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Q-Soft Normal Subgroups

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Abstaract — This paper contains some definitions and results in Q-soft normal subgroup theory and cosets. Also some results are introduced which have been used by homomorphism and anti-homomorphism of Q-soft normal subgroups. Next we prove the analogue of the Lagrange's theorem.

Keywords — *Q-soft subsets, group theory, Q-soft subgroups, Q-soft normal subgroups, homomorphism.*

1 Introduction

In mathematics and abstract algebra, group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra: other well-known algebraic structures, such as rings, fields, and vector spaces, can all be seen as groups endowed with additional operations and axioms. Groups recur throughout mathematics, and the methods of group theory have influenced many parts of algebra. Linear algebraic groups and Lie groups are two branches of group theory that have experienced advances and have become subject areas in their own right. Various physical systems, such as crystals and the hydrogen atom, may be modelled by symmetry groups. Thus group theory and the closely related representation theory have many important applications in physics, chemistry, and materials science. Group theory is also central to public key cryptography. Soft set theory is a generalization of fuzzy set theory, that was proposed by Molodtsov in 1999 to deal with uncertainty in a parametric manner [10]. A soft set is a parameterised family of sets - intuitively, this is "soft" because the boundary of the set depends on the parameters. Formally, a soft set, over a universal set X and set of parameters E is a pair (f, A) where A is a subset of E and f is a function from A to the power set of X. For each e in A, the set $f(e)$ is called the value set of e in (f, A). One of the most important steps for the new theory of soft sets was to define mappings on soft sets, which was achieved in 2009 by the mathematicians Athar Kharal and Bashir Ahmad, with the results published in 2011 [7]. Soft sets have also been applied to the problem of medical diagnosis for use in medical expert systems. In abstract algebra, a normal subgroup is a subgroup which is invariant under conjugation by members of the group of which it is a part. In other words, a subgroup H of a group G is normal in G if and only if $gH = Hg$ for all g in G. The definition of normal subgroup implies that the sets of left and right cosets coincide. In fact, a seemingly weaker condition that the sets of left and right cosets coincide also implies that the subgroup H of a group G is normal in G [6]. Normal subgroups (and only normal subgroups) can be used to construct quotient groups from a given group. Then Maji et al. [8] introduced several operations on soft sets. The works of the algebraic structure of soft sets was first started by Aktas and Cagman [1]. They presented the notion of the soft group and derived its some basic properties. For basic notions and the applications of soft sets, we incite to read [1, 2, 3, 4, 8, 9, 10, 11]. A. Solairaju and R. Nagarajan [14] introduced the new structures of Q-fuzzy groups. The author investigated soft Lie ideals and anti soft Lie ideals and extension of Q-soft ideals in semigroups [13, 12]. In [5] the author introduced the concept of Q-soft subgroups and discussed the characterisations Q-soft subgroups under homomorphism and anti-homomorphism. The purpose of this paper is to deal with the algebraic structure of Q-soft normal subgroups. The concept of Q-soft normal subgroups is introduced, their characterization and algebraic properties are investigated. The rest of this paper is organized as follows. In Section 2, we summarize some basic concepts which will be used throughout the paper. In Section 3, we introduce the concept of Q-soft normal subgroups and investigate some of their basic properties. Also we investigate Q-soft normal subgroups under homomorphism and anti-homomorphisms. Next we prove the analogue of the Lagrange's theorem.

2 Preliminary

In this section, we present basic definitions of soft sets and their operations. Throughout this work, Q is a non-empty set, U refers to an initial universe set, E is a set of parameters and $P(U)$ is the power set of U.

Definition 2.1. ([8, 10]) For any subset A of E, a Q-soft subset $f_{A\times Q}$ over U is a set, defined by a function $f_{A\times Q}$, representing a mapping $f_{A\times Q} : E \times Q \to P(U)$, such that $f_{A\times Q}(x, q) = \emptyset$ if $x \notin A$. A soft set over U can also be represented by the set of ordered pairs $f_{A\times Q} = \{((x, q), f_{A\times Q}(x, q)) \mid (x, q) \in E \times Q, f_{A\times Q}(x, q) \in P(U)\}$. Note that the set of all Q-soft subsets over U will be denoted by $QS(U)$. From here on, soft set will be used without over U.

