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A Quantitative Estimate for the Sampling Kantorovich Series in Terms of the Modulus of Continuity in Orlicz Spaces

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ABSTRACT. In the present paper we establish a quantitative estimate for the sampling Kantorovich operators with respect to the modulus of continuity in Orlicz spaces defined in terms of the modular functional. At the end of the paper, concrete examples are discussed, both for what concerns the kernels of the above operators, as well as for some concrete instances of Orlicz spaces.

Keywords: Sampling Kantorovich series, Orlicz spaces, Modulus of continuity, Quantitative estimates, Kernels.

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1. INTRODUCTION

The sampling Kantorovich operators S_w have been introduced by Bardaro, Butzer, Stens and Vinti in [8], in order to study an L^1-version of the so-called generalized sampling operators ([12, 32, 14]). The main peculiarity of the sampling Kantorovich operators is that they revealed to be suitable in order to reconstruct not necessarily continuous signals ([2]).

Indeed, in the original paper [8] the authors proved the modular convergence of the operators S_w in the general setting of Orlicz spaces, which include, as a special case, the L^p-spaces.

Later on, the operators S_w have been studied under different aspects, both from theoretical ([17, 5, 23]) and applications point of view ([6, 7]). For instance, in [6, 7] some applications to energy engineering have been developed applying an algorithm for image reconstruction and enhancement based on the multivariate version of the operators S_w for the processing of thermographic images.

The order of approximation for the sampling Kantorovich operators has been also studied in [21]; this has been done assuming the function f in suitable Lipschitz classes, both in the space of uniformly continuous and bounded functions (i.e., in $C(\mathbb{R})$) and in Orlicz spaces (i.e., in $L^\varphi(\mathbb{R})$). For other results concerning the order of approximation for the above operators, see, e.g., [31, 11].

The above problem has been faced in $C(\mathbb{R})$ also from the quantitative point of view in [9], by using the modulus of continuity of the function being approximated.

Currently, the study of quantitative estimates in the setting of Orlicz spaces in terms of the modulus of continuity is still an open problem.

For the latter reason, in this paper we establish the quantitative rate of convergence for the sampling Kantorovich operators; in order to do this we firstly recall the notion of the modulus of continuity in $L^\varphi(\mathbb{R})$ which is based on the modular functional of the space ([10]).

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At the end of the paper, several examples of kernels and concrete cases of Orlicz spaces are recalled. For instance, the L^p-spaces, with $1 \leq p < +\infty$, are included in the present general theory, together with other well-known examples of Orlicz spaces.

2. Notation and preliminaries

We begin this section by recalling some basic facts concerning Orlicz spaces.

A function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be a φ-function if it satisfies the following conditions:
$(\Phi 1)$ φ is a non decreasing and continuous function;
$(\Phi 2)$ $\varphi(0) = 0$, $\varphi(u) > 0$ if $u > 0$ and $\lim_{u \to +\infty} \varphi(u) = +\infty$.

Let us now consider the functional I^φ associated to the φ-function φ and defined by

$$I^\varphi[f] := \int_{\mathbb{R}} \varphi(|f(x)|)\, dx,$$

for every $f \in M(\mathbb{R})$, i.e., for every (Lebesgue) measurable function $f : \mathbb{R} \to \mathbb{R}$. As it is well-known, I^φ is a modular functional (see e.g. [29, 10]), and the Orlicz space generated by φ is defined by

$$L^\varphi(\mathbb{R}) := \{ f \in M(\mathbb{R}) \ : \ I^\varphi[\lambda f] < \infty, \text{ for some } \lambda > 0 \}.$$

A notion of convergence in Orlicz spaces, called *modular convergence*, was introduced in [30].

We will say that a net of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R})$ is modularly convergent to $f \in L^\varphi(\mathbb{R})$, if there exists $\lambda > 0$ such that

$$I^\varphi[\lambda(f_w - f)] = \int_{\mathbb{R}} \varphi(\lambda |f_w(x) - f(x)|)\, dx \longrightarrow 0, \qquad w \to +\infty. \tag{2.1}$$

Moreover we recall, for the sake of completeness, that in $L^\varphi(\mathbb{R})$ it can be also given a strong notion of convergence, i.e. the Luxemburg-norm convergence, see e.g. [29, 10]. We will say that a net of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R})$ is convergent to $f \in L^\varphi(\mathbb{R})$ with respect to the Luxemburg norm if (2.1) holds for every $\lambda > 0$. Definition (2.1) induces a topology in $L^\varphi(\mathbb{R})$, called *modular topology*. Obviously, the modular convergence and the Luxemburg norm convergence coincide if and only if the well-known Δ_2-condition on φ is satisfied, see, e.g., [29, 10].

