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Quantitative Estimates for L^p-Approximation by Bernstein-Kantorovich-Choquet Polynomials with Respect to Distorted Lebesgue Measures

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ABSTRACT. For the univariate Bernstein-Kantorovich-Choquet polynomials written in terms of the Choquet integral with respect to a distorted probability Lebesgue measure, we obtain quantitative approximation estimates for the L^p-norm, $1 \leq p < +\infty$, in terms of a K-functional.

Keywords: Monotone and submodular set function, Choquet integral, Bernstein-Kantorovich-Choquet polynomial, L^p quantitative estimates, K-functional, Distorted Lebesgue measure.

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1. INTRODUCTION

Recently, in a series of papers we have started the study of the approximation properties of some nonlinear integral operators obtained from the linear ones by replacing the classical Lebesgue integral by its nonlinear extension called Choquet integral with respect to a monotone and submodular set function. Thus, qualitative and quantitative results of approximation by Bernstein-Durrmeyer-Choquet polynomials written in terms of Choquet integrals with respect to monotone and submodular set functions were obtained in the papers [7], [9], [10], [14]. Qualitative and quantitative approximation results for other Choquet integral operators obtained by using a Feller kind scheme (and including discrete Bernstein-Choquet operators and Picard-Choquet operators) were obtained in [8]. For large classes of functions, all these nonlinear operators give better estimates of approximation than their classical correspondents. Quantitative results of uniform and pointwise approximation by Bernstein-Kantorovich-Choquet polynomials, better in large classes of functions than those obtained by their classical correspondents, were obtained in the very recent paper [11]. Also, shape preserving properties of some Kantorovich-Choquet type operators were considered in [13].
It is worth to mention that implications of the concept of Choquet integral in other topics of mathematical analysis were obtained in the papers [12], [15], [16].
The aim of the present paper is to to obtain quantitative estimates for L^p-approximation, $1 \leq p < +\infty$, by Bernstein-Kantorovich-Choquet polynomials.
Section 2 contains some preliminaries on the Choquet integral. In Section 3, in the case when the Choquet integral is taken with respect to the so called distorted Lebesgue measures, quantitative estimates in terms of a K-functional for the L^p approximation, $1 \leq p < \infty$, are obtained.

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2. Preliminaries

In this section we present some concepts and results on the Choquet integral which will be used in the main section.

Definition 2.1. *Let Ω be a nonempty set and $\mathcal{C}$ be a σ-algebra of subsets in Ω.*
(i) (see, e.g., [21], p. 63) Let $\mu : \mathcal{C} \to [0, +\infty]$. If $\mu(\emptyset) = 0$ and $A, B \in \mathcal{C}$, with $A \subset B$, implies $\mu(A) \leq \mu(B)$, then μ is called a monotone set function (or capacity). Also, if

$$\mu(A \bigcup B) + \mu(A \bigcap B) \leq \mu(A) + \mu(B), \text{ for all } A, B \in \mathcal{C},$$

then μ is called submodular. If $\mu(\Omega) = 1$, then μ is called normalized.
(ii) (see [5], or [21], p. 233, or [19]) Let μ be a normalized, monotone set function on $\mathcal{C}$.
If $f : \Omega \to \mathbb{R}$ is $\mathcal{C}$-measurable, i.e. for any Borel subset $B \subset \mathbb{R}$ we have $f^{-1}(B) \in \mathcal{C}$, then for any $A \in \mathcal{C}$, the Choquet integral is defined by

$$(C) \int_A f d\mu = \int_0^{+\infty} \mu(F_\beta(f) \bigcap A) d\beta + \int_{-\infty}^0 [\mu(F_\beta(f) \bigcap A) - \mu(A)] d\beta,$$

where $F_\beta(f) = \{\omega \in \Omega; f(\omega) \geq \beta\}$. If $(C) \int_A f d\mu \in \mathbb{R}$, then f is called Choquet integrable on A. Notice that if $f \geq 0$ on A, then in the above formula we get $\int_{-\infty}^0 = 0$.
If μ is the Lebesgue measure, then the Choquet integral $(C) \int_A f d\mu$ reduces to the Lebesgue integral.

In what follows, we list some known properties of the Choquet integral.