Definition 2.2. ([8, 10]) Let $f_{A\times Q}, f_{B\times Q} \in QS(U)$. Then,

(1) $f_{A\times Q}$ is called an empty Q-soft subset, denoted by $\Phi_{A\times Q}$, if $f_{A\times Q}(x, q) = \emptyset$ for all $(x, q) \in E \times Q$,

(2) $f_{A\times Q}$ is called a $A \times Q$-universal soft set, denoted by $f_{A\tilde{\times}Q}$, if $f_{A\times Q}(x, q) = U$ for all $(x, q) \in A \times Q$,

(3) $f_{A\times Q}$ is called a universal Q-soft subset, denoted by $f_{E\tilde{\times}Q}$, if $f_{A\times Q}(x, q) = U$ for all $(x, q) \in E \times Q$,

(4) the set $Im(f_{A\times Q}) = \{f_{A\times Q}(x,q) : (x,q) \in A\times Q\}$ is called image of $f_{A\times Q}$ and if $A\times Q = E\times Q$, then $Im(f_{E\times Q})$ is called image of $E\times Q$ under $f_{A\times Q}$.

(5) $f_{A\times Q}$ is a Q-soft subset of $f_{B\times Q}$, denoted by $f_{A\times Q}\tilde{\subseteq} f_{B\times Q}$, if $f_{A\times Q}(x,q) \subseteq f_{B\times Q}(x,q)$ for all $(x,q)\in E\times Q$,

(6) $f_{A\times Q}$ and $f_{B\times Q}$ are soft equal, denoted by $f_{A\times Q} = f_{B\times Q}$, if and only if $f_{A\times Q}(x,q) = f_{B\times Q}(x,q)$ for all $(x,q)\in E\times Q$,

(7) the set $(f_{A\times Q}\tilde{\cup} f_{B\times Q})(x,q) = f_{A\times Q}(x,q)\cup f_{B\times Q}(x,q)$ for all $(x,q)\in E\times Q$ is called union of $f_{A\times Q}$ and $f_{B\times Q}$,

(8) the set $(f_{A\times Q}\tilde{\cap} f_{B\times Q})(x,q) = f_{A\times Q}(x,q)\cap f_{B\times Q}(x,q)$ for all $(x,q)\in E\times Q$ is called intersection of $f_{A\times Q}$ and $f_{B\times Q}$.

Example 2.3. Let $U = \{u_1,u_2,u_3,u_4,u_5\}$ be an initial universe set and $E = \{x_1,x_2,x_3,x_4,x_5\}$ be a set of parameters. Let $Q=\{q\}$, $A=\{x_1,x_2\}$, $B=\{x_2,x_3\}$, $C=\{x_4\}$, $D=\{x_5\}$, $F=\{x_1,x_2,x_3\}$. Define

$$f_{A\times Q}(x,q) = \begin{cases} \{u_1,u_2,u_3\} & \text{if } x=x_1 \\ \{u_1,u_5\} & \text{if } x=x_2 \end{cases}$$

$$f_{B\times Q}(x,q) = \begin{cases} \{u_1,u_2\} & \text{if } x=x_2 \\ \{u_2,u_4\} & \text{if } x=x_3 \end{cases}$$

$$f_{F\times Q}(x,q) = \begin{cases} \{u_1,u_2,u_3,u_4\} & \text{if } x=x_1 \\ \{u_1,u_2,u_5\} & \text{if } x=x_2 \\ \{u_2,u_4\} & \text{if } x=x_3 \end{cases}$$

$f_{C\times Q}(x_4,q) = U$ and $f_{D\times Q}(x_5,q)=\{\emptyset\}$. Then we will have

$$(f_{A\times Q}\tilde{\cup} f_{B\times Q})(x,q) = \begin{cases} \{u_1,u_2,u_3\} & \text{if } x=x_1 \\ \{u_1,u_2,u_5\} & \text{if } x=x_2 \\ \{u_2,u_4\} & \text{if } x=x_3 \end{cases}$$

$$(f_{A\times Q}\tilde{\cap} f_{B\times Q})(x,q) = \begin{cases} \{u_1\} & \text{if } x=x_2 \\ \{\} & \text{if } x\neq x_2 \end{cases}$$

