Communication in Mathematical Modeling and Applications
http://ntmsci.com/cmma

Modelling and optimization of multi-item solid transportation problems with uncertain variables and uncertain entropy function

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Received: 31 October 2017, Accepted: 30 December 2017
Published online: 29 January 2018.

Abstract: This paper concentrates on modeling of the uncertain entropy multi item solid transportation problem, in which the supply capacities, demands, conveyances and transportation capacities are thought to be uncertain variables due to the obvious uncertainty of information. In general, the transportation cost in the transportation problem is employed by the optimization aim, while the dispersals of trips among sources, destinations, and conveyances are often ignored. In order to minimize both transportation penalties and maximize entropy value which guarantees uniform transportation of products from sources to destinations via conveyances, this paper holds entropy function of dispersals of trips among sources, destinations, and conveyances as a second objective function. Inside the construction of uncertainty theory, the uncertain entropy function for transportation models is first proposed here. Thus the model is turned into its crisp equivalent by using uncertainty theory, which can be solved by applying minimizing distance optimization method. Finally, a numerical experiment is given to illustrate the models.

Keywords: Uncertain entropy function, multi-item solid transportation problem, uncertain multiobjective optimization, uncertainty theory.

1 Introduction

The requirement of improvement of classical Transportation Problem (TP) to Solid Transportation Problem (STP) occurs, when different species of conveyances are possible during the transportation of commodities to minimize time as well as cost. TP introduced by Shell [1], and then Haley [2] presented to solve a solution procedure. The multi-item solid transportation problem (MISTP) is the complicated form of the STP. It comprises collections of conveyances to carry one or more goods. Nowadays, it has been studied by some researchers [3-8] and so on.

It is usually difficult to predict the precise penalties for carrying the cost with a time, amount of demands, capacities of different conveyances due to the uncertainty of MISTP. Therefore the probability distributions of the variables cannot be observed. Thus, we can need some specialists to determine the belief degrees that each event will happen. It is an imprecision which is associated to as human uncertainty. In other words, a conventional view is to handle belief degrees, like probability distributions. Liu [9] demonstrated that it is inappropriate to determine belief degrees, it will give rise to unacceptable ends.

In order to handle these types of uncertainties, Liu [10] presented uncertainty theory which is a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity and product measure axioms. Then the uncertainty theory was redefined by Liu[11] in 2010. Since then, studies on uncertainty theory have started to increase in

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both theory and practice. Moreover, it has been used in many areas such as game theory [12-15], finance [16,17,18], differential equation [19] , regression [20], optimization [21-23], graph theory [24-27] and so on.

STP with uncertain variables has been examined by some scholars. Cui and Sheng [28] presented its uncertain programming models. Zhang et al. [29] studied the fixed charge solid transportation problem applying uncertainty theory to the crisp STP. Chen et al.[30] presented its uncertain bi-criteria models. They studied an entropy based STP with Shannon entropy in [31]. Uncertain MISTP models for carrying multiple product were studied by Dalman [4] and Liu et al.[6].

Entropy is employed to produce a quantitative measure of the degree of uncertainty. Based on the Shannon entropy of random variables [32], fuzzy entropy was first introduced by Zadeh [33] to quantify the fuzziness, who represented the entropy of a fuzzy event as a weighted Shannon entropy. Based on the uncertainty theory, the concept of entropy of uncertain variables is proposed by Liu [34] in 2009 to characterize the uncertainty of uncertain variables resulting from information deficiency. Chen and Dai [35] investigated the maximum entropy principle of uncertainty distribution for uncertain variables. The entropy of a function of uncertain variables is presented by Dai and Chen [36]. Uncertain entropy function describes a measure of dispersals of trips from sources to destinations via conveyances. It is useful to minimize the uncertain transportation penalties as well as to maximize uncertain entropy amount. That ensures consistent distribution of commodities between origins and destinations via different conveyances. Therefore an entropy based MISTP in this paper is formulated. Thus uncertain entropy function of dispersal of trips between origins and destinations is considered as a second objective function. So the single objective MISTP transformed to multiobjective MISTP, and then it is solved by using some mathematical programming methods.

The rest of the paper is constructed as follows. Section 2 and Section 3 offer basic definitions and theorems for modeling an entropy based MISTP with uncertain variables. The structure of the MISTP is presented in Sect. 4. An entropy based model is developed in Sect 5. Section 6 includes numerical experiment. The study is concluded in Sect. 7.

