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QUASI 2-ABSORBING SECOND MODULES

H. ANSARI-TOROGHY AND F. FARSHADIFAR

Abstract. In this paper, we will introduce the notion of quasi 2-absorbing second modules over a commutative ring and obtain some basic properties of this class of modules.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and "$\subset$" will denote the strict inclusion. Further, $\mathbb{Z}$ will denote the ring of integers.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [14]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \stackrel{a}{\to} S$ is either surjective or zero [20]. More information about this class of modules can be found in [3, 4, 5, 6, 11, 12]. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$ [15].

The notion of 2-absorbing ideals as a generalization of prime ideals was introduced and studied in [8]. Also, various generalizations of primary ideals are introduced and studied in [9, 19]. A proper ideal I of R is a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. The notion of 2-absorbing ideals was extended to 2-absorbing submodules in [13] and [17]. A proper submodule N of M is called a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

In [7], the authors introduced the dual notion of 2-absorbing submodules (that is, *2-absorbing (resp. strongly 2-absorbing) second submodules*) of M and investigated some properties of these classes of modules. A non-zero submodule N of M is said to be a *2-absorbing second submodule of M* if whenever $a, b \in R$, L is a completely irreducible submodule of M, and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$

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or $ab \in Ann_R(N)$. A non-zero submodule N of M is said to be a *strongly 2-absorbing second submodule of* M if whenever $a, b \in R$, K is a submodule of M, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$.

The purpose of this paper is to introduce the concepts of quasi 2-absorbing second modules as a generalization of strongly 2-absorbing second modules and obtain some related results.

2. Main results

Definition 2.1. We say that a non-zero R-module M is a *quasi 2-absorbing second module* if $Ann_R(M)$ is a 2-absorbing ideal of R.

By a quasi 2-absorbing second submodule of a module we mean a submodule which is a quasi 2-absorbing second module.

Example 2.2. By [7, 3.5] every strongly 2-absorbing second module is a quasi 2-absorbing second module. But the converse is not true in general. For example, every submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ is a quasi 2-absorbing second module which is not a strongly 2-absorbing second module.

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [2].

Proposition 2.3. Let M be a comultiplication R-module. Then a submodule N of M is a strongly 2-absorbing second submodule of M if and only if it is a quasi 2-absorbing second submodule of M.

Proof. This follows from [7, 3.5] and [7, 3.10]. □

Proposition 2.4. Let M be an R-module and N_1, N_2 be two submodules of M with $Ann_R(N_1)$ and $Ann_R(N_2)$ prime ideals of R. Then $N_1 + N_2$ is a quasi 2-absorbing second submodule of M.

Proof. Since $Ann_R(N_1 + N_2) = Ann_R(N_1) \cap Ann_R(N_2)$, the result follows from [8]. □

For a submodule N of an R-module M the the *second radical* (or second socle) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [12] and [3]).

The set of all second submodules of an R-module M is called the *second spectrum* of M and denoted by $Spec^s(M)$. The map $\phi : Spec^s(M) \to Spec(R/Ann_R(M))$ defined by $\phi(S) = Ann_R(S)/Ann_R(M)$ for every $S \in Spec^s(M)$, is called the *natural map* of $Spec^s(M)$ [5].

Theorem 2.5. *Let M be an R-module and N be a quasi 2-absorbing second submodule of M. Then we have the following.*

(a) *IN is a quasi 2-absorbing second submodules of M for all ideals I of R with $I \not\subseteq Ann_R(N)$.*
(b) *If I is an ideal of R, then $Ann_R(I^n N) = Ann_R(I^{n+1}N)$, for all $n \geq 2$.*
(c) *If the natural map ϕ of $Spec^s(N)$ is surjective, then $sec(N)$ is a quasi 2-absorbing second submodule of M.*

Proof. (a) Let I be an ideal of R with $I \not\subseteq Ann_R(N)$. Then $Ann_R(IN)$ is a proper ideal of R. Now let $a, b, c \in R$ and $abcIN = 0$. Then $acN = 0$ or $cbIN = 0$ or $abIN = 0$. If $cbIN = 0$ or $abIN = 0$, then we are done. If $acN = 0$, then $Ann_R(N) \subseteq Ann_R(IN)$ implies that $acIN = 0$, as needed.

