Proceedings of International Conference on Mathematics
and Mathematics Education (ICMME 2018)
Turk. J. Math. Comput. Sci.
10(2018) 222–241

http://dergipark.gov.tr/tjmcs

MATDER

Laguerre Collocation Method for Solutions of The Systems of First Order Linear Differential Equations

Şuayip Yüzbaşı[a], Gamze Yildirim[a,*]

[a]*Department of Mathematics, Faculty of Science, Akdeniz University, TR-07058, Antalya, Turkey.*

Received: 09-08-2018 • Accepted: 10-12-2018

Abstract. In this paper, a collocation method based on Laguerre polynomials is presented to solve the systems of linear differential equations. The Laguerre polynomials, their derivatives, the system of differential equations and the conditions are written in the matrix form. Then, by using the constructed matrix forms, the collocation points and the matrix operations, the system of linear differential equations is transformed into a system of linear algebraic equations. The solution of this system gives the coefficients of the solutions forms. Thus, the solutions based on the Laguerre polynomials is found. Also, an error estimation is presented by using residual functions.

2010 AMS Classification: 34A30, 34K28, 65L05, 65L60, 65L80.

Keywords: Collocation method, collocation points, Laguerre collocation method, Laguerre polynomials, systems of linear differential equations.

1. Introduction

Many problems in the fields of engineering and science often involve systems of differential equations. Numerical methods are used when it is not possible to calculate exact solutions of these equation systems. The differential transformation method [1], the Bessel polynomial approach [19], the Taylor polynomial approach [13], the Adomian decomposition method [4], the differential transform method and the Laplace transform method [15], the improvement of He's variational iteration method [14] and the exponential Chebyshev collocation method [11] can be given as example to these numerical methods.

In addition, there are numerical methods for also Fredholm [6–8, 10, 16, 20], Fredholm-Volterra [2, 5, 18] and integro [3, 9, 12, 17] equation or equation systems.

In this study, we consider the approximate solution of systems of first order linear differential equations

$$\sum_{j=1}^{k} P_j(x)y_j^{'}(x) + \sum_{j=1}^{k} Q_j(x)y_j(x) = g_i(x), \quad i = 1, 2, ..., k \tag{1.1}$$

under the mixed condition

*Corresponding Author
Email addresses: syuzbasi@akdeniz.edu.tr (Ş. Yüzbaşı), yildirimgamze17@hotmail.com (G. Yıldırım)
Authors presented this study as an oral presentation in International Conference on Mathematics and Mathematics Education (ICMME-2018), 27-29 June 2018, Ordu, Turkey.

$$\sum_{i=1}^{k}(\alpha_i y_i(a)+\beta_i y_i(b))=\lambda_i, \quad i=1,2,...,k \tag{1.2}$$

where $P_j(x)$, $Q_j(x)$ and $g_i(x)$ are the known functions defined on interval $0 \le a \le x \le b$; $y_j(x)$, $i=1,2,...,k$ is an unknown function; α_i, β_i and λ_i are appropriate real constants.

Our aim in this study is to obtain an approximate solution under conditions (1.2) of system (1.1) expressed in the truncated Laguerre series form

$$y_{i,N}(x)=\sum_{n=0}^{N}a_{i,n}L_n(x), \quad i=1,2,...,k \tag{1.3}$$

where $a_{i,n}$ are Laguerre coefficients to be determined and N is any chosen positive integer.

2. Fundamental Matrix Derivative Relation

Firstly, we write the approximate solutions in (1.3) in matrix form

$$y_j(x)=\mathbf{L}(x)\mathbf{A}_j, \quad j=1,2,...,k. \tag{2.1}$$

Here,

$$\mathbf{A}_j=\begin{bmatrix} a_{j,0} & a_{j,1} & \cdots & a_{j,N} \end{bmatrix}^T, \quad j=1,2,...,k, \quad \mathbf{L}(x)=\begin{bmatrix} L_0(x) & L_1(x) & \cdots & L_N(x) \end{bmatrix}_{1\times(N+1)}$$

and

$$\mathbf{L}(x)=\mathbf{X}(x)\mathbf{D}^T \tag{2.2}$$

where

$$\mathbf{X}(x)=\begin{bmatrix} 1 & x & x^2 & \cdots & x^N \end{bmatrix}_{1\times(N+1)}, \quad \mathbf{D}=\begin{bmatrix} \frac{(-1)^0}{0!}\binom{0}{0} & 0 & 0 & \cdots & 0 \\ \frac{(-1)^0}{0!}\binom{1}{0} & \frac{(-1)^1}{1!}\binom{1}{1} & 0 & \cdots & 0 \\ \frac{(-1)^0}{0!}\binom{2}{0} & \frac{(-1)^1}{1!}\binom{2}{1} & \frac{(-1)^2}{2!}\binom{2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^0}{0!}\binom{N}{0} & \frac{(-1)^1}{1!}\binom{N}{1} & \frac{(-1)^2}{2!}\binom{N}{2} & \cdots & \frac{(-1)^N}{N!}\binom{N}{N} \end{bmatrix}_{(N+1)\times(N+1)}.$$

Secondly, let's find $\mathbf{X}'(x)$ that the first derivative of $\mathbf{X}(x)=\begin{bmatrix} 1 & x & x^2 & \cdots & x^N \end{bmatrix}_{1\times(N+1)}$. For this purpose, by taking the derivative of $\mathbf{X}(x)$, we obtain

$$\mathbf{X}'(x)=\begin{bmatrix} 0 & 1 & 2x & \cdots & Nx^{N-1} \end{bmatrix}_{1\times(N+1)}=\mathbf{X}(x)\mathbf{B}^T$$

where

$$\mathbf{B}^T=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \quad (\mathbf{B}^T)^0=[\mathbf{I}]_{(N+1)\times(N+1)} \quad \text{birim} \quad \text{matris.}$$

As the third, let's write $y_j'(x)$ that the first derivative of the solution $y_j(x)$ in the matrix form. by taking the derivative of (2.1) and by replacing $\mathbf{X}'(x)$, we obtain

$$y_j^{'}(x) = \mathbf{X}^{'}(x)\mathbf{D}^T\mathbf{A}_j = \mathbf{X}(x)\mathbf{B}^T\mathbf{D}^T\mathbf{A}_j, \quad j = 1, 2, ..., k.$$

Thus, the matrices $y^{'}(x)$ can be written as

$$\mathbf{y}^{'}(x) = \bar{\mathbf{X}}(x)\bar{\mathbf{B}}\bar{\mathbf{D}}\mathbf{A}, \tag{2.3}$$

where

$$\mathbf{y}^{'}(x) = \begin{bmatrix} y_1^{'}(x) \\ y_2^{'}(x) \\ \vdots \\ y_k^{'}(x) \end{bmatrix}_{k\times 1}, \quad \bar{\mathbf{X}}(x) = \begin{bmatrix} \mathbf{X}(x) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(x) \end{bmatrix}_{k\times k}, \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B}^T & 0 & \cdots & 0 \\ 0 & \mathbf{B}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}^T \end{bmatrix}_{k\times k},$$

$$\bar{\mathbf{D}} = \begin{bmatrix} \mathbf{D}^T & 0 & \cdots & 0 \\ 0 & \mathbf{D}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{D}^T \end{bmatrix}_{k\times k}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_k \end{bmatrix}_{k\times 1}.$$

Finally, the system given in (1.1) can be written in matrix form as

$$\mathbf{P}_1(x)\mathbf{y}^{'}(x) + \mathbf{Q}_0(x)\mathbf{y}(x) = \mathbf{g}(x)$$

where

$$\mathbf{P}_1(x) = \begin{bmatrix} P_{1,1}^{'}(x) & P_{1,2}^{'}(x) & \cdots & P_{1,k}^{'}(x) \\ P_{2,1}^{'}(x) & P_{2,2}^{'}(x) & \cdots & P_{2,k}^{'}(x) \\ \vdots & \vdots & \ddots & \vdots \\ P_{k,1}^{'}(x) & P_{k,2}^{'}(x) & \cdots & P_{k,k}^{'}(x) \end{bmatrix}_{k\times k}, \quad \mathbf{Q}_0(x) = \begin{bmatrix} Q_{1,1}(x) & Q_{1,2}(x) & \cdots & Q_{1,k}(x) \\ Q_{2,1}(x) & Q_{2,2}(x) & \cdots & Q_{2,k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{k,1}(x) & Q_{k,2}(x) & \cdots & Q_{k,k}(x) \end{bmatrix}_{k\times k},$$

$$\mathbf{y}^{'}(x) = \begin{bmatrix} y_1^{'}(x) \\ y_2^{'}(x) \\ \vdots \\ y_k^{'}(x) \end{bmatrix}_{k\times 1}, \quad \mathbf{g}(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_k(x) \end{bmatrix}_{k\times 1}.$$