2 Preliminary

In this section, basic definitions and theorems are given about the theory of uncertainty.

Definition 1. *(Liu [10]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}:\mathcal{L}\to[0,1]$ s referred to as an* uncertain measure *if it meets the following axioms:*

*Axiom 1. (*Normality Axiom*)* $\mathcal{M}\{\Gamma\}=1$;

*Axiom 2. (*Duality Axiom*)* $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}=1$ *for any* $\Lambda\in\mathcal{L}$;

*Axiom 3. (*Subadditivity Axiom*) For each numerable sequence of* $\{\Lambda_i\}\in\mathcal{L}$, *we obtain*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\}\le\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma,\mathcal{L},\mathcal{M})$ *is referred to as an* uncertainty space, *and each component* Λ *in* $\mathcal{L}$ *is referred to as an* event. *Also, in order to obtain an uncertain measure of compound event, a product uncertain measure is introduced by Liu [**?**] by the following product axiom:*

*Axiom 4. (*Product Axiom*) Let* $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ *be uncertainty spaces for* $k=1,2,\cdots$ *The product uncertain measure* $\mathcal{M}$ *is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\}=\bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}$$

where Λ_k *are arbitrarily chosen events from* $\mathcal{L}_k$ *for* $k=1,2,\cdots$*, respectively.*

Definition 2. *(Liu [10]) An* uncertain variable ξ *is a measurable function from an uncertainty space* $(\Gamma,\mathcal{L},\mathcal{M})$ *to the set of real numbers, i.e., for any Borel set B of real numbers, the set*

$$\{\xi\in B\}=\{\gamma\in\Gamma|\xi(\gamma)\in B\}$$

is an event.

Definition 3. *(Liu [10]) The* uncertainty distribution Φ *of an uncertain variable* ξ *is defined by*

$$\Phi(x)=\mathcal{M}\{\xi\le x\},\quad \forall x\in\Re.$$

Definition 4. *(Liu[10]) Let* ξ *be an uncertain variable. The* expected value *of* ξ *is defined by*

$$E[\xi]=\int_0^{+\infty}\mathcal{M}\{\xi\ge r\}\mathrm{d}r-\int_{-\infty}^0\mathcal{M}\{\xi\le r\}\mathrm{d}r$$

provided that at least one of the above two integrals is finite. *An uncertain variable* ξ *is called linear if it has a linear uncertainty distribution*

$$\Phi(x)=\begin{cases}0, & x\le a\\ (x-a)/(b-a), & a\le x\le b\\ 1, & x\ge b\end{cases}$$

denoted by $Ł(a,b)$ *where a and b are real numbers with* $a<b$*. Suppose that* ξ_1 *and* ξ_2 *are independent linear uncertain variables* $Ł(a_1,b_1)$ *and* $Ł(a_2,b_2)$*. Then the sum* $\xi_1+\xi_2$ *is also a linear uncertain variable* $Ł(a_1+a_2,b_1+b_2)$*.*

Definition 5. *(Liu[10]) Let* ξ *be an uncertain variable with a regular uncertainty distribution* $\Phi(x)$*. If the expected value is exist, then*

$$E[\xi]=\int_0^1\Phi^{-1}(\alpha)\mathrm{d}\alpha$$

where $\Phi^{-1}(\alpha)$ *is the* inverse uncertainty distribution *of* ξ*.*

Theorem 1. *(Liu [11]) Assume* $\xi_1,\xi_2,\cdots,\xi_n$ *are independent uncertain variables with regular uncertainty distributions* $\Phi_1,\Phi_2,\cdots,\Phi_n$*, respectively. If the function* $f(x_1,x_2,\cdots,x_n)$ *is strictly increasing with respect to* $x_1,x_2,\cdots,x_m$ *and strictly decreasing with respect to* $x_{m+1},x_{m+2},\cdots,x_n$*, then* $\xi=f(\xi_1,\xi_2,\cdots,\xi_n)$ *has an inverse uncertainty distribution*

$$\Psi^{-1}(\alpha)=f\left(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha)\right).$$

In addition, Liu and Ha [?] proved that the uncertain variable ξ *has an expected value*

$$E[\xi]=\int_0^1 f\left(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha)\right)\mathrm{d}\alpha.$$