Now, we recall the definition of the modulus of continuity in Orlicz spaces $L^\varphi(\mathbb{R})$, with respect to the modular I^φ. For any fixed $f \in L^\varphi(\mathbb{R})$, and for a suitable $\lambda > 0$, we denote:

$$\omega(f, \delta)_\varphi := \sup_{|t| \leq \delta} I^\varphi \left[\lambda \left(f(\cdot + t) - f(\cdot) \right) \right], \tag{2.2}$$

with $\delta > 0$.

For general references concerning Orlicz spaces and some of their generalizations, see, e.g., [28, 1, 24, 25, 18].

In order to define the considered operators, we need some additional notions.

Let $\Pi = (t_k)_{k \in \mathbb{Z}}$ be a sequence of real numbers such that $-\infty < t_k < t_{k+1} < +\infty$ for every $k \in \mathbb{Z}$, $\lim_{k \to \pm\infty} t_k = \pm\infty$ and there are two positive constants Δ, δ such that $\delta \leq \Delta_k := t_{k+1} - t_k \leq \Delta$, for every $k \in \mathbb{Z}$.
In what follows, a function $\chi : \mathbb{R} \to \mathbb{R}$ will be called a kernel if it satisfies the following properties:

- $(\chi 1)$ $\chi \in L^1(\mathbb{R})$ and is bounded in a neighborhood of 0;
- $(\chi 2)$ for every $u \in \mathbb{R}$

$$\sum_{k \in \mathbb{Z}} \chi(u - t_k) = 1;$$

- $(\chi 3)$ for some $\beta > 0$,

$$m_{\beta,\Pi}(\chi) \;:=\; \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-t_k)| \cdot |u-t_k|^{\beta} \;<\; +\infty.$$

Then, the sampling Kantorovich operators S_w for a given kernel χ are defined by:

$$(S_w f)(x) \;:=\; \sum_{k\in\mathbb{Z}} \chi(wx-t_k) \left[\frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\, du \right] \qquad (x\in\mathbb{R}), \tag{2.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally integrable function such that the series is convergent for every $x \in \mathbb{R}$.

There holds the following lemma.

Lemma 2.1 ([8]). *Under the assumptions $(\chi 1)$ and $(\chi 3)$ on the kernel χ, it turns out:*

$$m_{0,\Pi}(\chi) \;:=\; \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-t_k)| \;<\; +\infty.$$

Note that, it is easy to see that the discrete absolute moment $m_{0,\Pi}(\chi) > 0$.

3. The main result

We can prove the following quantitative estimate for the sampling Kantorovich operators by using the modulus of continuity in Orlicz spaces.

Theorem 3.1. *Let φ be a convex φ-function. Suppose that, for any fixed $0 < \alpha < 1$, we have:*

$$w \int_{|y|>1/w^{\alpha}} |\chi(wy)|\, dy \;\le\; M\, w^{-\gamma}, \quad \textit{as} \quad w \to +\infty, \tag{3.4}$$

for suitable positive constants M, γ depending on α and χ. Then, for $f \in L^{\varphi}(\mathbb{R})$, and $\lambda > 0$ there holds:

$$I^{\varphi}[\lambda\,(S_w f - f)] \;\le\; \frac{\|\chi\|_1}{2\,\delta\, m_{0,\Pi}(\chi)}\, \omega\left(2\, m_{0,\Pi}(\chi)\, f,\, \frac{1}{w^{\alpha}}\right)_{\varphi}$$

$$+\; \frac{M\, I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi)\, f\right]}{2\,\delta\, m_{0,\Pi}(\chi)}\, w^{-\gamma} \;+\; \frac{\Delta}{2\,\delta}\, \omega\left(2\, m_{0,\Pi}(\chi)\, f,\, \frac{1}{w}\right)_{\varphi},$$

for every sufficiently large $w > 0$, where $m_{0,\Pi}(\chi) < +\infty$ in view of Lemma 2.1. In particular, if $\lambda > 0$ is sufficiently small, the above inequality implies the modular convergence of the sampling Kantorovich operators $S_w f$ to f.