Also $f_{C\times Q} = f_{C\tilde{\times} Q}$ and $f_{D\times Q} = \Phi_{D\times Q}$. Note that the difinition of classical subset is not valid for the soft subset. For example $f_{A\times Q}\tilde{\subseteq} f_{F\times Q}$ does not imply that every element of $f_{A\times Q}$ is an element of $f_{F\times Q}$. Thus $f_{A\times Q}\tilde{\subseteq} f_{F\times Q}$ but $f_{A\times Q}\nsubseteq f_{F\times Q}$ as classical subset.

Definition 2.4. ([5]) Let $\varphi : A\to B$ be a function and $f_{A\times Q}, f_{B\times Q}\in QS(U)$. Then soft image $\varphi(f_{A\times Q})$ of $f_{A\times Q}$ under φ is defined by

$$\varphi(f_{A\times Q})(y,q) = \begin{cases} \cup\{f_{A\times Q}(x,q) \mid (x,q)\in A\times Q, \varphi(x)=y\} & \text{if } \varphi^{-1}(y)\neq\emptyset \\ \emptyset & \text{if } \varphi^{-1}(y)=\emptyset \end{cases}$$

and soft pre-image (or soft inverse image) of $f_{B\times Q}$ under φ is $\varphi^{-1}(f_{B\times Q})(x,q) = f_{B\times Q}(\varphi(x),q)$ for all $(x,q)\in A\times Q$.

Definition 2.5. ([5]) Let $(G,.)$ be a group and $f_{G\times Q}\in QS(U)$. Then, $f_{G\times Q}$ is called a Q-soft subgroup over U if $f_{G\times Q}(xy,q)\supseteq f_{G\times Q}(x,q)\cap f_{G\times Q}(y,q)$ and $f_G(x^{-1},q) = f_{G\times Q}(x,q)$ for all $x,y\in G, q\in Q$.

Throughout this paper, G denotes an arbitrary group with identity element e_G and the set of all Q-soft subgroup with parameter set G over U will be denoted by $S_{G\times Q}(U)$.

Definition 2.6. ([5]) Let $(G,.),(H,.)$ be any two groups and $f_{G\times Q}\in S_{G\times Q}(U), g_{H\times Q}\in S_{H\times Q}(U)$. The product of $f_{G\times Q}$ and $g_{H\times Q}$, denoted by $f_{G\times Q}\tilde{\times} g_{H\times Q}:(G\times H)\times Q\rightarrow P(U)$, is defined as $f_{G\times Q}\tilde{\times} g_{H\times Q}((x,y),q)=f_{G\times Q}(x,q)\cap g_{H\times Q}(y,q)$ for all $x\in G, y\in H, q\in Q$. Throughout this paper, H denotes an arbitrary group with identity element e_H.

Theorem 2.7. (Lagrange) ([6]) Let G be a finite group. Let H be a subgroup of G. Then the order of H divides the order of G.

Definition 2.8. ([6]) Let $(G,.),(H,.)$ be any two groups. The function $f:G\rightarrow H$ is called a homomorphism (anti-homomorphism) if $f(xy)=f(x)f(y)(f(xy)=f(y)f(x))$, for all $x,y\in G$.

Definition 2.9. ([6]) We call a group G, Hamiltonian if G is non-abelian and every subgroup of G is normal.

Definition 2.10. ([6]) A Dedekind group is one which is abelian or Hamiltonian.

3 Main Results

Definition 3.1. Let $f_{G\times Q}\in S_{G\times Q}(U)$ then $f_{G\times Q}$ is said to be a Q-soft normal subgroup of G if $f_{G\times Q}(xy,q)=f_{G\times Q}(yx,q)$, for all $x,y\in G$ and $q\in Q$. Throughout this paper, G denotes an arbitrary group with identity element e_G and the set of all Q-soft normal subgroup with parameter set G over U will be denoted by $NS_{G\times Q}(U)$.

