Proceedings of International Conference on Mathematics
and Mathematics Education (ICMME 2018)
Turk. J. Math. Comput. Sci.
10(2018) 242–248

http://dergipark.gov.tr/tjmcs

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Ricci Solitons On Ricci Pseudosymmetric a Normal Paracontact Metric Manifold

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Received: 14-08-2018 • Accepted: 10-12-2018

Abstract. The object of the present paper is to study some types of Ricci pseudosymmetric a normal paracontact metric manifolds whose metric is Ricci soliton such as concircilar Ricci pseudosymmetric, projective Ricci pseudosymmetric. Finally, we constructed an example of concircular Ricci pseudosymmetric a normal paracontact metric manifold whose metric is Ricci soliton.

2010 AMS Classification: 53C15, 53C44, 53D10.

Keywords: Normal paracontact metric manifold, Ricci pseudosymmetric manifold, Ricci soliton.

1. Introduction

Let M be an n-dimensional manifold, ϕ, ξ and η be a tensor field of type (1,1), a vector field, 1-form on M, respectively. If ϕ, ξ and η satisfy the conditions

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1,$$

for any vector field X on M, then M is called an almost contact manifold. Furthermore, M is said to be almost contact metric manifold if M is of a Riemannian metric tensor such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi),$$

for any $X, Y \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the set of the differentiable vector fields on M. On the other hand, we say that M is called a normal paracontact metric manifold if

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{1.1}$$

and

$$\nabla_X \xi = \phi X,$$

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where ∇ denote the Levi-Civita connection on M [1].
A normal paracontact metric manifold M is said to be have a constant sectional curvature-c if Riemannian curvature tensor R is of the form

$$\begin{aligned} R(X,Y)Z &= \frac{1}{4}(c+3)\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{1}{4}(c-1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &- 2g(\phi X,Y)\phi Z\}, \end{aligned}$$

for all $X,Y,Z \in \Gamma(TM)$.
The concircular curvature tensor, Projective curvature tensor, conformal curvature tensor, Quasi-conformal curvature tensor and Pseudo-projective curvature tensor of a normal paracontact metric manifold M^n are, respectively, defined by

$$\widetilde{Z}(X,Y)Z = R(X,Y)Z - \frac{\tau}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\} \tag{1.2}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\} \tag{1.3}$$

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\} \\ &+ \frac{\tau}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned}$$

$$\begin{aligned} \widetilde{C}(X,Y)Z &= \lambda R(X,Y)Z + \mu\{S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\} \\ &- \frac{\tau}{n}(\frac{\lambda}{n-1} + 2\mu)(g(Y,Z)X - g(X,Z)Y) \end{aligned}$$

$$\begin{aligned} \widetilde{C}(X,Y)Z &= \lambda R(X,Y)Z + \mu\{S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\} \\ &- \frac{\tau}{n}(\frac{\lambda}{n-1} + 2\mu)(g(Y,Z)X - g(X,Z)Y) \end{aligned}$$

and

$$\begin{aligned} \widetilde{P}(X,Y)Z &= \lambda R(X,Y)Z + \mu\{S(Y,Z)X - S(X,Z)Y\} \\ &- \frac{\tau}{n}(\frac{\lambda}{n-1} + \mu)(g(Y,Z)X - g(X,Z)Y), \end{aligned} \tag{1.4}$$

for all $X,Y,Z \in \Gamma(TM)$.
Also, in a normal paracontact metric manifold M^n, the following relations are satisfied:

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X \tag{1.5}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi \tag{1.6}$$

$$S(X,\xi) = -(n-1)\eta(X), \quad Q\xi = -(n-1)\xi.$$

2. Preliminaries

In [2], Hamilton introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows;

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci f low. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of the diffeomorphizms and scaling. In precisely, a Ricci soliton on a Riemannian manifold (M, g) is a triple (g, V, λ) satisfying

$$\ell_V g + 2S + 2\lambda g = 0, \tag{2.1}$$

where S denotes the Ricci tensor of M, ℓ is the Lie-derivative along the vector field V on M and $\lambda \in \mathbb{R}$. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$, respectively.