3. Fundamental Matrix Relation Based on Collocation Points

Now, let's define the collocation points that we use in the method. The collocation points x_s are written as

$$x_s = a + \frac{b-a}{N}s, \quad s = 0, 1, 2, ..., N$$

where

$$0 \le a \le x \le b \quad \text{and} \quad a = x_0 < x_1 < ... < x_n = b.$$

Firstly, if we substitute the collocation points into system given in (1.1), then we obtain the system of matrix equations

$$\mathbf{P}_1(x_s)\mathbf{y}_j^{'}(x_s) + \mathbf{Q}_0(x_s)\mathbf{y}_j(x_s) = \mathbf{g}(x_s), \quad s = 0, 1, 2, ..., N.$$

or briefly

$$\mathbf{P}_1\mathbf{Y}^{'} + \mathbf{Q}_0\mathbf{Y} = \mathbf{G}, \quad s = 0, 1, 2, ..., N. \tag{3.1}$$

where

$$\mathbf{P}_1=\begin{bmatrix}\mathbf{P}_1(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{P}_1(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{P}_1(x_N)\end{bmatrix},\mathbf{Q}_0=\begin{bmatrix}\mathbf{Q}_0(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{Q}_0(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{Q}_0(x_N)\end{bmatrix},\mathbf{Y}'=\begin{bmatrix}\mathbf{y}'(x_0)\\ \mathbf{y}'(x_1)\\ \vdots\\ \mathbf{y}'(x_N)\end{bmatrix},\mathbf{G}=\begin{bmatrix}\mathbf{g}(x_0)\\ \mathbf{g}(x_1)\\ \vdots\\ \mathbf{g}(x_N)\end{bmatrix}.$$

Later, if we substitute the collocation points into (2.3), then we obtain

$$\mathbf{y}'(x_s)=\bar{\mathbf{X}}(x_s)\bar{\mathbf{B}}\bar{\mathbf{D}}\mathbf{A}$$

or briefly

$$\mathbf{Y}'=\mathbf{X}\bar{\mathbf{B}}\bar{\mathbf{D}}\mathbf{A} \tag{3.2}$$

where

$$\mathbf{X}=\begin{bmatrix}\bar{\mathbf{X}}(x_0)\\ \bar{\mathbf{X}}(x_1)\\ \vdots\\ \bar{\mathbf{X}}(x_N)\end{bmatrix},\quad \bar{\mathbf{X}}(x_s)=\begin{bmatrix}\mathbf{X}(x_s) & 0 & \cdots & 0\\ 0 & \mathbf{X}(x_s) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{X}(x_s)\end{bmatrix}_{k\times k},\quad s=0,1,2,...,N.$$

4. The Collocation Method

Let's put the relation (3.2) into the relation (3.1) to obtain of the system matrix equation. Thus we obtain the system matrix equation as

$$\left\{\mathbf{P}_1\mathbf{X}\bar{\mathbf{B}}\bar{\mathbf{D}}+\mathbf{Q}_0\mathbf{X}\bar{\mathbf{D}}\right\}\mathbf{A}=\mathbf{G}. \tag{4.1}$$

When we write the dimensions of the matrices $\mathbf{P}_1$, $\mathbf{Q}_0$, $\mathbf{X}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{D}}$, $\mathbf{A}$ and $\mathbf{G}$ in system given in (4.1), we obtain $k(N+1)\times k(N+1)$, $k(N+1)\times k(N+1)$, $k(N+1)\times k(N+1)$, $k(N+1)\times k(N+1)$, $k(N+1)\times k(N+1)$, $k(N+1)\times 1$ and $k(N+1)\times 1$, respectively.

The equation system in (4.1) can be written as

$$\mathbf{W}\mathbf{A}=\mathbf{G}\quad \text{veya}\quad [\mathbf{W};\mathbf{G}] \tag{4.2}$$

where

$$\mathbf{W}=\mathbf{P}_1\mathbf{X}\bar{\mathbf{B}}\bar{\mathbf{D}}+\mathbf{Q}_0\mathbf{X}\bar{\mathbf{D}}.$$

Now, by means of the conditions (1.2) and the relations (2.3), we obtain the matrix form for the mixed conditions (1.2) as

$$[\alpha_0\bar{\mathbf{X}}(a)+\beta_0\bar{\mathbf{X}}(b)]\bar{\mathbf{D}}\mathbf{A}=\lambda$$

where

$$\alpha_0=\begin{bmatrix}\alpha_{0,0}^1 & 0 & \cdots & 0\\ 0 & \alpha_{0,0}^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \alpha_{0,0}^k\end{bmatrix},\quad \beta_0=\begin{bmatrix}\beta_{0,0}^1 & 0 & \cdots & 0\\ 0 & \beta_{0,0}^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \beta_{0,0}^k\end{bmatrix},\quad \lambda=\begin{bmatrix}\lambda_{1,0}\\ \lambda_{2,0}\\ \vdots\\ \lambda_{k,0}\end{bmatrix},$$

or briefly, we can write in the form

$$\mathbf{U}\mathbf{A}=\lambda\quad \text{veya}\quad [\mathbf{U};\lambda] \tag{4.3}$$

where

$$\mathbf{U}=[\alpha_0\bar{\mathbf{X}}(a)+\beta_0\bar{\mathbf{X}}(b)]\bar{\mathbf{D}}.$$

Finally for the method, when the rows of the matrices **U** and λ are replaced by the last k rows of the matrices **W** and **G**, respectively, the new augmented matrix is obtained as

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=\left[\begin{array}{cccccc} w_{1,1} & w_{1,2} & \cdots & w_{1,k(N+1)} & ; & g_1(x_0) \\ w_{2,1} & w_{2,2} & \cdots & w_{2,k(N+1)} & ; & g_2(x_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{k,1} & w_{k,2} & \cdots & w_{k,k(N+1)} & ; & g_k(x_0) \\ w_{k+1,1} & w_{k+1,2} & \cdots & w_{k+1,k(N+1)} & ; & g_1(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{kN,1} & w_{kN,2} & \cdots & w_{kN,k(N+1)} & ; & g_k(x_{N-1}) \\ u_{1,1} & u_{1,2} & \cdots & u_{1,k(N+1)} & ; & \lambda_{1,0} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,k(N+1)} & ; & \lambda_{2,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{k,1} & u_{k,2} & \cdots & u_{k,k(N+1)} & ; & \lambda_{k,0} \end{array}\right]$$

or briefly, we can write in the form

$$\widetilde{\mathbf{W}}\mathbf{A}=\widetilde{\mathbf{G}}. \tag{4.4}$$

If $rank\widetilde{\mathbf{W}}=rank[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=N+1$, the matrix (4.4) can be written as

$$\mathbf{A}=(\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}.$$

Hence, when this linear system is solved and $a_{i,0},a_{i,1},...,a_{i,N}$ $(i=1,2,...,k)$ is substituted in (1.3), the unknown the Laguerre coefficients matrix **A** is determined. Namely, the Laguerre polynomial solutions can be obtained as

$$y_{i,N}(x)=\sum_{n=0}^{N}a_{i,N}L_n(x),\quad i=1,2,...,k.$$

Furthermore, when $det(\widetilde{\mathbf{W}})=0$, if $rank\widetilde{\mathbf{W}}=rank[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]<N+1$, then a particular solution can be find. Otherwise if $rank\widetilde{\mathbf{W}}\neq rank[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]<N+1$, then it isn't a solution.

5. Residual Error Estimation and Improvement of Solutions

The actual error functions are defined as

$$e_{i,N}(x)=y_i(x)-y_{i,N}(x),\quad i=1,2,...,k$$

where y_i and $y_{i,N}$, respectively, represent the exact solutions and the approximate solutions.