Example 3.2. Let $U=\{u_1,u_2,u_3,u_4,u_5\}$ be an initial universe set and $(\mathbb{Z},+)$ be an additive group. Define $f_{\mathbb{Z}\times Q}:\mathbb{Z}\times Q\rightarrow P(U)$ as

$$f_{\mathbb{Z}\times Q}(x,q)=\begin{cases}\{u_1,u_2,u_3\} & \text{if } x\in\mathbb{Z}^{\geq 0}\\ \{u_2,u_4,u_5\} & \text{if } x\in\mathbb{Z}^{<0}\end{cases}$$

then $f_{\mathbb{Z}\times Q}\in NS_{\mathbb{Z}\times Q}(U)$.

Proposition 3.3. Let $f_{G\times Q}, g_{G\times Q}\in NS_{G\times Q}(U)$. Then $f_{G\times Q}\tilde{\cap} g_{G\times Q}\in NS_{G\times Q}(U)$.

Proof. By [5, Proposition 2.16] we have that $f_{G\times Q}\tilde{\cap} g_{G\times Q}\in S_{G\times Q}(U)$. Let $x,y\in G, q\in Q$. Then

$$\begin{aligned}(f_{G\times Q}\tilde{\cap} g_{G\times Q})(xy,q) &= f_{G\times Q}(xy,q)\cap g_{G\times Q}(xy,q)=f_{G\times Q}(yx,q)\cap g_{G\times Q}(yx,q)\\ &=(f_{G\times Q}\tilde{\cap} g_{G\times Q})(yx,q)\end{aligned}$$

and so $f_{G\times Q}\tilde{\cap} g_{G\times Q}\in NS_{G\times Q}(U)$. □

Corollary 3.4. The intersection of a family of Q-soft normal subgroups of a group G is a Q-soft subgroup of a group G.

Proposition 3.5. Let $f_{G\times Q} \in NS_{G\times Q}(U)$. Then $f_{G\times Q}(yxy^{-1}, q) = f_{G\times Q}(y^{-1}xy, q)$ for every $x, y \in G$ and $q \in Q$.

Proof. Let $x, y \in G$ and $q \in Q$. As $f_{G\times Q} \in NS_{G\times Q}(U)$ so

$$\begin{aligned} f_{G\times Q}(yxy^{-1}, q) &= f_{G\times Q}(y^{-1}yx, q) = f_{G\times Q}(ex, q) = f_{G\times Q}(x, q) = f_{G\times Q}(xyy^{-1}, q) \\ &= f_{G\times Q}(y^{-1}xy, q). \end{aligned}$$ □

Proposition 3.6. If every Q-soft subgroup of a group G is normal, then G is a Dedekind group.

Proof. Suppose that every Q-soft subgroup of a group G is normal. We have, consider a subgroup H of G. So H can be regarded as a Q-level subgroup of some Q-soft subgroup $f_{G\times Q}$ of G. By assumption, $f_{G\times Q}$ is a Q-soft normal subgroup of G. Now, it is easy to deduce that H is a normal subgroup of G. Thus G is a Dedekind group □

Proposition 3.7. If $f_{G\times Q} \in NS_{G\times Q}(U), g_{H\times Q} \in NS_{H\times Q}(U)$. Then $f_{G\times Q}\tilde{\times} g_{H\times Q} \in NS_{(G\times H)\times Q}(U)$.

Proof. From [5, Proposition 2.22] we have that $f_{G\times Q}\tilde{\times} g_{H\times Q} \in S_{(G\times H)\times Q}(U)$. Let $(x_1, y_1), (x_2, y_2) \in G \times H, q \in Q$. Then

$$\begin{aligned} f_{G\times Q}\tilde{\times} g_{H\times Q}((x_1, y_1)(x_2, y_2), q) &= f_{G\times Q}\tilde{\times} g_{H\times Q}((x_1x_2, y_1y_2), q) \\ &= f_{G\times Q}(x_1x_2, q) \cap g_{H\times Q}(y_1y_2, q) \\ &= f_{G\times Q}(x_2x_1, q) \cap g_{H\times Q}(y_2y_1, q) \\ &= f_{G\times Q}\tilde{\times} g_{H\times Q}((x_2x_1, y_2y_1), q) \\ &= f_{G\times Q}\tilde{\times} g_{H\times Q}((x_2, y_2)(x_1, y_1), q). \end{aligned}$$