Recently, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [5], Sharma studied the metric manifolds has been studied by various authors.

On the other hand, the notion of Ricci pseudosymmetric manifold was introduced by Deszcs. A geometrical interpretation of Ricci pseudosymmetric manifolds in the Riemannian case was given [4].

A Riemannian manifold (M^n, g) is called Ricci pseudosymmetric if the tensor $R.S$ and the Tachibana tensor $Q(g, S)$ are linearly dependent, that is,

$$(R(X, Y)S)(Z, U) = L_s Q(g, S)(Z, U, X, Y),$$

where

$$(R(X, Y).S)(Z, U) = -S((X, Y)Z, U) - S(Z, R(X, Y)U),$$

$$Q(g, S)(Z, U, X, Y) = -S((X\Lambda_g Y)Z, U) - S(Z, (X\Lambda_g Y)U),$$

and

$$(X\Lambda_g Y)Z = g(Y, Z)X - g(X, Z)Y,$$

for any $X, Y, Z, U \in \Gamma(TM)$, where R is the Riemannian curvature tensor, S is the Ricci tensor of M. Then (M^n, g) is the Ricci pseudosymmetric if and only if

$$(R(X, Y)S)(Z, U) = L_s Q(g, S)(Z, U; X, Y),$$

holds on $U_s = \{x \in M : S = \frac{\tau}{n} g \;\; at \;\; x\}$ for some function L_s on U_s. If $R.S = 0$, then M is called Ricci semi symmetric.

In each of the cases, we will find the value of L_S and hence it turns out that the condition that a Ricci soliton is shrinking, steady, or expanding depends on L_S being less that equal, or greater than certain value. We call it the Critical Value for L_S [3].

In each type of Ricci pseudosymmetric, concircular Ricci pseudosymmetric and projective Ricci pseudosymmetric for the critical value L_S is obtained and they are interpreted. Finally, we constructed a non-trivial example which is expanding. Let (g, ξ, λ) be a Ricci soliton on a normal paracontact metric manifold M^n. From (1.1) and (2.1), we have

$$\begin{aligned} (L_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= g(\phi X, Y) + g(X, \phi Y) = 2g(\phi X, Y). \end{aligned}$$

From (1.6), we arrive at

$$S(X, Y) = -\lambda g(X, Y) - g(\phi X, Y) \tag{2.2}$$

$$S(X, \xi) = -\lambda \eta(X)$$

$$\tau = -\lambda n - tr(\phi). \tag{2.3}$$

A normal paracontact metric manifold M^n is said to be concircular Ricci pseudosymmetric if its concircular tensor $\widetilde{Z}$ satisfies

$$(\widetilde{Z}(X,Y).S)(Z,U) = L_s Q(g,S)(Z,U;X,Y),$$

on $U_s = \{x \in M : S = \frac{\tau}{n} g \;\; at \;\; x\}$. Now, taking into account concircular Ricci pseudosymmetric a normal paracontact metric manifold whose metric is Ricci soliton, then we have

$$\begin{aligned} S(\widetilde{Z}(X,Y)Z,U) \quad &+ \quad S(Z,\widetilde{Z}(X,Y)U) = L_s\{g(Y,Z)S(X,U) \\ &- \quad g(X,Z)S(Y,U) + g(Y,U)S(X,Z) - g(X,U)S(Y,Z)\}, \end{aligned} \tag{2.4}$$

for any $X,Y,Z,U \in \Gamma(TM)$. Next, let $Z = \xi$ in (2.4), we obtain

$$\begin{aligned} S(\widetilde{Z}(X,Y)\xi,U) \quad &+ \quad S(\xi,\widetilde{Z}(X,Y)U) = L_s\{\eta(Y)S(X,U) \\ &- \quad \eta(X)S(Y,U) + g(Y,U)S(X,\xi) - g(X,U)S(Y,\xi)\}. \end{aligned}$$