Proof. Let $\lambda > 0$ be fixed. Using the convexity of φ, and since φ is non decreasing, we can write what follows:

$$I^{\varphi}[\lambda\,(S_w f - f)]$$

$$\le\; \frac{1}{2}\left\{ \int_{\mathbb{R}} \varphi\left(2\,\lambda \left| (S_w f)(x) - \sum_{k\in\mathbb{Z}} \chi(wx-t_k)\, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u + x - t_k/w)\, du \right| \right) dx \right.$$

$$\left. +\; \int_{\mathbb{R}} \varphi\left(2\,\lambda \left| \sum_{k\in\mathbb{Z}} \chi(wx-t_k)\, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u + x - t_k/w)\, du \;-\; f(x) \right| \right) dx \right\} \;=:\; I_1 + I_2,$$

$w > 0$. We estimate I_1. By using the Jensen inequality (see, e.g., [19]) twice, and the change of variable $y = x - t_k/w$, we obtain:

$$2\, I_1 \;\leq\; \int_{\mathbb{R}} \varphi\left(2\,\lambda \sum_{k\in\mathbb{Z}} |\chi(wx-t_k)|\, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + x - t_k/w)|\, du \right) dx$$

$$\leq\; \frac{1}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(wx-t_k)|\, \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi) \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + x - t_k/w)|\, du \right) dx$$

$$\leq\; \frac{1}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(wx-t_k)|\, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi) |f(u) - f(u + x - t_k/w)|\right) du\, dx$$

$$\leq\; \frac{\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} |\chi(wy)|\; w \sum_{k\in\mathbb{Z}} \int_{t_k/w}^{t_{k+1}/w} \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi) |f(u) - f(u + y)|\,\right) du\, dy$$

$$=\; \frac{\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} |\chi(wy)|\; w \int_{\mathbb{R}} \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi) |f(u) - f(u + y)|\right) du\, dy$$

$$=\; \frac{\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} w\, |\chi(wy)|\; I^{\varphi}\left[2\,\lambda\, m_{0,\Pi}(\chi) (f(\cdot) - f(\cdot + y))\right] dy \;=:\; J,$$

$w > 0$. Let now $0 < \alpha < 1$ be fixed. Thus we can split the above integral J as follows:

$$J \;:=\; \frac{w\,\delta^{-1}}{m_{0,\Pi}(\chi)} \times$$
$$\left\{ \int_{|y|\leq 1/w^{\alpha}} + \int_{|y|>1/w^{\alpha}} \right\} |\chi(wy)|\; I^{\varphi}\left[2\,\lambda\, m_{0,\Pi}(\chi) (f(\cdot) - f(\cdot + y))\right] dy \;=:\; J_1 \,+\, J_2.$$

For J_1, we have:

$$J_1 \;\leq\; \frac{w\,\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{|y|\leq 1/w^{\alpha}} |\chi(wy)|\, \omega\left(2\, m_{0,\Pi}(\chi)\, f,\, |y|\right)_{\varphi}\, dy$$

$$\leq\; \omega\left(2\, m_{0,\Pi}(\chi)\, f,\, 1/w^{\alpha}\right)_{\varphi} \frac{w\,\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{|y|\leq 1/w^{\alpha}} |\chi(wy)|\, dy$$

$$\leq\; \omega\left(2\, m_{0,\Pi}(\chi)\, f,\, 1/w^{\alpha}\right)_{\varphi} \frac{\delta^{-1}\, \|\chi\|_1}{m_{0,\Pi}(\chi)},$$

$w > 0$. Moreover, by using the convexity of φ, for J_2 we can obtain:

$$J_2 \;\leq\; \frac{w\,\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{|y|>1/w^{\alpha}} |\chi(wy)|\; \frac{1}{2} \left\{ I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi) f\right] + I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi) f(\cdot + y)\right] \right\} dy.$$

Obviously, it is easy to see that:

$$I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi) f\right] \;=\; I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi) f(\cdot + y)\right],$$

for every y. Then, by exploiting assumption (3.4), we finally obtain:

$$J_2 \;\leq\; \frac{w\,\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{|y|>1/w^{\alpha}} |\chi(wy)|\; I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi) f\right]\, dy$$

$$\leq\; \frac{\delta^{-1}}{m_{0,\Pi}(\chi)}\, I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi) f\right]\, M\, w^{-\gamma},$$

for $w > 0$ sufficiently large.