Theorem 2. *(Liu[11]) Let ξ and η be independent uncertain variables with finite expected values. Hence, for any real numbers a and b, we have*

$$\mathrm{E}[a\xi+b\eta]=a\mathrm{E}[\xi]+b\mathrm{E}[\eta].$$

Theorem 3. *(Liu[22]) Let $g(\{x,\xi_1,\xi_2,\ldots,\xi_n)$ be constraint function. This function is strictly increasing with respect to $\xi_1,\xi_2,\ldots,\xi_k$ and strictly decreasing with respect to ξ_{k+1}. $\xi_1,\xi_2,\ldots,\xi_k$ are also independent uncertain variables with uncertain distributions $\Phi_1,\Phi_2,\ldots,\Phi_n$, respectively, then the chance constraint*

$$\mathscr{M}\{g(\{x,\xi_1,\xi_2,\ldots,\xi_n)\leq 0\}\geq\alpha$$

holds if and only if

$$g\left(\{x,\Phi_1^{-1}(\alpha),\ldots,\Phi_k^{-1}(\alpha),\right.\\ \left.\Phi_{k+1}^{-1}(1-\alpha),\ldots,\Phi_n^{-1}(1-\alpha)\right)\leq 0.$$

The definition and theory of uncertain entropy function is introduced by Liu [15], as follows.

Theorem 4. *(Liu [34]) Let ξ be an uncertain variable and its entropy is determined as:*

$$H[\xi]=\int_{-\infty}^{\infty}S(\mathscr{M}\{\xi\leq x\})\,\mathrm{d}x,$$

where $S(t)=-t\ln t-(1-t)\ln(1-t)$.

Theorem 5. *Let ξ be an uncertain variable with regular uncertainty distribution Φ. If the entropy $H[\xi]$ exists, then*

$$H[\xi]=\int_0^1\Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha.$$

Let's examine the entropy of uncertain variables with the examples given below.

Example 1. Let $\Phi^{-1}(\alpha)=(1-\alpha)a+\alpha b$ be the inverse uncertainty distribution for linear uncertain variable $\mathscr{L}(a,b)$. From Theorem 2.5, its entropy is determined as follows:

$$H=\int_0^1\Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha=\frac{b-a}{2}.$$

Example 2. Let

$$\Phi^{-1}(\alpha)=\begin{cases}(1-2\alpha)a+2\alpha b, & \text{if}\alpha<0.5\\ (2-2\alpha)b+(2\alpha-1)c, & \text{if}\alpha\geq 0.5.\end{cases}$$

be the inverse uncertainty distribution of zigzag uncertain variable $\mathscr{Z}(a,b,c)$. Then Theorem 2.5, its entropy can be obtained as follows:

$$H=\int_0^1\Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha=\frac{c-a}{2}.$$

Example 3. Similarly, let $\Phi^{-1}(\alpha)=e+\frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$ be the inverse uncertainty distribution of normal uncertain variable $\mathscr{N}(e,\sigma)$. Applying Theorem 2.5, its entropy obtained is as follows:

$$H=\int_0^1\Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha=\frac{\pi\sigma}{\sqrt{3}}.$$

3 Uncertain programming model for multi-item solid transportation problem

In order to model an entropy based MISTP with uncertain variables, the formulation of MISTP is first presented, then it is transformed into the entropy based MISTP model with uncertain variables.

In a MISTP, there is a multi product to be carried from a set of origins to a set of destinations by a set of both similar or distinct conveyances. Every origin has a state which provides any of the destinations employing some of the conveyances, and every destination can receive its demand from some of the origins employing some of the conveyances. Therefore each origin can provide zero, one or more destinations, and the demand for each destination can be joined by at least one origin. Each conveyance is employed from the origins to the destinations for zero, one or more unlocked ways. A unit price is considered for carrying any quantity of products between the origins and the destinations via distinct conveyances. In actual, the goal of a MISTP is to minimize the total transportation cost by receiving an optimal result of products which are communicated in the unlocked directions by distinct conveyances.

Here the following notations are employed in all models.

M : the number of origins,

N : the number of destinations,

L : the number of conveyances,

R : the number of items,

i, j, k, p : the indexes used for source, destination and conveyance, respectively.

a_i^p : the capacity of products of item p at origins i ,

b_j^p : the demand of products of item p at destination j,

e_k : the total transportation capacity of conveyance k,

c_{ijk}^p is the unit cost of transporting one unit of item p from source i to destination j by conveyance k,

x_{ijk}^p is the amount of item p to be carried from source i to destination j by conveyance k.