From (1.2) and (1.5), we know

$$\widetilde{Z}(X,Y)\xi = (1 + \frac{\tau}{n(n-1)})(\eta(X)Y - \eta(Y)X).$$

Thus we have

$$\begin{aligned} (1 + \frac{\tau}{n(n-1)})S(\eta(X)Y \quad &- \quad \eta(Y)X,U) - \lambda\eta(\widetilde{Z}(X,Y)U) = L_s\{\eta(Y)S(X,U) \\ &- \quad \eta(X)S(Y,U) - g(Y,U)\lambda\eta(X) + g(X,U)\lambda\eta(Y)\}, \end{aligned}$$

that is,

$$\begin{aligned} (1 \quad &+ \quad \frac{\tau}{n(n-1)})(\eta(X)S(Y,U) - \eta(Y)S(X,U) - \lambda\eta(Y)g(X,U) + \lambda\eta(X)g(Y,U)) \\ &= \quad L_s\{\eta(Y)S(X,U) - \eta(X)S(Y,U) - \lambda\eta(X)g(Y,U) + \lambda\eta(Y)g(X,U)\}. \end{aligned}$$

Making use of (2.2), we have

$$(1 + \frac{\tau}{n(n-1)})(\eta(Y)g(\phi X,U) - \eta(X)g(\phi Y,U)) = L_s\{\eta(X)g(\phi Y,U) - \eta(Y)g(\phi X,U)\}.$$

Since $\tau = -\lambda n - g(\phi e_i, e_i) = -\lambda n - tr(\phi)$,

$$L_s = (\frac{\lambda n + tr(\phi)}{n(n-1)} - 1). \tag{2.5}$$

Thus we have the following theorem.

Theorem 2.1. *If (g,ξ,λ) is a Ricci soliton on a concircular Ricci pseudosymmetric normal paracontact metric manifold M^n, then $L_s = (\frac{\lambda n + tr(\phi)}{n(n-1)} - 1)$.*

From (2.5), we get

$$\lambda = (n-1)(1 + L_s) - \frac{tr(\phi)}{n}.$$

Corollary 2.2. *In a concircular Ricci pseudosymmetric normal paracontact metric manifold M^n, the Critical value for L_s is $\frac{tr(\phi)}{n(n-1)} - 1$.*

Let us take a Ricci pseudosymmetric normal paracontact metric manifold M^n whose metric is Ricci soliton. Then we have

$$\begin{aligned} S(R(X,Y)Z,U) \quad &+ \quad S(Z,R(X,Y)Z) = L_s\{g(Y,Z)S(X,U) \\ &- \quad g(Z,X)S(Y,U) + g(U,Y)S(X,Z) - g(X,U)S(Y,Z)\}, \end{aligned}$$

for all $X,Y,Z,U \in \Gamma(TM)$. For $Z = \xi$, we reach at

$$\begin{aligned} (1 + \frac{\tau}{n(n-1)})S(\eta(Y)X \quad &- \quad \eta(X)Y) - \lambda\eta(R(X,Y)U) = L_s\{\eta(Y)S(X,U) \\ &- \quad \eta(X)S(U,Y) + g(Y,U)S(X,\xi) - g(X,U)S(Y,\xi)\}, \end{aligned}$$

that is,

$$\begin{aligned}
& (1+\frac{\tau}{n(n-1)})(\eta(Y)S(X,U)-\eta(X)S(Y,U)-\lambda\eta(X)g(Y,U)\\
+ & \lambda\eta(Y)g(X,U)) = L_S\{\eta(Y)S(X,U)-\eta(X)S(Y,U)\\
- & \lambda\eta(X)g(Y,U)+\lambda\eta(Y)g(X,U)\},
\end{aligned}$$