Remark 2.1. *If $\mu : \mathcal{C} \to [0, +\infty]$ is a monotone set function, then the following properties hold :*
(i) For all $a \geq 0$ we have $(C) \int_A a f d\mu = a \cdot (C) \int_A f d\mu$ (if $f \geq 0$ then see, e.g., [21], Theorem 11.2, (5), p. 228 and if f is of arbitrary sign, then see, e.g., [6], p. 64, Proposition 5.1, (ii)).
(ii) For all $c \in \mathbb{R}$ and f of arbitrary sign, we have (see, e.g., [21], pp. 232-233, or [6], p. 65)

$$(C) \int_A (f + c) d\mu = (C) \int_A f d\mu + c \cdot \mu(A).$$

If μ is submodular too, then for all f, g of arbitrary sign and lower bounded we have (see, e.g., [6], p. 75, Theorem 6.3)

$$(C) \int_A (f + g) d\mu \leq (C) \int_A f d\mu + (C) \int_A g d\mu,$$

that is the Choquet integral is sublinear.
(iii) If $f \leq g$ on A then $(C) \int_A f d\mu \leq (C) \int_A g d\mu$ (see, e.g., [21], p. 228, Theorem 11.2, (3) if $f, g \geq 0$ and p. 232 if f, g are of arbitrary sign).
(iv) Let $f \geq 0$. By the definition of the Choquet integral, it is immediate that if $A \subset B$ then

$$(C) \int_A f d\mu \leq (C) \int_B f d\mu$$

and if, in addition, μ is finitely subadditive, then

$$(C) \int_{A \bigcup B} f d\mu \leq (C) \int_A f d\mu + (C) \int_B f d\mu.$$

(v) By the definition of the Choquet integral, it is immediate that

$$(C) \int_A 1 \cdot d\mu(t) = \mu(A).$$

(vi) The formula $\mu(A) = \gamma(M(A))$, where $\gamma : [0, 1] \to [0, 1]$ is an increasing and concave function, with $\gamma(0) = 0$, $\gamma(1) = 1$ and M is a probability measure (or only finitely additive) on a σ-algebra on Ω

(that is, $M(\emptyset) = 0$, $M(\Omega) = 1$ and M is countably additive), gives simple examples of monotone and submodular set functions (see, e.g., [6], pp. 16-17, Example 2.1). Such of set functions μ are also called distorsions of normalized and countably additive measures (or distorted measures).

3. L^p-APPROXIMATION

Denoting by $\mathcal{B}_{[0,1]}$ the sigma algebra of all Borel measurable subsets in $\mathcal{P}([0,1])$, everywhere in this section, $(\Gamma_{n,x})_{n\in\mathbb{N},x\in[0,1]}$, will be a collection of families $\Gamma_{n,x} = \{\mu_{n,k,x}\}_{k=0}^{n}$, of monotone, submodular and strictly positive set functions $\mu_{n,k,x}$ on $\mathcal{B}_{[0,1]}$. Note here that a set function on $\mathcal{B}_{[0,1]}$ is called strictly positive, if for any open subset $A \subset \mathbb{R}$ with $A \cap [0,1] \neq \emptyset$, we have $\mu(A \cap [0,1]) > 0$.
Suggested by the classical form of the linear and positive operators of Bernstein-Kantorovich (see, e.g., [17]), we can introduce the following.

Definition 3.2. *The Bernstein-Kantorovich-Choquet polynomials with respect to $\Gamma_{n,x} = \{\mu_{n,k,x}\}_{k=0}^{n}$, are defined by the formula*

$$K_{n,\Gamma_{n,x}}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/(n+1),(k+1)/(n+1)])},$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

In order to be well defined these operators, it is good enough if, for example, we suppose that $f : [0,1] \to \mathbb{R}_+$ is a $\mathcal{B}_{[0,1]}$-measurable function, bounded on $[0,1]$.