(b) It is enough to show that $Ann_R(I^2N) = Ann_R(I^3N)$. It is clear that $Ann_R(I^2N) \subseteq Ann_R(I^3N)$. Since N is quasi 2-absorbing second submodule, $Ann_R(I^3N)I^3N = 0$ implies that $Ann_R(I^3N)I^2N = 0$ or $I^2N = 0$. If $Ann_R(I^3N)I^2N = 0$, then $Ann_R(I^3N) \subseteq Ann_R(I^2N)$. If $I^2N = 0$, then $Ann_R(I^2N) = R = Ann_R(I^3N)$.

(c) Let the natural map ϕ of $Spec^s(N)$ be surjective. Then $Ann_R(sec(N)) = \sqrt{Ann_R(N)}$ by [6, 2.9]. Now the result follows from the fact that $\sqrt{Ann_R(N)}$ is a 2-absorbing ideal of R by [8, 2.1]. □

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [10].

Corollary 2.6. *Let M be a multiplication quasi 2-absorbing second R-module. Then every non-zero submodule of M is a quasi 2-absorbing second module.*

Proof. This follows from Theorem 2.5 (a). □

Corollary 2.7. *If R is a quasi 2-absorbing second R-module, then $Ann_R(I)$ is a 2-absorbing ideal of R for each non-zero ideal I of R.*

Proof. This follows from Corollary 2.6. □

Proposition 2.8. Let M be an R-module and $\{K_i\}_{i\in I}$ be a chain of quasi 2-absorbing second submodules of M. Then $\cup_{i\in I}K_i$ is a quasi 2-absorbing second submodule of M.

Proof. Clearly, $Ann_R(\cup_{i\in I}K_i) \neq R$. Let $a, b, c \in R$ and $abc \in Ann_R(\cup_{i\in I}K_i) = \cap_{i\in I}Ann_R(K_i)$. Assume contrary that $ab \notin \cap_{i\in I}Ann_R(K_i)$, $bc \notin \cap_{i\in I}Ann_R(K_i)$, and $ac \notin \cap_{i\in I}Ann_R(K_i)$. Then there are $m, n, t \in I$ where $ab \notin Ann_R(K_n)$, $bc \notin Ann_R(K_m)$, and $ac \notin Ann_R(K_t)$. Since $\{K_i\}_{i\in I}$ is a chain, we can assume that $K_m \subseteq K_n \subseteq K_t$. Then $Ann_R(K_t) \subseteq Ann_R(K_n) \subseteq Ann_R(K_m)$. As $abc \in Ann_R(K_t)$ and K_t is a quasi 2-absorbing second module, we have $ab \in Ann_R(K_t)$ or $ac \in Ann_R(K_t)$ or $bc \in Ann_R(K_t)$. In any cases, we have a contradiction. □

Definition 2.9. We say that a quasi 2-absorbing second submodule N of an R-module M is a *maximal quasi 2-absorbing second submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a quasi 2-absorbing second submodule T of M such that $N \subset T \subset K$.

Lemma 2.10. *Let M be an R-module. Then every quasi 2-absorbing second submodule of M is contained in a maximal quasi 2-absorbing second submodule of M.*

Proof. This is proved easily by using Zorn's Lemma and Proposition 2.8. □

Theorem 2.11. *Every Artinian R-module M has only a finite number of maximal quasi 2-absorbing second submodules.*