or,

$$\begin{aligned}
& (1+\frac{\tau}{n(n-1)})(\eta(Y)S(X,U)-\eta(X)S(Y,U)-\lambda\eta(X)g(Y,U)\\
+ & \lambda\eta(Y)g(X,U)) = L_S\{\eta(Y)S(X,U)-\eta(X)S(Y,U)-\lambda\eta(X)g(Y,U)\\
+ & \lambda\eta(Y)g(X,U)\},
\end{aligned}$$

which implies that

$$L_S = 1-\frac{\lambda n+tr(\phi)}{n(n-1)}.$$

Since $n > 1$, we obtain

$$\lambda = (n-1)(1-L_S)-\frac{tr(\phi)}{n}.$$

Theorem 2.3. *If (g,ξ,λ) is a Ricci soliton on a Ricci pseudosymmetric normal paracontact metric manifold M^n, then* $L_S = 1-\frac{\lambda n+tr(\phi)}{n(n-1)}$

Next, we will study of Ricci solitons on projective Ricci pseudosymmetric normal paracontact metric manifolds. The projective curvature tensor is an important concept of Riemannian geometry which one uses to calculate the basic geometric measurements on a manifold. In a normal paracontact metric manifold, the projective curvature tensor can be calculated from (1.3) and (1.5), we have

$$P(X,Y)\xi = (1-\frac{\lambda}{n-1})(\eta(X)X-\eta(Y)X)$$

and

$$\begin{aligned}
\eta(P(X,Y)Z) &= g(R(X,Y)Z,\xi)-\frac{1}{n-1}(\eta(X)S(Y,Z)-\eta(Y)S(X,Z))\\
&= \eta(Y)g(X,Z)-\eta(X)g(Y,Z)-\frac{1}{n-1}(\eta(X)[-\lambda g(Y,Z)\\
&- g(\phi Y,Z)]-\eta(Y)[-\lambda g(X,Z)-g(\phi X,Z)])\\
&= (1-\frac{\lambda}{n-1})(\eta(Y)g(X,Z)-\eta(X)g(Y,Z))\\
&+ \frac{1}{n-1}(\eta(Y)g(\phi X,Z)-\eta(X)g(\phi Y,Z)).
\end{aligned}$$

Let M^n be a Ricci soliton on a projective Ricci pseudosymmetric normal paracontact metric manifold. Then we have

$$\begin{aligned}
S(P(X,Y)Z,U) &+ S(Z,P(X,Y)U) = L_S\{g(Y,Z)S(X,U)\\
&- g(X,Z)S(Y,U)+g(Y,U)S(X,Z)-g(X,U)S(Y,Z)\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
- & \lambda g(P(X,Y)Z,U)-g(P(X,Y)Z,\phi U)-\lambda g(P(X,Y)U,Z)\\
- & g(P(X,Y)U,\phi Z) = L_S\{g(Y,Z)S(X,U)-g(X,Z)S(Y,U)\\
+ & g(Y,U)S(X,Z)-g(X,U)S(Y,Z)\}.
\end{aligned} \tag{2.6}$$

For $Z=\xi$ in (2.6), we have

$$\begin{aligned}
(1-\frac{\lambda}{n-1})(\eta(X)g(\phi Y,U)-\eta(Y)g(X,\phi U)) &= L_S\{\eta(X)g(Y,\phi U)\\
&- \eta(Y)g(X,\phi U)\}.
\end{aligned} \tag{2.7}$$

Thus from (2.7), we have the following theorem.