Now, we can estimate I_2. Using Jensen inequality twice (as above), the change of variable $y = u - t_k/w$, and Fubini-Tonelli theorem, we have:

$$
\begin{aligned}
& 2\, I_2 \\
\leq\ & \frac{1}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(wx - t_k)| \, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi) |f(u + x - t_k/w) - f(x)|\,\right) du\, dx \\
& \leq\ \frac{\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(wx - t_k)| \, w \int_0^{\Delta/w} \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi) |f(x+y) - f(x)|\,\right) dy\, dx \\
& \leq\ \delta^{-1} \int_{\mathbb{R}} w \int_0^{\Delta/w} \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi) |f(x+y) - f(x)|\,\right) dy\, dx \\
& \leq\ \delta^{-1} w \int_0^{\Delta/w} I^{\varphi}\left[2\,\lambda\, m_{0,\Pi}(\chi)\, (f(\cdot + y) - f(\cdot))\right] dy \\
& \leq\ \delta^{-1}\, \omega(2\, m_{0,\Pi}(\chi)\, f,\, 1/w)_{\varphi}\, w \int_0^{\Delta/w} dy\ =\ \delta^{-1}\, \Delta\, \omega(2\, m_{0,\Pi}(\chi)\, f,\, 1/w)_{\varphi},
\end{aligned}
$$

$w > 0$. This completes the proof. □

Remark 3.1. Note that, it is easy to show that for any kernels such that $\chi(u) = \mathcal{O}(|u|^{-\theta})$, as $|u| \to +\infty$, for $\theta > 1$, we have that assumption (3.4) is satisfied for some constant $M > 0$ and $\gamma = (1-\alpha)(\theta - 1) > 0$, for every fixed $0 < \alpha < 1$.

4. Examples

Examples of convex φ-functions generating remarkable Orlicz spaces, where the above result is valid are:
$\varphi_p(u) := u^p$, $1 \leq p < \infty$, $\varphi_{\alpha,\beta} := u^{\alpha} \log^{\beta}(u + e)$, for $\alpha \geq 1$, $\beta > 0$ and $\varphi_{\gamma}(u) = e^{u^{\gamma}} - 1$, for $\gamma > 0$, $u \geq 0$. It is well-known that φ_p generates the $L^p(\mathbb{R})$-space and the corresponding convex modular functional is given by $I^{\varphi_p}[f] := \|f\|_p^p$, while $\varphi_{\alpha,\beta}$ and φ_{γ} generate the $L^{\alpha} \log^{\beta} L$-spaces (or Zygmund spaces), largely used, e.g., in the theory of partial differential equations, and the exponential spaces respectively, e.g., used for embedding theorems between Sobolev spaces. The convex modular functionals corresponding to $\varphi_{\alpha,\beta}$ and φ_{γ} are

$$
I^{\varphi_{\alpha,\beta}}[f] := \int_{\mathbb{R}} |f(x)|^{\alpha} \log^{\beta}(e + |f(x)|)\, dx, \qquad (f \in M(\mathbb{R})),
$$

and

$$
I^{\varphi_{\gamma}}[f] := \int_{\mathbb{R}} (e^{|f(x)|^{\gamma}} - 1)\, dx, \qquad (f \in M(\mathbb{R})),
$$

respectively.

Now, we give a brief list of some well-known and important class of kernels which satisfy the above assumptions $(\chi 1) - (\chi 3)$, and for which Theorem 3.1 holds.

First of all, we recall the definition of the well-known central B-spline of order N (see e.g., [33, 3, 4]):

$$
\beta^N(x) \ := \ \frac{1}{(N-1)!} \sum_{i=0}^{N} (-1)^i \binom{N}{i} \left(\frac{N}{2} + x - i\right)_+^{N-1}, \qquad x \in \mathbb{R}. \tag{4.5}
$$

It is well-known that β^N have compact support, then (3.4) is obviously satisfied for every $\gamma > 0$.

Other important (band-limited) kernels are given by the so-called Jackson type kernels of order N, defined by:

$$J_N(x) \; := \; c_N \, \mathrm{sinc}^{2N}\left(\frac{x}{2N\pi\alpha}\right), \qquad x \in \mathbb{R}, \tag{4.6}$$

with $N \in \mathbb{N}$, $\alpha \geq 1$, and c_N is a non-zero normalization coefficient, given by:

$$c_N \; := \; \left[\int_{\mathbb{R}} \mathrm{sinc}^{2N}\left(\frac{u}{2N\pi\alpha}\right) du\right]^{-1}.$$

For J_N, assumption (3.4) turns out to be satisfied in view of what has been observed in Remark 3.1. For the sake of completeness, we recall that the well-known (above mentioned) $sinc$-function is that defined as $\sin(\pi x)/\pi x$, if $x \neq 0$, and 1 if $x = 0$, see e.g., [26, 27]. For other examples of kernels, see, e.g., [13, 20, 15, 22, 16].

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