Firstly, let's define the residual function. For this purpose, the Laguerre polynomial solutions are written in system (1.1) as

$$R_{i,N}(x)=\sum_{j=1}^{k}P_j(x)y'_{j,N}(x)+\sum_{j=1}^{k}Q_j(x)y_{j,N}(x)=g_i(x),\quad i=1,2,...,k$$

or briefly, we can write as

$$\sum_{j=1}^{k}P_j(x)y'_{j,N}(x)+\sum_{j=1}^{k}Q_j(x)y_{j,N}(x)=R_{i,N}(x)+g_i(x) \tag{5.1}$$

Similarly, the Laguerre polynomial solutions are written in condition (1.2) as

$$\sum_{i=1}^{k}(\alpha_i y_{i,N}(a)+\beta_i y_{i,N}(b))=\lambda_i,\quad i=1,2,...,k. \tag{5.2}$$

Secondly, if (5.1) is subtracted from (1.1), then the system of error differential equations is obtained as

$$\sum_{j=1}^{k} P_j(x)[y_j'(x) - y_{j,N}'(x)] + \sum_{j=1}^{k} Q_j(x)[y_j(x) - y_{j,N}(x)] = -R_{i,N}(x)$$

or briefly, we can write in the form

$$\sum_{j=1}^{k} P_j(x)[e_{j,N}'(x)] + \sum_{j=1}^{k} Q_j(x)[e_{j,N}(x)] = -R_{i,N}(x). \tag{5.3}$$

Afterward likewise, if (5.2) is subtracted from (1.2), then we obtain in the form

$$\sum_{i=1}^{k} (\alpha_i[y_i(a) - y_{i,N}(a)] + \beta_i[y_i(b) - y_{i,N}(b)]) = \lambda_i$$

or briefly, we can write in the form

$$\sum_{i=1}^{k} (\alpha_i[e_{i,N}(a)] + \beta_i[e_{i,N}(b)]) = \lambda_i. \tag{5.4}$$

Consequently, we obtain approximate solutions of the error problem (5.3) and (5.4) in the form

$$e_{i,N,M}(x) = \sum_{n=0}^{M} \widetilde{a_{i,n}}, L_n(x), \quad i = 1, 2, ..., k.$$

Thus, by adding approximate solution and approximate solution of the error problem, we obtain the improved approximate solution in the form

$$y_{i,N,M} = y_{i,N} + e_{i,N,M}.$$

Finally, we obtain the error function of the improved approximate solution in the form

$$E_{i,N,M} = y_i - y_{i,N,M}.$$

6. Illustrative Examples

In this section, numerical examples are given to understand that the method is effective. Numerical results are shown in tables and graphs and are compared with other methods. The values of the exact solutions, the approximate solutions, the corrected approximate solutions, the absolute error functions, the estimated error function and the corrected error function, respectively, is represented by $y_i(x)$, $y_{i,N}(x)$, $y_{i,N,M}(x)$, $e_{i,N}(x)$, $e_{i,N,M}(x)$ and $E_{i,N,M}(x)$ in tables and figures.

Example 6.1. Firstly, let's consider the system of first-order variable-coefficient linear differential equations

$$\begin{cases} y_1'(x) + y_2'(x) + y_1(x) + y_2(x) = 1 \\ y_2'(x) - 2y_1(x) - y_2(x) = 0 \end{cases}, 0 \le x \le 1 \tag{6.1}$$

with the conditions

$$y_1(0) = 0, \quad y_2(0) = 1 \tag{6.2}$$

and the exact solutions

$$y_1(x) = e^{-x} - 1 \quad ve \quad y_2(x) = 2 - e^{-x}.$$

Here;

$k = 2$, $m = 1$, $g_1(x) = 1$, $g_2(x) = 0$, $P^0_{1,1}(x) = 1$, $P^0_{1,2}(x) = 1$, $P^0_{2,1}(x) = -2$, $P^0_{2,2}(x) = -1$,

$P^1_{1,1}(x) = 1$, $P^1_{1,2}(x) = 1$, $P^1_{2,1}(x) = 0$, $P^1_{2,2}(x) = 1$.

Now, let's find solutions truncated Laguerre series expansions in the form

$$y_{i,N}(x) = \sum_{n=0}^{6} a_{i,n} L_n(x), \quad i = 1, 2$$

by using the Laguerre polynomials for $N = 6$. Here, the set of the collocation points for $N = 6$ is computed as

$$\{x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{1}{3}, x_3 = \frac{1}{2}, x_4 = \frac{2}{3}, x_5 = \frac{5}{6}, x_6 = 1\}$$

and thus the system of fundamental matrix equation of the problem is written as

$$\left\{\mathbf{P}_0\mathbf{X}\bar{\mathbf{D}} + \mathbf{P}_1\mathbf{X}\bar{\mathbf{B}}\bar{\mathbf{D}}\right\}\mathbf{A} = \mathbf{G}.$$

The augmented matrix for this system of fundamental matrix equations is calculated from Matlab. The matrix form of the conditions is also calculated. Then, instead of the last two lines of the augmented matrix, the matrix form of the conditions is written and so that the new augmented matrix $[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}]$ is obtained. Hence, this system is solved and the Laguerre coefficient matrix $\mathbf{A}$ is obtained. Finally, the matrix $\mathbf{A}$ is substituted into the (2.1)

$$y_{j,N}(x) = \mathbf{L}(x)\mathbf{A}_j \quad j = 1, 2$$

and thus we obtain approximate solutions as

$$\begin{aligned} y_{1,6}(x) = {} & 0.000923058514503x^6 - 0.00784599737327x^5 + 0.0414094305812x^4 \\ & -0.166599241611x^3 + 0.499992877635x^2 - x \end{aligned}$$

and

$$\begin{aligned} y_{2,6}(x) = {} & -0.000923058514503x^6 + 0.00784599737327x^5 - 0.0414094305812x^4 \\ & +0.166599241611x^3 - 0.499992877635x^2 + x + 1. \end{aligned}$$

Hence, for $N = 6$ the actual error functions are obtained as

$$\begin{aligned} e_{1,6}(x) &= y_1(x) - y_{1,6}(x) \\ &= e^{-x} - 1 - (0.000923058514503x^6 - 0.00784599737327x^5 \\ &\quad +0.0414094305812x^4 - 0.166599241611x^3 + 0.499992877635x^2 - 1.0x) \end{aligned}$$

and

$$\begin{aligned} e_{2,6}(x) &= y_2(x) - y_{2,6}(x) \\ &= 2 - e^{-x} - (-0.000923058514503x^6 + 0.00784599737327x^5 \\ &\quad -0.0414094305812x^4 + 0.166599241611x^3 - 0.499992877635x^2 + x + 1). \end{aligned}$$

Now, let's find the $R_{1,N}(x)$ and $R_{2,N}(x)$ residual functions: By writing the approximate solutions for $N = 6$ into the system (6.1), the residual functions are defined as

$$\begin{aligned} R_{1,6}(x) &= y'_{1,6}(x) + y'_{2,6}(x) + y_{1,6}(x) + y_{2,6}(x) - 1 \\ R_{2,6}(x) &= y'_{2,6}(x) - 2y_{1,6}(x) - y_{2,6}(x) - 0. \end{aligned} \tag{6.3}$$

(6.3) can also be written as

$$\begin{aligned} R_{1,6}(x) + 1 &= y'_{1,6}(x) + y'_{2,6}(x) + y_{1,6}(x) + y_{2,6}(x) \\ R_{2,6}(x) &= y'_{2,6}(x) - 2y_{1,6}(x) - y_{2,6}(x). \end{aligned} \tag{6.4}$$

If (6.4) is subtracted from (6.1), then the system of error differential equations is obtained as

$$\begin{aligned} -R_{1,6}(x) &= \left[y'_1(x) - y'_{1,6}(x)\right] + \left[y'_2(x) - y'_{2,6}(x)\right] + \left[y_1(x) - y_{1,6}(x)\right] + \left[y_2(x) - y_{2,6}(x)\right] \\ -R_{2,6}(x) &= \left[y'_2(x) - y'_{2,6}(x)\right] - 2\left[y_1(x) - y_{1,6}(x)\right] - \left[y_2(x) - y_{2,6}(x)\right]. \end{aligned} \tag{6.5}$$