Thus $f_{G\times Q}\tilde{\times} g_{H\times Q} \in NS_{(G\times H)\times Q}(U)$. □

Proposition 3.8. Let $f_{G\times Q}, g_{H\times Q} \in QS(U), f_{G\times Q}\tilde{\times} g_{H\times Q} \in NS_{(G\times H)\times Q}(U)$. Then at least one of the following two statements must hold.
(1) $g_{H\times Q}(e_H, q) \supseteq f_{G\times Q}(x, q)$, for all $x \in G, q \in Q$,
(2) $f_{G\times Q}(e_G, q) \supseteq g_{H\times Q}(y, q)$, for all $y \in H, q \in Q$.

Proof. Use [5, Proposition 2.23]. □

Proposition 3.9. Let $f_{G\times Q}, g_{H\times Q} \in QS(U), f_{G\times Q}\tilde{\times} g_{H\times Q} \in NS_{(G\times H)\times Q}(U)$. Then we have the following statements.
(1) If for all $x \in G, q \in Q, f_{G\times Q}(x, q) \subseteq g_{H\times Q}(e_H, q)$, then $f_{G\times Q} \in NS_{G\times Q}(U)$.
(2) If for all $x \in H, q \in Q, g_{H\times Q}(x, q) \subseteq f_{G\times Q}(e_G, q)$, then $g_{H\times Q} \in NS_{H\times Q}(U)$.
(3) Either $f_{G\times Q} \in NS_{G\times Q}(U)$ or $g_{H\times Q} \in S_{H\times Q}(U)$.

Proof. (1) Let $x, y \in G, q \in Q$. From [5, Proposition 2.24] we have that $f_{G\times Q} \in S_{G\times Q}(U)$. As $f_{G\times Q}(x, q) \subseteq g_{H\times Q}(e_H, q)$ so

$$\begin{aligned} f_{G\times Q}(xy, q) &= f_{G\times Q}(xy, q) \cap g_{H\times Q}(e_He_H, q) \\ &= f_{G\times Q}\tilde{\times} g_{H\times Q}((xy, e_He_H), q) \\ &= f_{G\times Q}\tilde{\times} g_{H\times Q}((x, e_H)(y, e_H), q) \\ &= f_{G\times Q}\tilde{\times} g_{H\times Q}((y, e_H)(x, e_H), q) \end{aligned}$$

$$\begin{aligned}
&= f_{G\times Q}\tilde{\times} g_{H\times Q}((yx, e_He_H), q)\\
&= f_{G\times Q}(yx, q) \cap g_{H\times Q}(e_He_H, q)\\
&= f_{G\times Q}(yx, q).
\end{aligned}$$

Thus $f_{G\times Q} \in NS_{G\times Q}(U)$.

(2) Let $x, y \in H, q \in Q$. By [5, Proposition 2.24] we get that $g_{H\times Q} \in S_{H\times Q}(U)$. Since $g_{H\times Q}(x, q) \subseteq f_{G\times Q}(e_G, q)$ so

$$\begin{aligned}
g_{H\times Q}(xy, q) &= f_{G\times Q}(e_Ge_G, q) \cap g_{H\times Q}(xy, q)\\
&= f_{G\times Q}\tilde{\times} g_{H\times Q}((e_Ge_G, xy), q)\\
&= f_{G\times Q}\tilde{\times} g_{H\times Q}((e_G, x)(e_G, y), q)\\
&= f_{G\times Q}\tilde{\times} g_{H\times Q}((e_G, y)(e_G, x), q)\\
&= f_{G\times Q}\tilde{\times} g_{H\times Q}((e_Ge_G, yx), q)\\
&= f_{G\times Q}(e_Ge_G, q) \cap g_{H\times Q}(yx, q)\\
&= g_{H\times Q}(yx, q).
\end{aligned}$$

Therefore $g_{H\times Q} \in NS_{H\times Q}(U)$.

(3) Straight forward. □

Recall that $(x) = \{y^{-1}xy : y \in G\}$ is called the conjugate class of x in G.