Under these notations, a conventional mathematical model of MISTP can be formulated as follows:

$$\begin{cases} f(x) = \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} c_{ijk}^p x_{ijk}^p & (a) \\ s.t. \begin{cases} \sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^p \leq a_i^p, \forall i \in M;\ \forall p \in R & (b) \\ \sum_{i=1}^{M}\sum_{k=1}^{L} x_{ijk}^p \geq b_j^p, \forall j \in N;\ \forall p \in R & (c) \\ \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{k=1}^{L} x_{ijk}^p \leq e_k, \forall k \in L & (d) \\ x_{ijk}^p \geq 0, \forall i \in M;\ \forall j \in N;\ \forall k \in L;\ \forall p \in R & (e) \end{cases} \end{cases} \tag{1}$$

The objective function (a) diminishes the total transportation cost which is the sum of each unit cost. Constraint (b) guarantee that the whole amount of product p carried from each origin to every destination should not be greater than the capacity of that origin. Constraint (c) explains that the demand for each destination should be answered. Constraint (d) shows the capacity of each conveyance, and constraint (e) represents nonnegative variables.

In order to obtain its uncertain programming model, let us consider that the per unit cost ξ_{ijk}^p, the capacity of each origin

$\tilde{a}_i^p$, that of each destination$\tilde{b}_j^p$ and each conveyance$\tilde{b}_j^p$ are all uncertain variables, respectively. Then uncertain programming model of the MISTP can be expressed as

$$\begin{cases} f(x,\xi) = \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p}\xi_{ijk}^{p} \\ s.t. \begin{cases} \sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p} \le \tilde{a}_i^p, \forall i \in M;\ \forall p \in R \\ \sum_{i=1}^{M}\sum_{k=1}^{L} x_{ijk}^{p} \ge \tilde{b}_j^p, \forall j \in N;\ \forall p \in R \\ \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{k=1}^{L} x_{ijk}^{p} \le \tilde{e}_k, \forall k \in L \\ x_{ijk}^{p} \ge 0, \forall i \in M;\ \forall j \in N;\ \forall k \in L;\ \forall p \in R \end{cases} \end{cases} \tag{2}$$

This problem can not be optimized in this form because it contains uncertain variables. In order to optimize the MISTP with uncertain variables given above, its expected value programming model should be formulated via chance constraint.

$$\begin{cases} E\left[f(x,\xi)\right] = \min \mathrm{E}\left[\sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L}\left(\xi_{ijk}^{p}x_{ijk}^{p}\right)\right] \\ s.t. \begin{cases} \mathrm{M}\left\{\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p} - \tilde{a}_i^p \le 0\right\} \ge \gamma_i^p,\ \forall i \in M; \forall p \in R \\ \mathrm{M}\left\{\tilde{b}_j - \sum_{k=1}^{M}\sum_{k=1}^{L} x_{ijk}^{p} \le 0\right\} \ge \beta_j^p,\ \forall j \in N; \forall p \in R \\ \mathrm{M}\left\{\sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{L} x_{ijk}^{p} - \tilde{e}_k \le 0\right\} \ge \delta_k, \forall k \in L \\ x_{ijk}^{p} \ge 0, \quad \forall i \in M; \forall j \in N; \forall k \in L; \forall p \in R \end{cases} \end{cases} \tag{3}$$

where α_i^p,β_j^p,δ_k, $\forall i \in M; \forall j \in N; \forall k \in L; \forall p \in R$ are show that the confidence levels of each of constraint.

Theorem 6. *Further, model (3) is converted into its equivalent form as:*

$$\begin{cases} \min \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p}\int_{0}^{1}\Phi_{\xi_{ijk}^{p}}^{-1}(\alpha)\,d\alpha \\ s.t. \begin{cases} \sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p} - \Phi_{\tilde{a}_i^p}^{-1}(1-\alpha_i^p) \le 0,\ \forall i \in M; \forall p \in R \\ \Phi_{\tilde{b}_j^p}^{-1}\left(\beta_j^p\right) - \sum_{k=1}^{M}\sum_{k=1}^{L} x_{ijk}^{p} \le 0,\ \forall j \in N; \forall p \in R \\ \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N} x_{ijk}^{p} - \Phi_{\tilde{c}_k}^{-1}(1-\delta_k) \le 0,\ \forall k \in L \\ x_{ijk}^{p} \ge 0,\ \forall i \in M; \forall j \in N; \forall p \in R \end{cases} \end{cases} \tag{4}$$

where $\xi_{ijk}^{p}, \tilde{a}_i^p, \tilde{b}_j^p, \tilde{e}_k, \forall ijk$ *are independent uncertain variables with uncertainty distributions* $\Phi_{\xi_{ijk}^{p}}, \Phi_{\tilde{a}_i^p}, \Phi_{\tilde{b}_j^p}, \Phi_{\tilde{e}_k}$.