Afterward likewise, since the Laguerre polynomial solutions provides conditions (6.2), we can write as

$$y_{1,6}(0) = 0, \quad y_{2,6}(0) = 1. \tag{6.6}$$

If (6.6) is subtracted from (6.2), then we obtain as

$$y_1(0) - y_{1,6}(0) = 0, \quad y_2(0) - y_{2,6}(0) = 0. \tag{6.7}$$

In conclusion, for $N = 6$ and $M = 8$, approximate solutions of the error problem (6.5) and (6.7) are obtained as

$$\begin{aligned} e_{1,6,8}(x) = {} & 0.0000161071982352x^8 - 0.000185232779705x^7 + 0.000454610284555x^6 \\ & -0.000481675789317x^5 + 0.000255553792544x^4 - 0.000067150954037x^3 \\ & +0.00000710301019424x^2 - 2.58608757181e - 37x + 9.99170198199e - 38 \\ e_{2,6,8}(x) = {} & -0.0000161071982352x^8 + 0.000185232779705x^7 - 0.000454610284555x^6 \\ & +0.000481675789317x^5 - 0.000255553792544x^4 + 0.000067150954037x^3 \\ & -0.00000710301019424x^2 + 2.58608757181e - 37x - 9.99170198199e - 38. \end{aligned}$$

Hence, approximate solution and approximate solution of the error problem are added and the improved approximate solution is obtained as

$$\begin{aligned} y_{1,6,8}(x) = {} & 0.0000161071982352x^8 - 0.000185232779705x^7 + 0.00137766879906x^6 \\ & -0.00832767316259x^5 + 0.0416649843737x^4 - 0.166666392565x^3 \\ & +0.499999980645x^2 - x + 9.99170198199e - 38 \\ y_{2,6,8}(x) = {} & -0.0000161071982352x^8 + 0.000185232779705x^7 - 0.00137766879906x^6 \\ & +0.00832767316259x^5 - 0.0416649843737x^4 + 0.166666392565x^3 \\ & -0.499999980645x^2 + x + 1. \end{aligned}$$

Finally, the improved error functions is obtained as

$$\begin{aligned} E_{1,6,8}(x) = {} & -0.0000161071982352x^8 + 0.000185232779705x^7 - 0.00137766879906x^6 \\ & +0.00832767316259x^5 - 0.0416649843737x^4 + 0.166666392565x^3 \\ & -0.499999980645x^2 + x - 1 + e^{-x} \\ E_{2,6,8}(x) = {} & +0.0000161071982352x^8 - 0.000185232779705x^7 + 0.00137766879906x^6 \\ & -0.00832767316259x^5 + 0.0416649843737x^4 - 0.166666392565x^3 \\ & +0.499999980645x^2 - x + 1 - e^{-x}. \end{aligned}$$

Now, the exact solutions, the approximate solutions, the improved approximate solutions and the absolute errors will be given in tables and graphs and will be compared with other methods available in the literature.

In Table 2, for $N = 6$ almost the same results are obtained in Laguerre and Bessel methods and better results are obtained than the other method. The results for $N = 10$ are better than those for $N = 6$. From this, it is seen that approximate solutions are closer to exact solution with increasing N values. When we examine the actual absolute errors in Table 3, it can be seen that as the N value increases, smaller results are obtained. From this, it is seen that approximate solutions are closer to exact solution with increasing N values. Looking at the numerical results of actual and estimated absolute errors, it seems that they are very close to each other. The numerical results of the improved absolute errors have yielded better results than the actual and estimated absolute errors and have not moved away from the actual errors. From this, it is seen that the residual error estimation method is effective.

From Figures (1)-(2), it can be seen that approximate solutions are closer to the exact solution with increasing N values. From Figure 3, it can be seen that for $N = 6$, the Laguerre and Bessel methods yield almost the same results and better results than the other method. It can be seen that better numerical results are obtained with increasing N values in Figures (4)-(5). From Figures (6)-(7)-(8)-(9), it is seen that the method is effective.

TABLE 1. Comparison numerical results of the exact solutions and approximate solutions with other methods for $N = 6$ of Eq. (6.1)

x_i	Exact solutions $y_1(x_i) = e^{-x_i} - 1$	Bessel [19] $N = 6, y_{1,6}(x_i)$	DTM [15] $N = 6, y_{1,6}(x_i)$	Laguerre $N = 6, y_{1,6}(x_i)$
0	0	0	0	0
0.2	-0.18126924692	-0.18126927538	-0.16569377777	-0.18126927538
0.4	-0.32967995396	-0.32967997178	-0.27847111111	-0.32967997178
0.6	-0.45118836390	-0.45118837657	-0.34969200000	-0.45118837657
0.8	-0.55067103588	-0.55067106942	-0.38436977777	-0.55067106942
1	-0.63212055882	-0.63211987225	-0.39861111111	-0.63211987225
x_i	$y_2(x_i) = 2 - e^{-x_i}$	$N = 6, y_{2,6}(x_i)$	$N = 6, y_{2,6}(x_i)$	$N = 6, y_{2,6}(x_i)$
0	1	1	1	1
0.2	1.18126924692	1.18126927538	1.18359546667	1.18126927538
0.4	1.32967995396	1.32967997178	1.34654720000	1.32967997178
0.6	1.45118836390	1.45118837657	1.50568720000	1.45118837657
0.8	1.55067103588	1.55067106942	1.68261546667	1.55067106942
1	1.63212055882	1.63211987225	1.91250000000	1.63211987225

TABLE 2. Comparison numerical results of actual errors with other methods for $N = 6, 10$ and Laguerre collocation method for $N = 12$ of Eq. (6.1)

x_i	Bessel [19] $e_{1,6}(x_i)$	DTM [15] $e_{1,6}(x_i)$	Laguerre $e_{1,6}(x_i)$	Laguerre $e_{1,10}(x_i)$	Bessel [19] $e_{1,10}(x_i)$	Laguerre $e_{1,12}(x_i)$
0	0	0	0	0	0	0
0.2	2.8460e-008	1.5575e-002	2.8460e-08	3.0309e-14	2.6645e-014	0
0.4	1.7820e-008	5.1209e-002	1.7820e-08	2.5646e-14	2.0650e-014	0
0.6	1.2668e-008	1.0150e-001	1.2668e-08	2.0761e-14	3.1530e-014	1.1102e-16
0.8	3.3538e-008	1.6630e-001	3.3537e-08	1.2101e-14	7.1498e-014	0
1	6.8657e-007	2.3351e-001	6.8658e-07	1.7061e-12	1.5510e-012	1.4988e-15
x_i	$e_{2,6}(x_i)$	$e_{2,6}(x_i)$	$e_{2,6}(x_i)$	$e_{2,10}(x_i)$	$e_{2,10}(x_i)$	$e_{2,12}(x_i)$
0	0	0	0	0	0	0
0.2	2.8460e-008	2.3262e-003	2.8460e-08	3.0309e-14	2.9421e-014	0
0.4	1.7820e-008	1.6867e-002	1.7820e-08	2.5646e-14	3.6082e-014	0
0.6	1.2668e-008	5.4499e-021	1.2668e-08	2.0761e-14	7.4274e-014	1.1102e-16
0.8	3.3538e-008	1.3194e-001	3.3537e-08	1.2101e-14	1.6065e-013	0
1	6.8657e-007	2.8038e-001	6.8658e-07	1.7061e-12	1.3884e-012	1.4988e-15

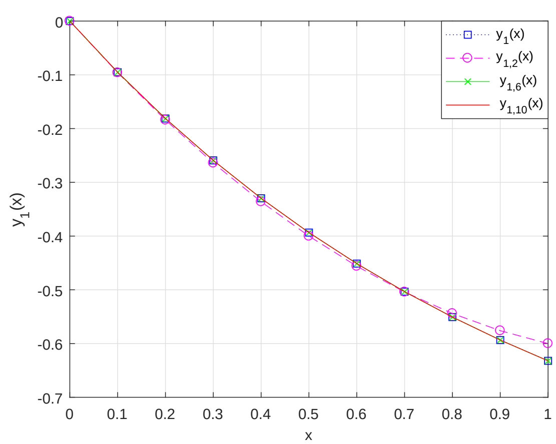

FIGURE 1. Comparison of $y_1(x)$ exact solution and approximate solutions for $N = 2, 6, 10$ of Eq. (6.1)

