Luc, D.T., Theory of Vector Optimization, Springer-Verlag, Berlin, 1989. 1

[11] Mangasarian, O.L., *Pseudo-Convex functions*, SIAM Journal on Control, **3**(1965), 281–290. 1

[12] Özdemir, M.E., Avci, M., Kavurmaci, H., *Hermite-Hadamard-type inequalities via (α;m)-convexity*, Comput. Math. Appl., **61(9)**(2011), 2614–2620 1

[13] Pečarić, J.E., Proschan, F., Tong, Y.L., Convex functions, Partial Orderings and Statistical Applications, Academic Press, Boston, 1992. 1

[14] Polyak, B.T., *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl., **7**(1966) 72–75. 1

[15] Rajba, T., *On strong delta-convexity and Hermite-Hadamard type inequalities for delta convex functions of higher order*, Math. Inequal. Appl., **18(1)**(2015), 267–293 1

[16] Robert, A.W., Varbeg, D.E., Convex Functions, Academic Press, 1973. 1