Remark 3.2. *It is clear that if $\mu_{n,k,x} = M$, for all n, k and x, where M is the Lebesgue measure, then the above polynomials become the classical ones.*
Also, if $\mu_{n,k,x} = \delta_{k/n}$ (the Dirac measures), since $k/n \in (k/(n+1), (k+1)/(n+1))$, it is immediate that $K_{n,\Gamma_{n,x}}(f)(x)$ become the Bernstein polynomials. This fact shows the great flexibility of the formulas of these operators. More exactly, we can generate very many kinds of approximation operators, by choosing for some $\mu_{n,k,x}$ the Lebesgue measure, for some others $\mu_{n,k,x}$, the Dirac measures and for the others $\mu_{n,k,x}$, some Choquet measures.

Note that pointwise and uniform approximation by $K_{n,\Gamma_{n,x}}(f)(x)$ were studied in [11].
In this section we study quantitative L^p-approximation results, $1 \le p < \infty$, for the Bernstein-Kantorovich-Choquet polynomials $K_{n,\Gamma_{n,x}}(f)(x)$ when $\Gamma_{n,x} = \{\mu\}$. In this case, we denote them by $K_{n,\mu}$.
But as in the case of Bernstein-Durrmeyer-Choquet polynomials studied in [10], even in the simple case when, for example $p = 1$, for $f \in L^1_\mu$ (meaning that f is $\mathcal{B}_{[0,1]}$-measurable and $\|f\|_{L^1_\mu} = (C)\int_0^1 |f(t)|d\mu(t) < \infty$), considering for example the operator $K_{n,\mu}$, we easily get

$$\|K_{n,\mu}(f)\|_{L^1_\mu} \le \sum_{k=0}^{n} (C)\int_0^1 p_{n,k}(x)d\mu(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu(t)}{\mu([k/(n+1),(k+1)/(n+1)])}$$
$$\le \sum_{k=0}^{n} (C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu(t) \le (n+1) \cdot \|f\|_{L^1_\mu}.$$

This is due to the fact that $(C)\int_0^1 f d\mu$ is not, in general, additive as function of f (it is only subadditive).
Therefore, quantitative estimates for L^p-approximation by Bernstein-Kantorovich-Choquet polynomials, remain, for the general case, an open question.

However, in what follows, for a large class of distorted Lebesgue measures (see Remark 2.1, (vi)), we will be able to prove L^p-approximation results.
If $\mu : \mathcal{B}_{[0,1]} \to [0, +\infty)$ is a monotone set function and $1 \leq p < +\infty$, then we make the following notations :

$$L^p_\mu[0,1] = \{f : [0,1] \to \mathbb{R}; f \text{ is } \mathcal{B}_{[0,1]}\text{-measurable and } (C)\int_0^1 |f(t)|^p d\mu(t) < +\infty\},$$

$$L^p_{\mu,+}[0,1] = L^p_\mu[0,1] \bigcap \{f : [0,1] \to \mathbb{R}_+\},$$

$$C^1_+[0,1] = \{g : [0,1] \to [0,+\infty); g \text{ is differentiable on } [0,1]\},$$

$$K\,(f;t)_{L^p_\mu[0,1]} = \inf_{g\in C^1_+[0,1]}\{\|f-g\|_{L^p_\mu} + t\|g'\|_{C[0,1]}\},$$

$$\text{where } \|F\|_{L^p_\mu[0,1]} = \left(\int_0^1 |F(t)|^p d\mu(t)\right)^{1/p}, \|F\|_{C[0,1]} = \sup\{|F(t)|; t \in [0,1]\},$$

$$IC[0,1] = \{g : [0,1] \to [0,1] : g(0) = 0, g(1) = 1, g \text{ is concave and strictly increasing on } [0,1] \text{ and there exists } g'(0) < +\infty\}.$$

Also, denote by $\mathcal{D}(\mathcal{B}_{[0,1]})$ the class of all set functions $\mu : \mathcal{B}_{[0,1]} \to [0, +\infty)$ of the form $\mu(A) = g(M(A))$, for all $A \in \mathcal{B}_{[0,1]}$, where $g \in IC[0,1]$ and M is the Lebesgue measure on $\mathcal{B}_{[0,1]}$. In the words of Remark 2.1, (vi), any such a μ is a distorted Lebesgue measure.