Table 3. Comparison of actual, estimated and improved absolute errors for $N = 3, 5, 8$ and $M = 4, 5, 6, 7, 9, 10$ of Eq. (6.1)

	Absolute absolute errors	Estimated absolute errors	Improved absolute errors
x_i	$\lvert e_{1,3}(x_i)\rvert$	$\lvert e_{1,3,5}(x_i)\rvert$	$\lvert E_{1,3,4}(x_i)\rvert$
0	0	0	1.1020e-39
0.2	2.4427e-04	2.4354e-04	1.5250e-05
0.4	2.6599e-04	2.6570e-04	7.0814e-06
0.6	5.3229e-05	5.3772e-05	7.2843e-06
0.8	2.4788e-04	2.4818e-04	1.3280e-05
1	3.0146e-03	3.0012e-03	2.0759e-04
	$\lvert e_{1,5}(x_i)\rvert$	$\lvert e_{1,5,7}(x_i)\rvert$	$\lvert E_{1,5,6}(x_i)\rvert$
0	0	4.3714e-38	0
0.2	7.2618e-07	7.2519e-07	2.8460e-08
0.4	2.9591e-07	2.9511e-07	1.7820e-08
0.6	5.4324e-07	5.4271e-07	1.2668e-08
0.8	3.0087e-07	3.0095e-07	3.3537e-08
1	1.3335e-05	1.3302e-05	6.8658e-07
	$\lvert e_{1,8}(x_i)\rvert$	$\lvert e_{1,8,10}(x_i)\rvert$	$\lvert E_{1,8,9}(x_i)\rvert$
0	0	9.1101e-38	0
0.2	3.2050e-11	3.2020e-11	1.0056e-12
0.4	2.7746e-11	2.7720e-11	8.5747e-13
0.6	2.3806e-11	2.3785e-11	7.1331e-13
0.8	2.0218e-11	2.0206e-11	7.1421e-13
1	1.3378e-09	1.3361e-09	5.1067e-11
x_i	$\lvert e_{2,3}(x_i)\rvert$	$\lvert e_{2,3,5}(x_i)\rvert$	$\lvert E_{2,3,4}(x_i)\rvert$
0	0	0	1.1020e-39
0.2	2.4427e-04	2.4354e-04	1.5250e-05
0.4	2.6599e-04	2.6570e-04	7.0814e-06
0.6	5.3229e-05	5.3772e-05	7.2843e-06
0.8	2.4788e-04	2.4818e-04	1.3280e-05
1	3.0146e-03	3.0012e-03	2.0759e-04
	$\lvert e_{2,5}(x_i)\rvert$	$\lvert e_{2,5,7}(x_i)\rvert$	$\lvert E_{2,5,6}(x_i)\rvert$
0	0	4.3714e-38	0
0.2	7.2618e-07	7.2519e-07	2.8460e-08
0.4	2.9591e-07	2.9511e-07	1.7820e-08
0.6	5.4324e-07	5.4271e-07	1.2668e-08
0.8	3.0087e-07	3.0095e-07	3.3537e-08
1	1.3335e-05	1.3302e-05	6.8658e-07
	$\lvert e_{2,8}(x_i)\rvert$	$\lvert e_{2,8,10}(x_i)\rvert$	$\lvert E_{2,8,9}(x_i)\rvert$
0	0	9.1101e-38	0
0.2	3.2050e-11	3.2020e-11	1.0056e-12
0.4	2.7746e-11	2.7720e-11	8.5747e-13
0.6	2.3806e-11	2.3785e-11	7.1331e-13
0.8	2.0218e-11	2.0206e-11	7.1421e-13
1	1.3378e-09	1.3361e-09	5.1067e-11

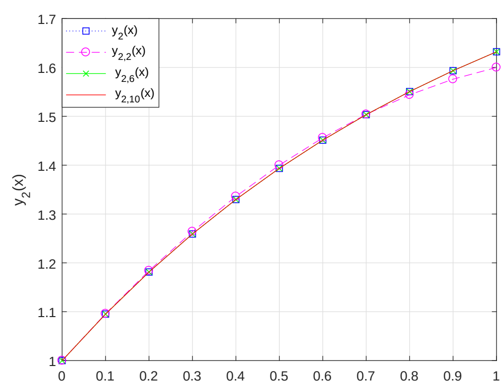

Figure 2. Comparison of $y_2(x)$ exact solution and approximate solutions for $N = 2, 6, 10$ of Eq. (6.1)

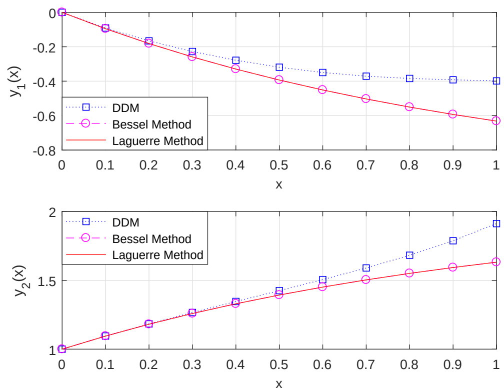

FIGURE 3. Comparison numerical results of approximate solutions with other methods for $N = 6$ of Eq. (6.1)

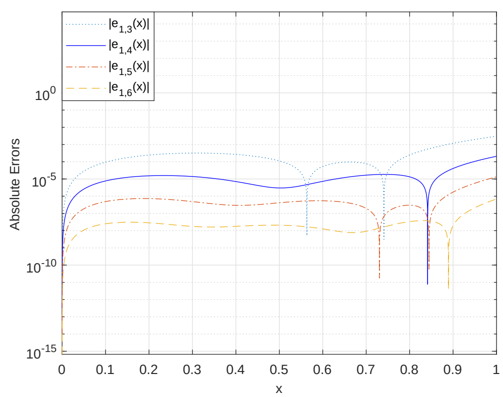

FIGURE 4. Comparison numerical results of actual absolute errors of $y_1(x)$ solution of Eq. (6.1)

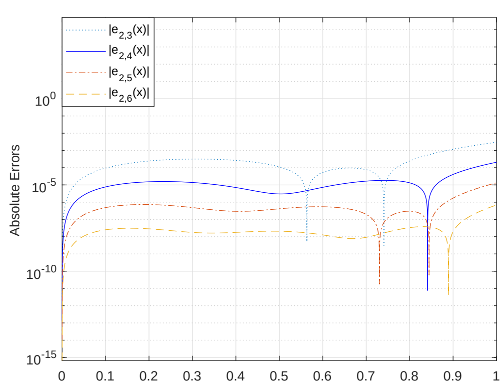

FIGURE 5. Comparison numerical results of actual absolute errors of $y_2(x)$ solution of Eq. (6.1)

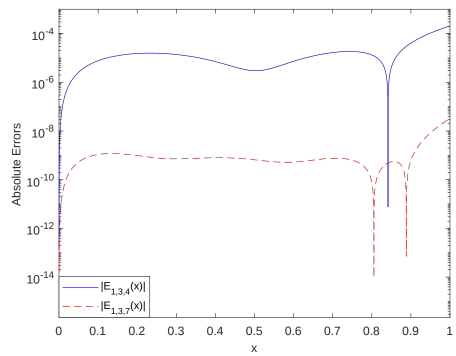

FIGURE 6. Comparison numerical results of improved absolute errors for $N = 3$, $M = 4,7$ of $y_1(x)$ solution of Eq. (6.1)

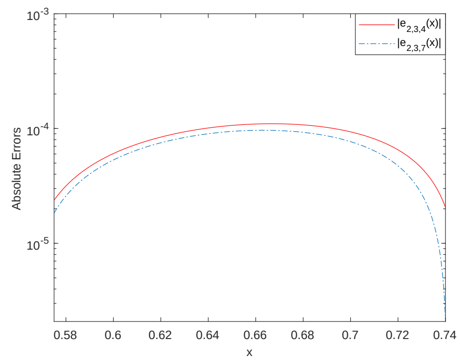

FIGURE 7. Comparison numerical results of estimated absolute errors for $N = 3$, $M = 4,7$ of $y_2(x)$ solution of Eq. (6.1)