Proposition 1. *Since* ξ_{ijk}^{p} *has a regular uncertainty distribution* Φ_{ijk}^{p}, *from Theorem 1 and Theorem 2, we write*

$$\mathrm{E}\left[\sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L}\left(\xi_{ijk}^{p}x_{ijk}^{p}\right)\right] = \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p}\mathrm{E}\left[\xi_{ijk}^{p}\right] = \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p}\int_{0}^{1}\Phi_{\xi_{ijk}^{p}}^{-1}.$$

Applying Theorem 3 to the constraints of model (3), we have

$$\mathrm{M}\left\{\sum_{j=1}^{N}\sum_{k=1}^{L}x_{ijk}^{p}-\tilde{a}_{i}^{p}\leq 0\right\}\geq\gamma_{i}^{p}\Leftrightarrow\sum_{j=1}^{N}\sum_{k=1}^{L}x_{ijk}^{p}-\Phi_{\tilde{a}_{i}^{p}}^{-1}(1-\alpha_{i}^{p})\leq 0,\forall i\in M;\forall p\in R$$
$$\mathrm{M}\left\{\tilde{b}_{j}-\sum_{k=1}^{M}\sum_{k=1}^{L}x_{ijk}^{p}\leq 0\right\}\geq\beta_{j}^{p}\Leftrightarrow\Phi_{\tilde{b}_{j}^{p}}^{-1}\left(\beta_{j}^{p}\right)-\sum_{k=1}^{M}\sum_{k=1}^{L}x_{ijk}^{p}\leq 0,\ \forall j\in N;\forall p\in R$$
$$\mathrm{M}\left\{\sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{L}x_{ijk}^{p}-\tilde{e}_{k}\leq 0\right\}\geq\delta_{k}\Leftrightarrow\sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}x_{ijk}^{p}-\Phi_{\tilde{c}_{k}}^{-1}\left(1-\delta_{k}\right)\leq 0,\forall k\in L$$

The above operations show that model (3) is equivalent to model (4).

4 Entropy based MISTP models with uncertain variables

In a conventional MISTP, some products are carried from origin to some distinct destinations through some conveyances by minimizing the total costs. At the same time, the cost is expensive without entropy function because we need to try to arrive all directions in a transportation network, if possible. It helps to reach all directions and reduce cost. This shows the necessity of the entropy function in the MISTP. Therefore an uncertain entropy function is presented here as an additional objective.

By employing Theorem 4 and Theorem 5, the objective function of model (4) can be converted to the following uncertain entropy function.

Lemma 1. *Suppose ξ_{ijk}^{p} is an uncertain variable with regular uncertainty distribution Φ_{ijk}^{p}. If $f:\mathscr{R}^{n}\to\mathscr{R}$ is a strictly increasing function with respect to x_{ijk}^{p} then the uncertain function $f(x,\xi)$ has an entropy, i.e.,*

$$H[x,\xi]=\sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L}x_{ijk}^{p}\int_{0}^{1}\Phi_{\xi_{ijk}^{p}}^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha$$

It is obvious that if $f:\mathscr{R}^{n}\to\mathscr{R}$ is a strictly decreasing function with respect to x_{ijk}^{p}, then the uncertain function $f(x,\xi)$ has an entropy

$$H[x,\xi]=\sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L}x_{ijk}^{p}\int_{0}^{1}\Phi_{\xi_{ijk}^{p}}^{-1}(1-\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha.$$

Proposition 2. *Since ξ_{ijk}^{p} has a regular uncertainty distribution Φ_{ijk}^{p}, we obtain*

$$H[x,\xi]=\sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L}x_{ijk}^{p}\int_{-\infty}^{\infty}f\left(\Phi_{\xi_{ijk}^{p}}(x)\right)dx$$

From Theorem 2.5, this equality can be rewriten as:

$$H[x,\xi]=\sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L}x_{ijk}^{p}\int_{-\infty}^{0}\int_{0}^{\Phi_{\xi_{ijk}^{p}}(x)}f'(\alpha)\,\mathrm{d}\alpha dx+\sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L}x_{ijk}^{p}\int_{-\infty}^{0}\int_{\Phi_{\xi_{ijk}^{p}}(x)}^{\infty}-f'(\alpha)\,\mathrm{d}\alpha dx$$

where $f'(\alpha) = (-\alpha\ln\alpha - (1-\alpha)\ln(1-\alpha))' = -\ln\frac{\alpha}{1-\alpha}$.