Remark 3.3. *According to Remark 2.1, (vi), any $\mu \in \mathcal{D}(\mathcal{B}_{[0,1]})$ is a normalized, monotone, strictly positive and submodular set function. Simple examples of $\mu \in \mathcal{D}(\mathcal{B}_{[0,1]})$ are $\mu(A) = \sin[M(A)]/\sin(1)$ or $\mu(A) = g[M(A)]$, for all $A \in \mathcal{B}_{[0,1]}$, where M denotes the Lebesgue measure and $g(x) = \frac{2x}{1+x}$.*

We can state the following.

Theorem 3.1. *Let $1 \leq p < \infty$. If $\mu \in \mathcal{D}(\mathcal{B}_{[0,1]})$, then for all $f \in L^p_{\mu,+}[0,1]$, $n \in \mathbb{N}$, we have*

$$\|f - K_{n,\mu}(f)\|_{L^p_\mu} \leq c_p \cdot K\left(f; \frac{1}{2\sqrt{n+1}}\right)_{L^p_\mu},$$

where $c_p = 1 + g'(0)^{(p+1)/p}$.

Proof. Let $\mu(A) = g[M(A)]$ with $\mu \in \mathcal{D}(\mathcal{B}_{[0,1]})$. The main ideas used several times in the proof are that the Choquet integral with respect to m reduces to the classical Lebesgue integral and that if μ and ν are two monotone set functions satisfying $\mu(A) \leq c \cdot \nu(A)$ for all A, with $c > 0$ a constant independent of A, then $(C)\int_0^1 F d\mu \leq c \cdot (C)\int_0^1 F d\nu$, for any $F \geq 0$.
Firstly, by $g(0) = 0$, $g(1) = 1$ and by the concavity of g, we immediately obtain the inequalities

$$x \leq g(x) \leq g'(0)x, \text{ for all } x \in [0,1], \tag{3.1}$$

which clearly implies

$$M(A) \leq \mu(A) \leq g'(0)M(A), \text{ for all } A \in \mathcal{B}_{[0,1]}. \tag{3.2}$$

Indeed, the inequalities in (3.1) hold since all the points of the segment passing through the points $(0, g(0))$ and $(1, g(1))$ are below the graph of g and since all the points of the tangent to the graph of g at $(0, g(0))$ are above the graph of g.
We make the proof in three steps.
Step 1. For $f \in L^p_{\mu,+}[0,1]$ we obtain

$$\|K_{n,\mu}(f)\|_{L^p_\mu} \leq [g'(0)]^{(p+1)/p} \cdot \|f\|_{L^p_\mu}. \tag{3.3}$$

Indeed, by $\|K_{n,M}(f)\|_{L^p_M} \le \|f\|_{L^p_M}$ (see, e.g. [3]) combined with (3.2), it follows $\|f\|_{L^p_M} \le \|f\|_{L^p_\mu}$ and

$$\|K_{n,M}(f)\|_{L^p_M} \le \|f\|_{L^p_\mu}. \tag{3.4}$$

On the other hand, by (3.2), we obtain

$$\|K_{n,M}(f)\|_{L^p_M}$$
$$= \left(\int_0^1 \left[\sum_{k=0}^n p_{n,k}(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dM(t)}{M([k/(n+1),(k+1)/(n+1)])} \right]^p dM(x) \right)^{1/p}$$
$$\ge \frac{1}{g'(0)^{1/p}} \cdot \left((C)\int_0^1 \left[\sum_{k=0}^n p_{n,k}(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dM(t)}{M([k/(n+1),(k+1)/(n+1)])} \right]^p d\mu(x) \right)^{1/p}$$
$$\ge \frac{1}{g'(0)^{1/p}}$$
$$\cdot \left((C)\int_0^1 \left[\sum_{k=0}^n p_{n,k}(x) \cdot \frac{1}{g'(0)} \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu(t)}{\mu([k/(n+1),(k+1)/(n+1)])} \right]^p d\mu(x) \right)^{1/p}$$
$$= \frac{1}{[g'(0)]^{(p+1)/p}} \cdot \|K_{n,\mu}(f)\|_{L^p_\mu},$$

which combined with (3.4), implies (3.3).