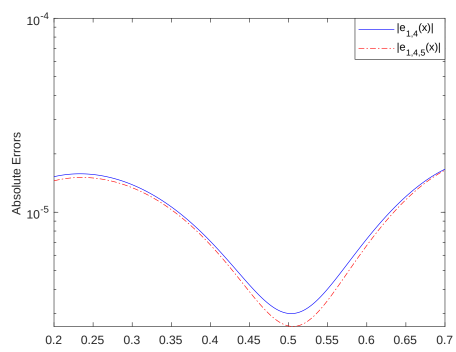

FIGURE 8. Comparison numerical results of actual and estimated absolute errors for $N = 4$, $M = 5$ of $y_1(x)$ solution of Eq. (6.1)

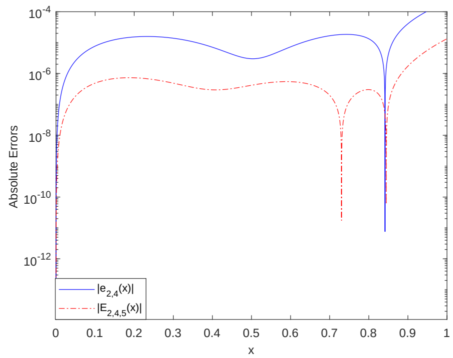

FIGURE 9. Comparison numerical results of actual and improved absolute errors for $N = 4$, $M = 5$ of $y_2(x)$ solution of Eq. (6.1)

Example 6.2. Firstly, let's consider the system of first-order variable-coefficient linear differential equations

$$\begin{cases} y_1'(x) - y_3(x) = -cos(x) \\ y_2'(x) - y_3(x) = -e^x \\ y_3'(x) - y_1(x) + y_2(x) = 0 \end{cases} , 0 \le x \le 1 \tag{6.8}$$

with the conditions

$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 2 \tag{6.9}$$

and the exact solutions

$$y_1(x) = e^x, \quad y_2(x) = sin(x) \quad ve \quad y_3(x) = e^x + cos(x).$$

Here;

$k = 3$, $m = 1$, $g_1(x) = -cos(x)$, $g_2(x) = -e^x$, $g_3(x) = 0$, $P^0_{1,1}(x) = 0$, $P^0_{1,2}(x) = 0$, $P^0_{1,3}(x) = -1$, $P^0_{2,1}(x) = 0$, $P^0_{2,2}(x) = 0$, $P^0_{2,3}(x) = -1$, $P^0_{3,1}(x) = -1$, $P^0_{3,2}(x) = 1$, $P^0_{3,3}(x) = 0$, $P^1_{1,1}(x) = 1$, $P^1_{1,2}(x) = 0$, $P^1_{1,3}(x) = 0$, $P^1_{2,1}(x) = 0$, $P^1_{2,2}(x) = 1$, $P^1_{2,3}(x) = 0$, $P^1_{3,1}(x) = 0$, $P^1_{3,2}(x) = 0$, $P^1_{3,3}(x) = 1$.

Now, let's find solutions truncated Laguerre series expansions in the form

$$y_{i,N}(x) = \sum_{n=0}^{5} a_{i,n} L_n(x), \quad i = 1, 2, 3$$

by using the Laguerre polynomials for $N = 5$. Here, the set of the collocation points for $N = 5$ is computed as

$$\{x_0 = 0, x_1 = \frac{1}{5}, x_2 = \frac{2}{5}, x_3 = \frac{3}{5}, x_4 = \frac{4}{5}, x_5 = 1\}$$

and thus the system of fundamental matrix equation of the problem is written as

$$\left\{\mathbf{P}_0\mathbf{X}\bar{\mathbf{D}} + \mathbf{P}_1\mathbf{X}\bar{\mathbf{B}}\bar{\mathbf{D}}\right\}\mathbf{A} = \mathbf{G}.$$

The augmented matrix for this system of fundamental matrix equations is calculated from Matlab. The matrix form of the conditions is also calculated. Then, instead of the last three lines of the augmented matrix, the matrix form of the conditions is written thus the new augmented matrix $[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}]$ is obtained. Hence, this system is solved and the Laguerre coefficient matrix $\mathbf{A}$ is obtained. Finally, the matrix $\mathbf{A}$ is substituted into the (2.1)

$$y_{j,N}(x) = \mathbf{L}(x)\mathbf{A}_j, \quad j = 1, 2, 3$$

and thus we obtain approximate solutions as

$$y_{1,5}(x) = 0.0125314366248x^5 + 0.0377210999073x^4 + 0.168225455271x^3 + 0.499768584831x^2 + x + 1,$$

$$y_{2,5}(x) = 0.00764092737182x^5 + 0.000782824607297x^4 - 0.16698894934x^3 + 0.0000453276257103x^2 + x - 6.10622663544e - 16$$

and

$$y_{3,5}(x) = 0.0093438587612x^5 + 0.0820919229142x^4 + 0.167226486969x^3 - 0.0000938977776579x^2 + x + 2.$$

Thus, for $N = 6$ the actual error functions are obtained as

$$\begin{aligned} e_{1,5}(x) &= y_1(x) - y_{1,5}(x) \\ &= e^x - (0.0125314366248x^5 + 0.0377210999073x^4 + 0.168225455271x^3 \\ &\quad + 0.499768584831x^2 + x + 1), \end{aligned}$$

$$\begin{aligned} e_{2,5}(x) &= y_2(x) - y_{2,5}(x) \\ &= sin(x) - (0.00764092737182x^5 + 0.000782824607297x^4 - 0.16698894934x^3 \\ &\quad + 0.0000453276257103x^2 + x - 6.10622663544e - 16) \end{aligned}$$

ve

$$\begin{aligned} e_{3,5}(x) &= y_3(x) - y_{3,5}(x) \\ &= e^x + cos(x) - (0.0093438587612x^5 + 0.0820919229142x^4 \\ &\quad + 0.167226486969x^3 - 0.0000938977776579x^2 + x + 2). \end{aligned}$$

The exact solutions, the approximate solutions and the actual absolute errors of Eq. (6.8) for $N = 5$ are shown in Table 4. In Table 6, the actual absolute errors of Eq. (6.8) are compared with other method for $N = 6$. The results for Laguerre method are better than for other method. The actual absolute errors and the estimated absolute errors of Eq. (6.8) for $N = 5$ and $M = 7$ are compared in Table 5.

In Figures (10)-(11)-(12), the exact solution and approximate solutions are compared for $N = 3, 5, 10$. In Figures (13)-(14)-(15), actual absolute errors are compared for $N = 3, 4, 5, 6, 7, 8$. From figures it can be seen that as the N value increases, the errors become decrease. In Figures (16)-(17)-(18)-(19), absolute errors are compared for various N and M. From figures, it is seen that the method is effective.

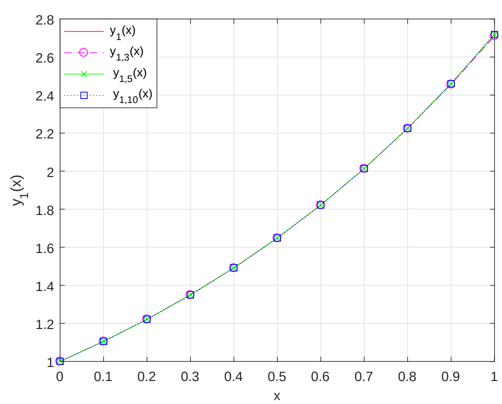

FIGURE 10. Comparison numerical results of the $y_1(x)$ exact solutions and approximate solutions for $N = 3, 5, 10$ of Eq. (6.8)

Table 4. Comparison numerical results of the exact solutions, approximate solutions and actual absolute errors for $N = 5$ of Eq. (6.8)