By applying the Fubini Theorem to the above function, we obtain

$$H[x,\xi] = \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p} \int_{-\infty}^{0}\int_{\Phi_{\xi_{ijk}^{p}}^{-1}(x)}^{0} f'(\alpha)\,\mathrm{d}\alpha dx + \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p} \int_{\Phi_{\xi_{ijk}^{p}}(0)}^{1}\int_{0}^{\Phi_{\xi_{ijk}^{p}}(x)} -f'(\alpha)\,\mathrm{d}\alpha dx.$$

$$= \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p} \int_{0}^{1} \Phi_{\xi_{ijk}^{p}}^{-1}(\alpha) f'(\alpha)\,\mathrm{d}\alpha = \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p} \int_{0}^{1} \Phi_{\xi_{ijk}^{p}}^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha.$$

Adding the entropy function to model (4), it turns to the following multiobjective model.

$$\begin{cases} \min \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p} \int_{0}^{1} \Phi_{\xi_{ijk}^{p}}^{-1}(\alpha)\, d\alpha \\ \max \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p} \int_{0}^{1} \Phi_{\xi_{ijk}^{p}}^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha \\ s.t. \begin{cases} \sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p} - \Phi_{\tilde{a}_i^p}^{-1}(1-\alpha_i^p) \le 0, \forall i \in M; \forall p \in R \\ \Phi_{\tilde{b}_j^p}^{-1}\left(\beta_j^p\right) - \sum_{k=1}^{M}\sum_{k=1}^{L} x_{ijk}^{p} \le 0, \forall j \in N; \forall p \in R \\ \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N} x_{ijk}^{p} - \Phi_{\tilde{c}_k}^{-1}(1-\delta_k) \le 0, \forall k \in L \\ x_{ijk}^{p} \ge 0, \forall i \in M; \forall j \in N; \forall p \in R \end{cases} \end{cases} \tag{5}$$

In the multiobjective MISTP model given above, the entropy function serves as a measure of dispersal of trips among origins, destinations, and conveyances. It results more valuable to produce minimum transportation costs with maximum entropy amount.

Example 4. Let us consider the MISTP function and suppose x_{ijk}^{p} are independent uncertain variables with linear distribution $L\left(a_{ijk}^{p}, b_{ijk}^{p}\right)$. Then the MISTP function has the following entropy function

$$H[x,\xi] = \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p} \int_{0}^{1} \Phi_{\xi_{ijk}^{p}}^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha = \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p}\left(\frac{b_{ijk}^{p}-a_{ijk}^{p}}{2}\right)$$

where

$$\Phi_{\xi_{ijk}^{p}}^{-1}(\alpha) = (1-\alpha)a_{ijk}^{p} + \alpha b_{ijk}^{p}.$$

Example 5. Suppose x_{ijk}^{p} are independent uncertain variables with with zigzag distribution $Z\left(a_{ijk}^{p}, b_{ijk}^{p}, c_{ijk}^{p}\right)$. Then the MISTP function has the following entropy function

$$H[x,\xi] = \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p} \int_{0}^{1} \Phi_{\xi_{ijk}^{p}}^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha = \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x_{ijk}^{p}\left(\frac{c_{ijk}^{p}-a_{ijk}^{p}}{2}\right)$$

where

$$\Phi^{-1}_{\xi^p_{ijk}}(\alpha) = \begin{cases} (1-2\alpha)\, a^p_{ijk} + 2\alpha b^p_{ijk}, & \text{if} \alpha < 0.5 \\ (2-2\alpha)\, b^p_{ijk} + (2\alpha - 1)\, c^p_{ijk}, & \text{if} \alpha \geq 0.5. \end{cases}$$

Example 6. Suppose x^p_{ijk} are independent uncertain variables with with normal uncertain distribution $N\left(e^p_{ijk}, \sigma^p_{ijk}\right)$. Then the MISTP function has the following entropy function

$$H[x,\xi] = \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x^p_{ijk} \int_0^1 \Phi^{-1}_{\xi^p_{ijk}}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha = \sum_{p=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{L} x^p_{ijk} \left(\frac{\pi \sigma^p_{ijk}}{\sqrt{3}}\right)$$

where $\Phi^{-1}_{\xi^p_{ijk}}(\alpha) = e^p_{ijk} + \frac{\sigma^p_{ijk}\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$.