Step 2. For $n \in \mathbb{N}$ and $0 \le k \le n$ arbitrary fixed, let us define $T_{n,k} : L^p_{\mu,+}[0,1] \to \mathbb{R}_+$ by

$$T_{n,k}(f) = (C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu(t), f \in L^p_{\mu,+}([0,1]).$$

From $L^p_{M,+}[0,1] \subset L^1_{M,+}[0,1]$ and since from (3.2) we clearly have $f \in L^p_{M,+}[0,1]$ if and only if $f \in L^p_{\mu,+}[0,1]$, it follows that $L^p_{\mu,+}[0,1] \subset L^1_{\mu,+}[0,1]$, for all $1 \le p < +\infty$.
Also, $0 \le (C)\int_{k/(n+1)}^{(k+1)/(n+1)} f^p(t)d\mu(t) \le (C)\int_0^1 f^p(t)d\mu(t) < \infty$, for any $f \in L^p_{\mu,+}[0,1]$.
Based on the Remark 3.3 and Remark 2.1, (i), (ii), (iii), by similar reasonings with those in the proof of Lemma 3.1 in [7], we obtain $|T_{n,k}(f) - T_{n,k}(g)| \le T_{n,k}(|f-g|)$. Also, since $T_{n,k}$ is positively homogeneous, sublinear and monotonically increasing, it is immediate that $K_{n,\mu}$ keeps the same properties, Consequently, it follows

$$|K_{n,\mu}(f)(x) - K_{n,\mu}(g)(x)| \le K_{n,\mu}(|f-g|)(x),\ f,g \in L^p_{\mu,+}[0,1], \tag{3.5}$$

$K_{n,\mu}(\lambda f) = \lambda K_{n,\mu}(f)$, $K_{n,\mu}(f+g) \le K_{n,\mu}(f) + K_{n,\mu}(g)$ and that $f \le g$ on $[0,1]$ implies $K_{n,\mu}(f) \le K_{n,\mu}(g)$ on $[0,1]$, for all $\lambda \ge 0$, $f,g \in L^p_{\mu,+}[0,1]$, $n \in \mathbb{N}$.
Now, from (3.5) we get

$$\|K_{n,\mu}(f) - K_{n,\mu}(g)\|_{L^p_\mu} \le \|K_{n,\mu}(|f-g|)\|_{L^p_\mu}. \tag{3.6}$$

Step 3. Let $f,g \in L^p_{\mu,+}[0,1]$. We will apply the Minkowski's inequality in the Choquet integral (see. e.g., Theorem 3.7 in [20] or Theorem 2 in [4]). It is worth mentioning that the proof of Minkowski's inequality in [20] or [4] is based on the Hölder's inequality

$$(C)\int |fg| \le \left((C)\int |f|d\mu\right)^{1/p} \cdot \left((C)\int |g|d\mu\right)^{1/q}, 1/p + 1/q = 1,$$

where the proof is performed under the supposition that $(C)\int |f|d\mu \neq 0$ and $(C)\int |g|d\mu \neq 0$. But from (3.2), it easily follows that the Hölder's inequality immediately holds even if $(C)\int |f|d\mu = 0$ or $(C)\int |g|d\mu = 0$. Therefore, under the hypothesis of the theorem, the Minkowski's inequality holds in its full generality.
So, we get

$$\|f - K_{n,\mu}(f)\|_{L^p_\mu} = \|(f-g) + (g - K_{n,\mu}(g)) + (K_{n,\mu}(g) - K_{n,\mu}(f))\|_{L^p_\mu}$$

$$\leq \|f-g\|_{L^p_\mu} + \|g - K_{n,\mu}(g)\|_{L^p_\mu} + \|K_{n,\mu}(g) - K_{n,\mu}(f)\|_{L^p_\mu}. \tag{3.7}$$

By (3.6) and (3.3), we obtain

$$\|K_{n,\mu}(g) - K_{n,\mu}(f)\|_{L^p_\mu} \leq [g'(0)]^{(p+1)/p} \cdot \|f-g\|_{L^p_\mu}. \tag{3.8}$$

Now, let us estimate $\|g - K_{n,\mu}(g)\|_{L^p_\mu}$ for $g \in C^1_+[0,1]$. Thus, by (3.5) and $K_{n,\mu}(e_0)(x) = e_0(x) = 1$, we get

$$|g(x) - K_{n,\mu}(g)(x)| = |K_{n,\mu}(g(x))(x) - K_{n,\mu}(g(t))(x)| \leq K_{n,\mu}(|g(x) - g(\cdot)|)(x).$$