	Exact solutions	Approximate solutions	Absolute errors
x_i	$y_1(x_i) = e^x$	$N = 5, \quad y_{1,5}(x_i)$	$N = 5, \quad e_{1,5}(x_i)$
0	1	1	0
0.2	1.221402758160170	1.221400910854970	1.847305199862804e-06
0.4	1.491824697641270	1.491823384778939	1.312862331769250e-06
0.6	1.822118800390509	1.822116487937574	2.312452935093101e-06
0.8	2.225540928492467	2.225540191065770	7.374266971753343e-07
1	2.718281828459045	2.718246576634042	3.525182500358197e-05
x_i	$y_2(x_i) = sin(x)$	$N = 5, \quad y_{2,5}(x_i)$	$N = 5, \quad e_{2,5}(x_i)$
0	0	-6.10622663543836e-16	6.106226635438361e-16
0.2	0.198669330795061	0.198669599126442	2.683313805533364e-07
0.4	0.389418342308650	0.389418243068613	9.924003740269377e-08
0.6	0.564642473395035	0.56464231746944	1.559255952202472e-07
0.8	0.717356090899523	0.717355091658926	9.992405972177166e-07
1	0.841470984807897	0.841480130265226	9.145457329101270e-06
x_i	$y_3(x_i) = e^x + cos(x)$	$N = 5, \quad y_{3,5}(x_i)$	$N = 5, \quad e_{3,5}(x_i)$
0	2	2	0
0.2	2.20146933600141	2.20146839309611	9.429053016268426e-07
0.4	2.41288569164416	2.41288470586189	9.857822643174806e-07
0.6	2.64745441530019	2.64745280965224	1.605647951360405e-06
0.8	2.92224763783963	2.9222465140148	1.123824833127563e-06
1	3.25858413432718	3.25856837086643	1.576346075300203e-05

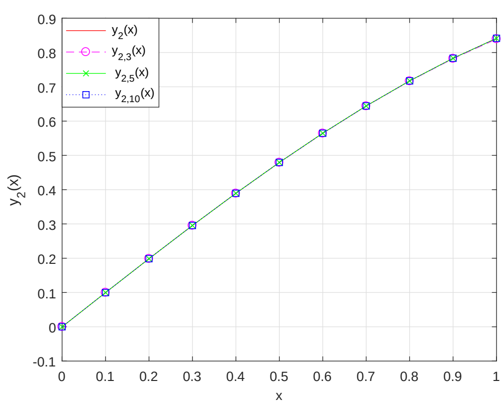

Figure 11. Comparison numerical results of the $y_2(x)$ exact solutions and approximate solutions for $N = 3, 5, 10$ of Eq. (6.8)

Table 5. Comparison numerical results of the actual absolute errors and estimated absolute errors for $N = 5$ and $M = 7$ of Eq. (6.8)

	Actual absolute errors	Estimated absolute errors	Improved absolute errors
x_i	$\lvert e_{1,5}(x_i)\rvert$	$\lvert e_{1,5,7}(x_i)\rvert$	$\lvert E_{1,5,7}(x_i)\rvert$
0	7.4246e-16	2.3882e-32	7.4246e-16
0.2	1.8473e-06	1.8438e-06	3.4814e-09
0.4	1.3129e-06	1.3086e-06	4.2826e-09
0.6	2.3125e-06	2.3075e-06	4.9058e-09
0.8	7.3743e-07	7.3220e-07	5.2255e-09
1	3.5252e-05	3.5165e-05	8.7058e-08
	$\lvert e_{2,5}(x_i)\rvert$	$\lvert e_{2,5,7}(x_i)\rvert$	$\lvert E_{2,5,7}(x_i)\rvert$
0	6.1062e-16	8.0119e-32	6.1062e-16
0.2	2.6833e-07	2.6988e-07	1.5476e-09
0.4	9.9240e-08	9.6850e-08	2.3897e-09
0.6	1.5593e-07	1.5267e-07	3.2564e-09
0.8	9.9924e-07	9.9507e-07	4.1675e-09
1	9.1455e-06	9.1704e-06	2.4990e-08
	$\lvert e_{3,5}(x_i)\rvert$	$\lvert e_{3,5,7}(x_i)\rvert$	$\lvert E_{3,5,7}(x_i)\rvert$
0	6.9389e-18	8.3357e-34	6.9389e-18
0.2	9.4291e-07	9.3830e-07	4.6012e-09
0.4	9.8578e-07	9.8093e-07	4.8527e-09
0.6	1.6056e-06	1.6010e-06	4.6801e-09
0.8	1.1238e-06	1.1200e-06	3.8183e-09
1	1.5763e-05	1.5631e-05	1.3274e-07

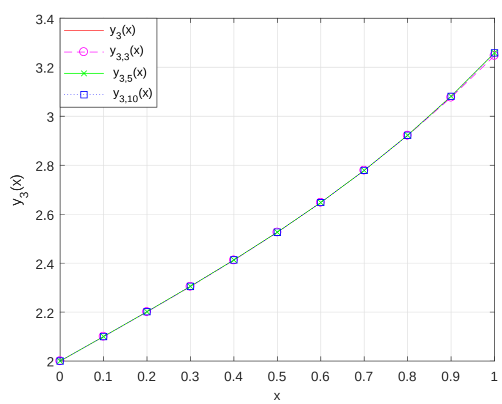

Figure 12. Comparison numerical results of the $y_3(x)$ exact solutions and approximate solutions for $N = 3, 5, 10$ of Eq. (6.8)

Table 6. Comparison numerical results of actual errors with other method for $N = 6$ Eq. (6.8)

x_i	Laguerre $\lvert e_{1,6}(x_i)\rvert$	DTM [15] $\lvert e_{1,6}(x_i)\rvert$	Laguerre $\lvert e_{2,6}(x_i)\rvert$	DTM [15] $\lvert e_{2,6}(x_i)\rvert$	Laguerre $\lvert e_{3,6}(x_i)\rvert$	DTM [15] $\lvert e_{3,6}(x_i)\rvert$
0	1.7319e-14	0	5.8398e-14	0	2.7756e-17	0
0.2	8.9691e-08	2.6046e-09	2.6784e-08	2.5383e-09	1.1003e-07	2.6681e-09
0.4	9.4300e-08	3.4209e-07	1.9940e-09	3.2436e-07	1.0917e-07	3.5831e-07
0.6	1.1665e-07	6.0004e-06	2.9723e-08	5.5266e-06	1.2560e-07	6.4153e-06
0.8	1.9907e-07	4.6173e-05	2.3020e-08	4.1242e-05	2.1407e-07	5.0305e-05
1	1.5209e-06	2.2627e-04	1.1272e-06	1.9568e-04	2.0229e-06	2.5080e-04

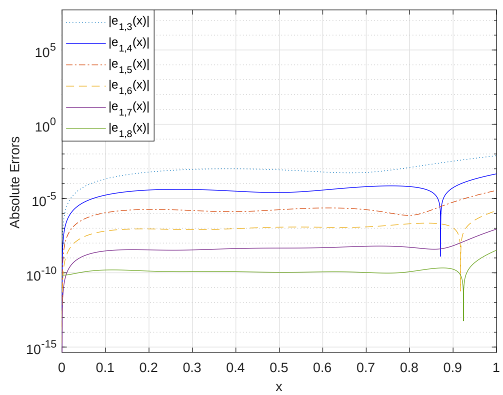

Figure 13. Comparison numerical results of actual absolute errors of $y_1(x)$ solution for $N = 3, 4, 5, 6, 7, 8$ of Eq. (6.8)

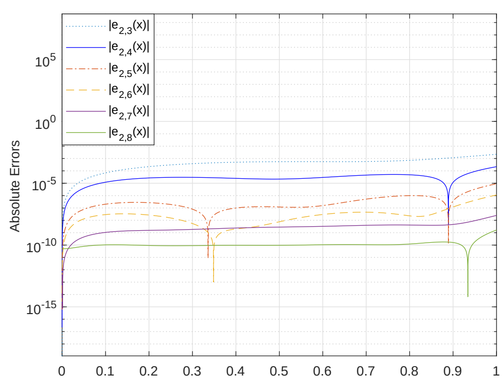

Figure 14. Comparison numerical results of actual absolute errors of $y_2(x)$ solution for $N = 3, 4, 5, 6, 7, 8$ of Eq. (6.8)