4.1 Minimizing distance function

This method combines multiple objectives

$$(E[f_1(x,\xi)], E[f_2(x,\xi)], \cdots, E[f_q(x,\xi)])$$

employing the distance metric of any solution from the ideal solution $(E^*_1, E^*_2, ..., E^*_q)$ where E^*_i for each $i = 1, 2, ..., q$,are the optimal values of the $i-$th objective function.

$$\begin{cases} \min\limits_{x} \left(\sqrt{(E[f_1(x,\xi)] - E^*_1)^2 + (E[f_2(x,\xi)] - E^*_2)^2 +, \cdots, +(E[f_q(x,\xi)] - E^*_q)^2}\right) \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_j(x,\xi) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p \end{cases} \tag{6}$$

Theorem 7. *Let $x*$ be an optimal solution of model (6).Then $x*$ should a Pareto optimal solution of model (5).*

Proof of the above theorem is given by Dalman[4].

5 A numerical example

In order to show the employment of these models, we confer a numerical experiment in this section. Suppose that two products (items) are to be carried through two distinct conveyances (C_1, C_2) from three origins(O_1, O_2, O_3) to four destinations(D_1, D_2, D_3, D_4).

The schematic illustration for the considered MISTP is given with Fig. 1 where the hosts display all the potential directions for carrying two different products from the origins to the destinations via conveyances.

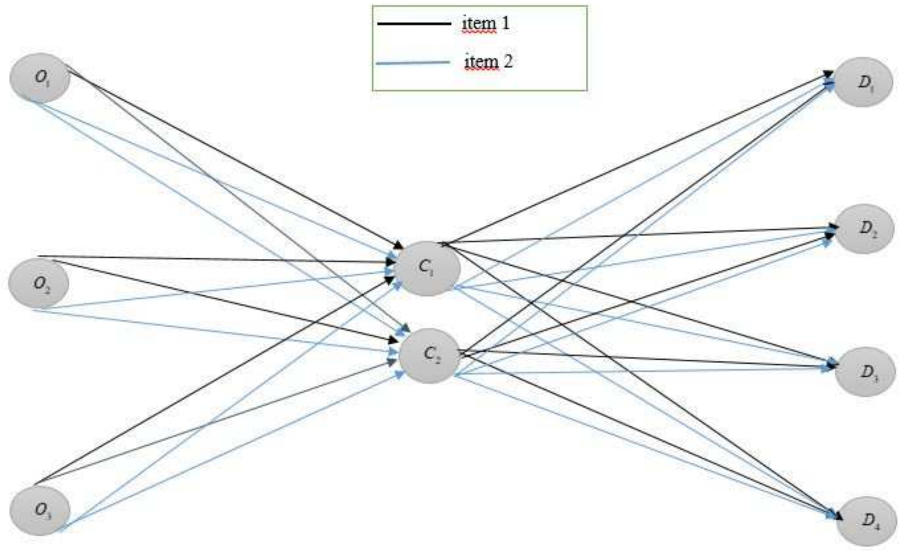

Fig. 1: The representation of network for the considered MISTP

Let us consider that variables of the objective function are normal uncertain variables and also, variables of constraints are linear uncertain variables, respectively. All data for the model is presented in Tables 1, 2, 3, 4 and 5, respectively.

$$\begin{aligned} &\xi_{ijk}^{p} \to \mathrm{N}\left(e_{ijk}^{p}, \sigma_{ijk}^{p}\right), && i \in [1,3];\, j \in [1,4];\, k \in [1,2];\, p \in [1,2] \\ &\tilde{a}_i^p \to L\left(a_i^p, b_i^p\right), && i \in [1,3];\, p \in [1,2] \\ &\tilde{b}_j^p \to L\left(b_j^p, b_j^p\right), && j \in [1,4];\, p \in [1,2] \\ &\tilde{e}_k \to L\left(b_j^p, b_j^p\right), && k \in [1,2] \end{aligned}$$