Proposition 3.10. $f_{G\times Q} \in NS_{G\times Q}(U)$ if and only if $f_{G\times Q}$ is constant on the conjugate classes of G.

Proof. Let $x, y \in G$ and $q \in Q$. If $f_{G\times Q} \in NS_{G\times Q}(U)$, then

$$f_{G\times Q}(y^{-1}xy, q) = f_{G\times Q}(xyy^{-1}, q) = f_{G\times Q}(x, q)$$

Therefore $f_{G\times Q}$ is constant on the conjugate classes of G. Conversely, let $f_{G\times Q}$ is constant on the conjugate classes of G. Then

$$f_{G\times Q}(xy, q) = f_{G\times Q}(x^{-1}(xy)x, q) = f_{G\times Q}((x^{-1}x)yx, q) = f_{G\times Q}(yx, q)$$

and so $f_{G\times Q} \in NS_{G\times Q}(U)$. □

In the following propositions, we prove many results in homomorphism and anti-homomorphism in normal Q-soft subgroups.

Proposition 3.11. Let φ be an epimorphism from group G into group H. If $f_{G\times Q} \in NS_{G\times Q}(U)$, then $\varphi(f_{G\times Q}) \in NS_{H\times Q}(U)$

Proof. By [5, Proposition 4.3] we have that $\varphi(f_{G\times Q}) \in S_{H\times Q}(U)$. Let $h_1, h_2 \in H$ and $q \in Q$ then

$$\begin{aligned}
\varphi(f_{G\times Q})(h_1h_2, q) &= \cup\{f_{G\times Q}(g_1g_2, q) \mid g_1, g_2 \in G, \varphi(g_1g_2) = h_1h_2\}\\
&= \cup\{f_{G\times Q}(g_1g_2, q) \mid g_1, g_2 \in G, \varphi(g_1)\varphi(g_2) = h_1h_2\}\\
&= \cup\{f_{G\times Q}(g_1g_2, q) \mid g_1, g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\}\\
&= \cup\{f_{G\times Q}(g_2g_1, q) \mid g_2, g_1 \in G, \varphi(g_2) = h_2, \varphi(g_1) = h_1\}
\end{aligned}$$

$$
\begin{aligned}
&= \cup\{f_{G\times Q}(g_2g_1,q) \mid g_2,g_1 \in G, \varphi(g_2)\varphi(g_1) = h_2h_1\}\\
&= \cup\{f_{G\times Q}(g_2g_1,q) \mid g_2,g_1 \in G, \varphi(g_2g_1) = h_2h_1\}\\
&= \varphi(f_{G\times Q})(h_2h_1,q).
\end{aligned}
$$

Thus $\varphi(f_{G\times Q}) \in NS_{H\times Q}(U)$. □

Proposition 3.12. Let φ be a homorphism from group G into group H. If $g_{H\times Q} \in NS_{H\times Q}(U)$, then $\varphi^{-1}(g_{H\times Q}) \in NS_{G\times Q}(U)$.

Proof. By [5, Proposition 4.5] we have that $\varphi^{-1}(g_{H\times Q}) \in S_{G\times Q}(U)$. Let $g_1, g_2 \in G$ and $q \in Q$. Then

$$
\begin{aligned}
\varphi^{-1}(g_{H\times Q})(g_1g_2,q) &= g_{H\times Q}(\varphi(g_1g_2),q)\\
&= g_{H\times Q}(\varphi(g_1)\varphi(g_2),q)\\
&= g_{H\times Q}(\varphi(g_2)\varphi(g_1),q)\\
&= g_{H\times Q}(\varphi(g_2g_1),q)\\
&= \varphi^{-1}(g_{H\times Q})(g_2g_1,q).
\end{aligned}
$$

Therefore $\varphi^{-1}(g_{H\times Q}) \in NS_{G\times Q}(U)$. □

Proposition 3.13. Let φ be an anti-epimorphism from group G into group H. If $f_{G\times Q} \in NS_{G\times Q}(U)$, then $\varphi(f_{G\times Q}) \in NS_{H\times Q}(U)$.