Proof. Suppose that there exists a non-zero submodule N of M such that it has an infinite number of maximal quasi 2-absorbing second submodules. Let S be a submodule of M chosen minimal such that S has an infinite number of maximal quasi 2-absorbing second submodules. Then S is not a quasi 2-absorbing second submodule. Thus there exist $a, b, c \in R$ such that $abcS = 0$ but $abS \neq 0$, $acS \neq 0$, and $bcS \neq 0$. Let V be a maximal quasi 2-absorbing second submodule of M contained in S. Then $abV = 0$ or $acV = 0$ or $bcV = 0$. Thus $V \subseteq (0 :_M ab)$ or $V \subseteq (0 :_M ac)$ or $V \subseteq (0 :_M bc)$. Therefore, $V \subseteq (0 :_S ab)$ or $V \subseteq (0 :_S ac)$ or $V \subseteq (0 :_S bc)$. By the choice of S, the modules $(0 :_S ab)$, $(0 :_S ac)$, and $(0 :_S bc)$ have only finitely many maximal quasi 2-absorbing second submodules. Therefore, there is only a finite number of possibilities for the module S, which is a contradiction. □

Proposition 2.12. Let M be a comultiplication R-module, $N \subset K$ be two submodules of M, and K be a quasi 2-absorbing second submodule of M. Then K/N is a quasi 2-absorbing second submodule of M/N.

Proof. Let $a, b, c \in R$ such that $abc(K/N) = 0$. Then $abcK \subseteq N$ and so that $Ann_R(N)abcK = 0$. Thus $Ann_R(N)abK = 0$ or $Ann_R(N)acK = 0$ or $bcK = 0$. If $bcK = 0$, then $bc(K/N) = 0$ and we are done. If $Ann_R(N)abK = 0$ or $Ann_R(N)acK = 0$, then $abK \subseteq (0 :_M Ann_R(N))$ or $acK \subseteq (0 :_M Ann_R(N))$. Now as M is a comultiplication module, $N = (0 :_M Ann_R(N))$ and the result follows from this. □

The following example shows that the condition M is a "comultiplication R-module" in Proposition 2.12 can not be omitted.

Example 2.13. The $\mathbb{Z}$-module $\mathbb{Z}$ is a quasi 2-absorbing second module which is not a comultiplication $\mathbb{Z}$-module and $12\mathbb{Z} \subset \mathbb{Z}$. But $\mathbb{Z}/\nVdash\nvDash\mathbb{Z}$ is not a quasi 2-absorbing second module.

Recall that $Z(R)$ denotes the set of zero divisors of R.

Proposition 2.14. Let M be a finitely generated R-module and S be a multiplicatively closed subset of R. If M is a quasi 2-absorbing second module and $Ann_R(M) \cap S = \emptyset$, then $S^{-1}M$ is a quasi 2-absorbing second $S^{-1}R$-module. Furthermore, if $S^{-1}M$ is a quasi 2-absorbing second $S^{-1}R$-module and $S \cap Z(R/Ann_R(M)) = \emptyset$, then M is a quasi 2-absorbing second module.

Proof. As M is a finitely generated R-module, $Ann_{S^{-1}R}(S^{-1}M) = S^{-1}(Ann_R(M))$ by [18, 9.12]. Now the result follows from [16, 1.3]. □

Proposition 2.15. Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then N is a quasi 2-absorbing second module if and only if $f(N)$ is a quasi 2-absorbing second module.

Proof. This follows from the fact that $Ann_R(N) = Ann_R(f(N))$. □

Theorem 2.16. *Let E be an injective cogenerator of R and let N be a submodule of an R-module M. Then $Ann_R(M/N)$ is a 2-absorbing ideal of R if and only if $Hom_R(M/N, E)$ is a quasi 2-absorbing second module.*

Proof. Since E is an injective cogenerator of R, $Ann_R(M/N) \neq R$ if and only if $Ann_R(Hom_R(M/N, E)) \neq R$. Now let $Ann_R(M/N)$ be a 2-absorbing ideal of R and $a, b, c \in R$ such that $abc \in Ann_R(Hom_R(M/N, E))$. Then by using [1, 3.13 (a)], we have $Hom_R(M/(N :_M abc), E) = abcHom_R(M/N, E) = 0$. Thus as E is an injective cogenerator of R, $M/(N :_M abc) = 0$. Hence $abc \in Ann_R(M/N)$. By assumption, we can assume that $ab \in Ann_R(M/N)$. This in turn implies that $ab \in Ann_R(Hom_R(M/N, E))$ as needed. The proof of sufficiency is similar. □