Since for $g \in C^1_+[0,1]$ and $x, t \in [0,1]$, it follows (see, e.g., [18], formula (2.5), or [2])

$$|g(x) - g(t)| \leq \|g'\|_{C[0,1]} \cdot |x-t| = \|g'\|_{C[0,1]} \cdot \varphi_x(t),$$

applying $K_{n,\mu}$, which is subadditive as function of f, it follows $K_{n,\mu}(|g(x) - g(\cdot)|)(x) \leq \|g'\|_{C[0,1]} K_{n,\mu}(\varphi_x)$.
Taking to the power p and integrating above with respect to x and μ, we obtain

$$\|g - K_{n,\mu}(g)\|_{L^p_\mu} \leq \|g'\|_{C[0,1]} \cdot \|K_{n,\mu}(\varphi_x)\|_{L^p_\mu}. \tag{3.9}$$

Denoting $c_p = 1 + g'(0)^{(p+1)/p}$, from (3.8) and (3.9) replaced in (3.7), it follows

$$\|f - K_{n,\mu}(f)\|_{L^p_\mu} \leq c_p \left(\|f-g\|_{L^p_\mu} + \|g'\|_{C[0,1]} \cdot \Delta_{n,p}/c_p\right),$$

where $\Delta_{n,p} := \|K_{n,\mu}(\varphi_x)\|_{L^p_\mu}$, $\varphi_x(t) = |x-t|$ for $x, t \in [0,1]$.
Finally, the reasonings from Step 1 lead to the estimate

$$\Delta_{n,p}/c_p \leq \frac{[g'(0)]^{(p+1)/p}}{c_p} \cdot \|K_{n,M}(\varphi_x)\|_{L^p_M} \leq \frac{[g'(0)]^{(p+1)/p}}{c_p} \cdot \|K_{n,M}(\varphi_x)\|_{C[0,1]}$$

$$\leq \frac{[g'(0)]^{(p+1)/p}}{c_p} \cdot \frac{1}{2\sqrt{n+1}} \leq \frac{1}{2\sqrt{n+1}}.$$

(we have used above the inequality in, e.g., [1], p. 334, $|K_{n,M}(\varphi_x)(x)| \leq \frac{\sqrt{(n-1)x(1-x)}}{n+1}$).
This immediately proves the required conclusion. □

Remark 3.4. *Note that the order of L^p-approximation $K\left(f; \frac{1}{2\sqrt{n+1}}\right)_{L^p_\mu}$ in Theorem 3.1 is, in some sense, similar with the order of L^p-approximation for the classical Bernstein-Kantorovich operators, $\tau\left(f; \frac{1}{\sqrt{n+1}}\right)_p$, where $\tau(f;\delta)_p$ is the L^p-averaged modulus of smoothness of Sendov-Popov (see, e.g.,* [3]*, p. 279).*

Remark 3.5. *For f of arbitrary sign and lower bounded on $[0,1]$ with $f(x) - m \geq 0$, for all $x \in [0,1]$, Theorem 3.1 still take place for the slightly modified operator*

$$K^*_{n,\mu}(f)(x) = K_{n,\mu}(f-m)(x) + m.$$

Indeed, we have $K_{n,\mu}^*(f)(x) - f(x) = K_{n,\mu}(f-m)(x) - (f(x)-m)$ *and since we may consider here that* $m < 0$, *we immediately get*

$$K(f-m;t)_{L^p_\mu} = \inf_{g\in C^1_+[0,1]}\{\|f-(g+m)\|_{L^p_\mu} + t\|\nabla g\|_{C[0,1]}\}$$
$$= \inf_{g\in C^1_+[0,1]}\{\|f-(g+m)\|_{L^p_\mu} + t\|\nabla (g+m)\|_{C[0,1]}\}$$
$$= \inf_{h\in C^1[0,1],\, h\ge m}\{\|f-h\|_{L^p_\mu} + t\|\nabla h\|_{C[0,1]}\}.$$

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