Proof. By [5, Proposition 4.3] we have that $\varphi(f_{G\times Q}) \in S_{H\times Q}(U)$. Let $h_1, h_2 \in H$ and $q \in Q$ then

$$
\begin{aligned}
\varphi(f_{G\times Q})(h_1h_2,q) &= \cup\{f_{G\times Q}(g_1g_2,q) \mid g_1,g_2 \in G, \varphi(g_1g_2) = h_1h_2\}\\
&= \cup\{f_{G\times Q}(g_1g_2,q) \mid g_1,g_2 \in G, \varphi(g_2)\varphi(g_1) = h_1h_2\}\\
&= \cup\{f_{G\times Q}(g_1g_2,q) \mid g_1,g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\}\\
&= \cup\{f_{G\times Q}(g_2g_1,q) \mid g_2,g_1 \in G, \varphi(g_2) = h_2, \varphi(g_1) = h_1\}\\
&= \cup\{f_{G\times Q}(g_2g_1,q) \mid g_2,g_1 \in G, \varphi(g_2)\varphi(g_1) = h_2h_1\}\\
&= \cup\{f_{G\times Q}(g_2g_1,q) \mid g_2,g_1 \in G, \varphi(g_1g_2) = h_2h_1\}\\
&= \varphi(f_{G\times Q})(h_2h_1,q).
\end{aligned}
$$

Thus $\varphi(f_{G\times Q}) \in NS_{H\times Q}(U)$. □

Proposition 3.14. Let φ be an anti-homorphism from group G into group H. If $g_{H\times Q} \in NS_{H\times Q}(U)$, then $\varphi^{-1}(g_{H\times Q}) \in NS_{G\times Q}(U)$.

Proof. By [5, Proposition 4.8] we have that $\varphi^{-1}(g_{H\times Q}) \in S_{G\times Q}(U)$. Let $g_1, g_2 \in G$ and $q \in Q$. Then

$$
\begin{aligned}
\varphi^{-1}(g_{H\times Q})(g_1g_2,q) &= g_{H\times Q}(\varphi(g_1g_2),q)\\
&= g_{H\times Q}(\varphi(g_2)\varphi(g_1),q)\\
&= g_{H\times Q}(\varphi(g_1)\varphi(g_2),q)\\
&= g_{H\times Q}(\varphi(g_2g_1),q)\\
&= \varphi^{-1}(g_{H\times Q})(g_2g_1,q).
\end{aligned}
$$

Therefore $\varphi^{-1}(g_{H\times Q}) \in NS_{G\times Q}(U)$. □

Remark 3.15. In what follows the symbol $\circ$ stands for the composition operation of functions.

Proposition 3.16. Let φ be an isomorphism from group G into group H. If $f_{H\times Q} \in S_{H\times Q}(U)$, then we have the following:
(1) $f_{H\times Q} \circ \varphi \in S_{G\times Q}(U)$.
(2) If $f_{H\times Q} \in NS_{H\times Q}(U)$, then $f_{H\times Q} \circ \varphi \in NS_{G\times Q}(U)$.

Proof. (1) Let $x, y \in G$ and $q \in Q$.

$$\begin{aligned}
(f_{H\times Q} \circ \varphi)(xy^{-1}, q) &= f_{H\times Q}(\varphi(xy^{-1}), q) \\
&= f_{H\times Q}(\varphi(x)\varphi(y^{-1})), q) \\
&= f_{H\times Q}(\varphi(x)\varphi(y)^{-1}, q) \\
&\supseteq f_{H\times Q}(\varphi(x), q) \cap f_{H\times Q}(\varphi(y), q) \quad (as\ f_{H\times Q} \in S_{H\times Q}(U)) \\
&= (f_{H\times Q} \circ \varphi)(x, q) \cap (f_{H\times Q} \circ \varphi)(y, q)
\end{aligned}$$

and then $f_{H\times Q} \circ \varphi \in S_{G\times Q}(U)$.