Lemma 2.17. [2, 3.3] *Let S be a submodule of a comultiplication R-module M. Then S is a second submodule if and only if $Ann_R(S)$ is a prime ideal of R.*

Let R_i be a commutative ring with identity and M_i be an R_i-module for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Theorem 2.18. *Let $R = R_1 \times R_2$ be a decomposable ring and let $M = M_1 \times M_2$ be an R-module, where M_1 is a comultiplication R_1-module and M_2 is a comultiplication R_2-module. Suppose that $N = N_1 \times N_2$ is a non-zero submodule of M. Then the following conditions are equivalent:*

(a) *N is a quasi 2-absorbing second submodule of M;*
(b) *Either $N_1 = 0$ and N_2 is a quasi 2-absorbing second submodule of M_2 or $N_2 = 0$ and N_1 is a quasi 2-absorbing second submodule of M_1 or N_1, N_2 are second submodules of M_1, M_2, respectively.*

Proof. Since $Ann_R(N) = Ann_{R_1}(N_1) \times Ann_{R_2}(N_2)$, the result follows from [16, 1.2] and Lemma 2.17. □

Theorem 2.19. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ ($2 \leq n < \infty$) be a decomposable ring and $M = M_1 \times M_2 \cdots \times M_n$ be an R-module, where for every $1 \leq i \leq n$, M_i is a comultiplication R_i-module, respectively. Then for a non-zero submodule N of M the following conditions are equivalent:*

(a) *N is a quasi 2-absorbing second submodule of M;*

(b) *Either* $N = \times_{i=1}^{n} N_i$ *such that for some* $k \in \{1, 2, ..., n\}$, N_k *is a quasi 2-absorbing second submodule of* M_k, *and* $N_i = 0$ *for every* $i \in \{1, 2, ..., n\} \setminus \{k\}$ *or* $N = \times_{i=1}^{n} N_i$ *such that for some* $k, m \in \{1, 2, ..., n\}$, N_k *is a second submodule of* M_k, N_m *is a second submodule of* M_m, *and* $N_i = 0$ *for every* $i \in \{1, 2, ..., n\} \setminus \{k, m\}$.

Proof. We use induction on n. For $n = 2$ the result holds by Theorem 2.18. Now let $3 \leq n < \infty$ and suppose that the result is valid when $K = M_1 \times \cdots \times M_{n-1}$. We show that the result holds when $M = K \times M_n$. By Theorem 2.18, N is a quasi 2-absorbing second submodule of M if and only if either $N = L \times 0$ for some quasi 2-absorbing second submodule L of K or $N = 0 \times L_n$ for some quasi 2-absorbing second submodule L_n of M_n or $N = L \times L_n$ for some second submodule L of K and some second submodule L_n of M_n. Note that a non-zero submodule L of K is a second submodule of K if and only if $L = \times_{i=1}^{n-1} N_i$ such that for some $k \in \{1, 2, ..., n-1\}$, N_k is a second submodule of M_k, and $N_i = 0$ for every $i \in \{1, 2, ..., n-1\} \setminus \{k\}$. Consequently we reach the claim. □

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Current address: H. Ansari-Toroghy: Department of pure Mathematics, Faculty of mathematical Sciences, University of Guilan, P. O. Box 41335-19141, Rasht, Iran.

E-mail address: ansari@guilan.ac.ir

ORCID Address: http://orcid.org/0000-0001-7310-0924

Current address: F. Farshadifar (Corresponding Author): Assistant Professor, Department of Mathematics, Farhangian University, Tehran, Iran.

E-mail address: f.farshadifar@cfu.ac.ir

ORCID Address: http://orcid.org/0000-0001-7600-994X