Theorem 2.4. *If (g, ξ, λ) is a Ricci soliton on a projective Ricci pseudosymmetric normal paracontact manifold M^n, then $L_S = 1 - \frac{\lambda}{n-1}$.*

Example 2.5. Let us consider a 7-dimensional manifold
$M^7 = \{(x_1, x_2, x_3, y_1, y_2, y_3, z) \in \mathbb{R}^7\}$, where $(x_1, x_2, x_3, y_1, y_2, y_3, z)$ are standard coordinates in $\mathbb{R}^7$. Taking the vector fields

$$e_i = e^z \frac{\partial}{\partial x_i}, \quad e_j = e^z \frac{\partial}{\partial y_j}, \quad 1 \leq i, j \leq 3, \quad e_7 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Let g be the Riemannian metric on M defined by

$$g = e^{-2z} \sum_{i=1}^{3} \{dxi \otimes dxi + dy_i \otimes dy_i\} + dz \otimes dz.$$

We note that $g(e_i, e_j) = \delta_{ij}$. Thus the set $\{e_i\}$, $1 \leq i, j \leq 7$, is an orthonormal basis of M. let

$$X = \sum_{i=1}^{3} (X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + Z \frac{\partial}{\partial z}$$

be a vector field on M. We define the almost paracontact structure ϕ and 1-form η as

$$\phi X = \sum_{i=1}^{3} (-X_i \frac{\partial}{\partial x_i} - Y_i \frac{\partial}{\partial y_i}) \ \ and \ \ \eta(X) = g(X, e_7).$$

Thus we have

$$\phi e_i = -e_i, \phi e_7 = 0, \quad 1 \leq i \leq 6.$$

It is easy to see that

$$\phi^2 X = X - \eta(X) e_7, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad and \quad \eta(e_7) = 1,$$

for any $X, Y \in \Gamma(TM)$. Thus $(\phi, \xi = e_7, \eta, g)$ is an almost paracontact metric structure on M. By direct calculations, we have

$$[e_i, e_7] = -e_i, \quad 1 \leq i \leq 6, \quad [e_i, e_j] = 0, \quad 1 \leq i, j \leq 6.$$

By using Kozsul formula, we can easily to find that

$$\begin{aligned} \nabla_{e_i} e_i &= e_7, \ \ \nabla_{e_i} e_j = 0, \ \ i \neq j, \ \ 1 \leq i, j \leq 6 \\ \nabla_{e_i} e_7 &= = \phi e_i = -e_i, \ \ \nabla_{e_7} e_7 = 0, \ \ \nabla_{e_7} e_i = 0, \ \ 1 \leq i \leq 6. \end{aligned}$$

Using the Kozsul's formula, we get

$$(\nabla_X \phi) Y = -g(X, Y)\xi - \eta(Y) X + 2\eta(X)\eta(Y)\xi$$

for any $X, Y \in \Gamma(TM)$. Thus $M^7(\phi, \xi, \eta, g)$ is a normal paracontact metric manifold. By R we denote the Riemannian curvature tensor of M, it can be easily too seen that

$$\begin{aligned} R(e_i, e_j) e_j &= -e_i, \ \ 1 \leq i \neq j \leq 7 \\ R(e_i, e_j) e_k &= 0, \ \ 1 \leq i, j, k \leq 6, \ \ i \neq j \neq k. \end{aligned}$$

Let $X = X_i e_i$, $Y = Y_j e_j$ and $Z = Z_k e_k$, $1 \leq i, j, k \leq n$, be vector fields on M. By using the properties of R, we get

$$\begin{aligned} R(X, Y) Z &= X_i Y_j Z_k R(e_i, e_j) e_k = Y_j Z_j X_i R(e_i, e_j) e_j \\ &+ X_i Y_j Z_i R(e_i, e_j) e_i \\ &= -Y_j Z_j X_i e_i + X_i Z_i Y_j e_j = -\{g(Y, Z) X - g(X, Z) Y\}, \end{aligned}$$

that is, M has a constant curvature -1 and

$$S(X, Y) = -(n-1) g(X, Y) = -6 g(X, Y), \quad \tau = -42.$$

It can be easily to seen that $tr\phi = -6$. From (2.3), we conclude that $\lambda = \frac{48}{7}$, that is, normal paracontact metric structure is expanding

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