(2) Let $f_{H\times Q} \in NS_{H\times Q}(U)$ then

$$\begin{aligned}
(f_{H\times Q} \circ \varphi)(xy, q) &= f_{H\times Q}(\varphi(x)\varphi(y), q) \\
&= f_{H\times Q}(\varphi(y)\varphi(x), q) \\
&= f_{H\times Q}(\varphi(yx), q) \\
&= (f_{H\times Q} \circ \varphi)(yx, q).
\end{aligned}$$

Therefore $f_{H\times Q} \circ \varphi \in NS_{G\times Q}(U)$. □

Proposition 3.17. Let φ be an anti-isomorphism from group G into group H. If $f_{H\times Q} \in S_{H\times Q}(U)$, then we have the following:
(1) $f_{H\times Q} \circ \varphi \in S_{G\times Q}(U)$.
(2) If $f_{H\times Q} \in NS_{H\times Q}(U)$, then $f_{H\times Q} \circ \varphi \in NS_{G\times Q}(U)$.

Proof. (1) Let $x, y \in G$ and $q \in Q$.

$$\begin{aligned}
(f_{H\times Q} \circ \varphi)(xy^{-1}, q) &= f_{H\times Q}(\varphi(xy^{-1}), q) \\
&= f_{H\times Q}(\varphi(y^{-1})\varphi(x)), q) \\
&= f_{H\times Q}(\varphi(y)^{-1}\varphi(x), q) \\
&\supseteq f_{H\times Q}(\varphi(x), q) \cap f_{H\times Q}(\varphi(y), q) \quad (as\ f_{H\times Q} \in S_{H\times Q}(U)) \\
&= (f_{H\times Q} \circ \varphi)(x, q) \cap (f_{H\times Q} \circ \varphi)(y, q)
\end{aligned}$$

and then $(f_{H\times Q} \circ \varphi \in S_{G\times Q}(U)$.

(2) Let $f_{H\times Q} \in NS_{H\times Q}(U)$ then

$$\begin{aligned}
f_{H\times Q} \circ \varphi)(xy, q) &= f_{H\times Q}(\varphi(y)\varphi(x), q) = f_{H\times Q}(\varphi(x)\varphi(y), q) \\
&= f_{H\times Q}(\varphi(yx), q) = (f_{H\times Q} \circ \varphi)(yx, q).
\end{aligned}$$

Therefore $f_{H\times Q} \circ \varphi \in NS_{G\times Q}(U)$. □

This motivated us to examine the results for Q-soft cosets. We have found out that the results perfectly fit with Q-soft cosets.

Definition 3.18. Let $f_{G\times Q} \in S_{G\times Q}(U)$ and $H = \{x \in G : f_{G\times Q}(x, q) = f_{G\times Q}(e, q)\}$, then $O(f_{G\times Q})$, the order of $f_{G\times Q}$ is defined as $O(f_{G\times Q}) = O(H)$.

Proposition 3.19. Let $f_{G\times Q}$ be a Q-soft subgroup of a finite group G, then $O(f_{G\times Q}) \mid O(G)$.

Proof. Let $f_{G\times Q}$ be a Q-soft subgroup of a finite group G with e as its identity element. Clearly $H = \{x \in G : f_{G\times Q}(x, q) = f_{G\times Q}(e, q)\}$ is a subgroup of G for H is a Q-level subset of G. By Lagranges theorem $O(H) \mid O(G)$. Hence by the definition of the order of the Q-soft subgroup of G, we have $O(f_{G\times Q}) \mid O(G)$. □

Proposition 3.20. Let $f_{G\times Q}$ and $g_{G\times Q}$ be two Q-soft subgroups of normal group G. Then $O(f_{G\times Q}) = O(g_{G\times Q})$.

Proof. Let $f_{G\times Q}$ and $g_{G\times Q}$ be conjugate Q-soft subgroups of G. Now

$$\begin{aligned} O(f_{G\times Q}) &= \textit{order of } \{x \in G : f_{G\times Q}(x, q) = f_{G\times Q}(e, q)\} \\ &= \textit{order of } \{x \in G : g_{G\times Q}(y^{-1}xy, q) = g_{G\times Q}((y^{-1}ey, q)\} \\ &= \textit{order of } \{x \in G : g_{G\times Q}(x, q) = g_{G\times Q}((e, q)\} = O(g_{G\times Q}). \end{aligned}$$

Hence $O(f_{G\times Q}) = O(g_{G\times Q})$. □

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