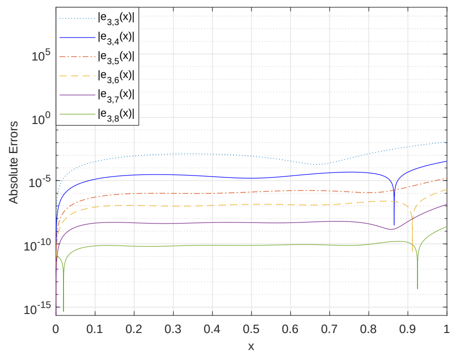

FIGURE 15. Comparison numerical results of actual absolute errors of $y_3(x)$ solution for $N = 3, 4, 5, 6, 7, 8$ of Eq. (6.8)

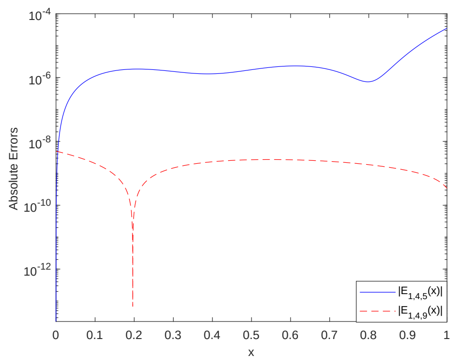

FIGURE 16. Comparison numerical results of improved absolute errors for $N = 4$ and $M = 5, 9$ of $y_1(x)$ solution of Eq. (6.8)

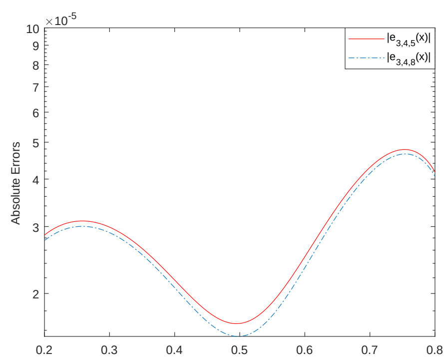

FIGURE 17. Comparison numerical results of estimated absolute errors for $N = 4$ and $M = 5, 8$ of $y_3(x)$ solution of Eq. (6.8)

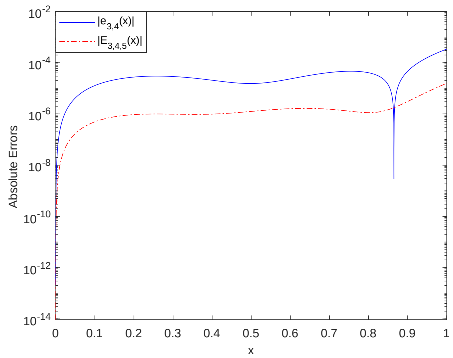

FIGURE 18. Comparison numerical results of actual and improved absolute errors for $N = 4$ and $M = 5$ of $y_3(x)$ solution of Eq. (6.8)

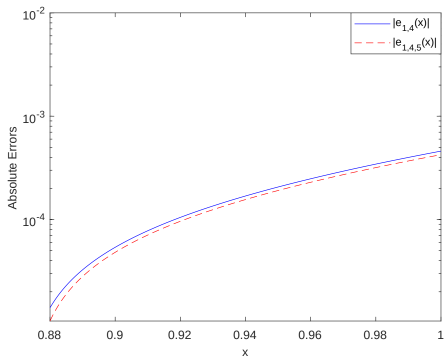

FIGURE 19. Comparison numerical results of actual and estimated absolute errors for $N = 4$ and $M = 5$ of $y_1(x)$ solution of Eq. (6.8)

7. CONCLUSIONS

In this work, systems of first order differential equations are numerically solved by using Laguerre polynomials and Laguerre polynomial solutions are improved by using residual error estimation technique. Numerical examples are made for the method and the results are shown in tables and graphs. In addition, numerical results are compared with other methods. It has been shown that very good results are obtained by using Laguerre collocation method. It seems from tables and graphs that the errors reduced as N is increased. The numerical results of the method are calculated by writing codes in matlab program and thus the numerical results are obtained in a short time.

Acknowledgement: The authors is supported by the Scientific Research Project Administration of Akdeniz University (project code FYL-2018-3110). The author would like to thank the editor and anonymous referees for their valuable comments which helped improve this manuscript.

REFERENCES

[1] Abdel, I.H., Hassan, H., *Application to differential transformation method for solving systems of differential equations*, Appl. Math. Modell., **32**(2008), 2552–2559. 1

[2] Akyüz Daşcıoğlu, A., Sezer, M., *Chebyshev polynomial solutions of systems of higher-order linear Fredholm-Volterra integro-differential equations*, Journal of the Franklin Inst., **342**(2005), 688–701. 1

[3] Arikoglu, A., Ozkol, I., *Solutions of integral and integro-differential equation systems by using differential transform method*, Comput Math Appl., **56**(2008), 2411–2417. 1

[4] Biazar, J., Babolian, E., Islam, R., *Solution of the system of ordinary differential equations by Adomian decomposition method*, Applied Mathematics and Computation, **147**(2004), 713–719. 1

[5] Gülsu, M., Sezer, M., *Taylor collocation method for solution of systems of high-order linear Fredholm Volterra integro-differential equations*, Intern. J. Computer Math., **83**(2006), 429–448. 1
[6] Javidi, M., *Modified homotopy perturbation method for solving system of linear Fredholm integral equations*, Mathematical and Computer Modelling, **50**(2009), 159–165. 1
[7] Kurt, N., Sezer, M., *Polynomial solution of high-order linear Fredholm integro-differential equations with constant coefficients*, Journal of the Frankin Institute., **345**(2008), 839–850. 1
[8] Maleknejad, K., Mirzaee, F., *Numerical solution of linear Fredholm integral equations system by rationalized Haar functions method*, Int. J. Comput. Math., **80**(2003), 1397–1405. 1
[9] Maleknejad, K., Mirzaee, F., Abbasbandy, S., *Solving linear integro-differential equations system by using rationalized Haar functions method*, Appl. Math. Comput., **155**(2004), 317–328. 1
[10] Pour-Mahmoud, J., Rahimi-Ardabili, M.Y., Shahmorad, S., *Numerical solution of the system of Fredholm integro-differential equations by the Tau method*, Appl. Math. Comput., **168**(2005), 465–478. 1
[11] Ramadan, M.A., Abd El Salam, M.A., *Solving systems of ordinary differential equations in unbounded domains by exponential Chebyshev collocation method*, Journal of Abstract and Computational Mathematics, **1**(2016), 33–46. 1
[12] Saberi-Nadjafi, J., Tamamgar, M., *The variational iteration method: A highly promising method for solving the system of integro differential equations*, Comput. Math. Appl., **56**(2008), 346–351. 1
[13] Sezer, M., Karamete, A., G´ulsu, M., *Taylor polynomial solutions of systems of linear differential equations with variable coefficients*, Int. J. Comput. Math., **82**(2005), 755–764. 1
[14] Tatari, M., Dehghan, M., *Improvement of He's variational iteration method for solving systems of differential equations*, Comput. Math. Appl., **58**(2009), 2160–2166. 1
[15] Thongmoon, M., Pusjuso, S., *The numerical solutions of differential transform method and the Laplace transform method for a system of differential equations*, Nonlinear Analysis: Hybrid Systems, **4**(2009), 425–431. 1, 1, 2, 6
[16] Yalçınbaş, S., Sezer, M., Sorkun, H.H., *Legendre polynomial solutions of high-order linear Fredholm integro-differential equations*, Appl. Math. Comput., **210**(2009), 334–349. 1
[17] Yusufoğlu, E., *An efficient algorithm for solving integro-differential equations system*, Appl. Math. Comput., **192**(2007), 51–55. 1
[18] Yüksel, G., Gülsu, M., Sezer, M., *A Chebyshev polynomial approach for high-order linear Fredholm-Volterra integro-differential equations*, Gazi University Journal of Science, **25**(2012), 393–401. 1
[19] Yüzbaşı, Ş., Şahin, N., Sezer, M., *A numerical approach for solving linear differential equation systems*, Journal of Advanced Research in Differential Equations, **3**(2011), 8–32. 1, 1, 2
[20] Zarali, B., Rabbani, M., *Solution of Fredholm integro-differential equations system by modified decomposition method*, The Journal of Mathematics and Computer Science, **4**(2012), 258–264. 1