Table 1 :The transportation cost ξ_{ijk}^{p} for item $p=1$

ξ_{ij1}^{1}	1	2	3	4	ξ_{ij2}^{1}	1	2	3	4
1	$\mathcal{N}(10,2)$	$\mathcal{N}(9,1.5)$	$\mathcal{N}(12,2)$	$\mathcal{N}(8,1.5)$	1	$\mathcal{N}(7,2)$	$\mathcal{N}(5,1.5)$	$\mathcal{N}(5,2)$	$\mathcal{N}(6,1.5)$
2	$\mathcal{N}(8,1)$	$\mathcal{N}(9,1.5)$	$\mathcal{N}(11,2)$	$\mathcal{N}(10,1)$	2	$\mathcal{N}(5,1)$	$\mathcal{N}(6,1.5)$	$\mathcal{N}(4,2)$	$\mathcal{N}(6,1)$
3	$\mathcal{N}(8,1.5)$	$\mathcal{N}(17,1.5)$	$\mathcal{N}(6,1.5)$	$\mathcal{N}(10,1.5)$	3	$\mathcal{N}(4,1.5)$	$\mathcal{N}(7,1.5)$	$\mathcal{N}(6,1.5)$	$\mathcal{N}(5,1.5)$

Table 2 :The transportation cost ξ_{ijk}^{p} for item $p=2$

ξ_{ij1}^{2}	1	2	3	4	ξ_{ij2}^{2}	1	2	3	4
1	$\mathcal{N}(9,2)$	$\mathcal{N}(6,1.5)$	$\mathcal{N}(3,2)$	$\mathcal{N}(7,1.5)$	1	$\mathcal{N}(5,2)$	$\mathcal{N}(6,1.5)$	$\mathcal{N}(7,2)$	$\mathcal{N}(9,1.5)$
2	$\mathcal{N}(9,1)$	$\mathcal{N}(10,1.5)$	$\mathcal{N}(11,2)$	$\mathcal{N}(10,1)$	2	$\mathcal{N}(5,1)$	$\mathcal{N}(5,1.5)$	$\mathcal{N}(3,2)$	$\mathcal{N}(3,1)$
3	$\mathcal{N}(6,1.5)$	$\mathcal{N}(18,1.5)$	$\mathcal{N}(8,1.5)$	$\mathcal{N}(12,1.5)$	3	$\mathcal{N}(2,1.5)$	$\mathcal{N}(8,1.5)$	$\mathcal{N}(7,1.5)$	$\mathcal{N}(3,1.5)$

Table 3 The sources $\tilde{a}_i^p$

i	1	2	3		1	2	3
$\tilde{a}_i^1$	Ł(200,300)	Ł(250,350)	Ł(100,150)	$\tilde{a}_i^2$	Ł(100,150)	Ł(125,250)	Ł(110,160)

Table 4 :The demands $\tilde{b}_j^p$

j	1	2	3	4		1	2	3	4
$\tilde{b}_j^1$	Ł(75,125)	Ł(30,75)	Ł(15,35)	Ł(25,50)	$\tilde{b}_j^2$	Ł(55,100)	Ł(60,100)	Ł(20,30)	Ł(10,20)

Table 5 :The transportation capacities $\tilde{e}_k$

k	1	2
$\tilde{e}_k$	Ł(250,300)	Ł(350,500)

Following from the above data tables, expected value function for the MISTP is defined as;

$$\sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p}\int_{0}^{1}\Phi_{\xi_{ijk}^{p}}^{-1}(\alpha)\,d\alpha=\sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p}e_{ijk}^{p}$$

and from Lemma 5.1, it has an entropy

$$\sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p}\int_{0}^{1}\Phi_{\xi_{ijk}^{p}}^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}d\alpha=\sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p}\left(\frac{\pi\sigma_{ijk}^{p}}{\sqrt{3}}\right)$$

Asume that the expected value function and its entropy function are $f_1(x,\xi)$, $f_2(x,\xi)$,respectively. Because the aim of this problem is to reach maximum roots with minimum costs, then the expected value programming model for the entropy based MISTP model is formulated as follows:

$$\begin{cases} \mathrm{E}[f_1(x,\xi)]=\min\sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p}e_{ijk}^{p} \\ \mathrm{E}[f_2(x,\xi)]=\max\sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p}\left(\frac{\pi\sigma_{ijk}^{p}}{\sqrt{3}}\right) \\ s.t.\begin{cases} \sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p}-\Phi_{\tilde{a}_i^p}^{-1}(1-\alpha_i^p)\le 0, i\in[1,3]; p\in[1,2] \\ \Phi_{\tilde{b}_j^p}^{-1}\left(\beta_j^p\right)-\sum_{k=1}^{3}\sum_{k=1}^{2} x_{ijk}^{p}\le 0, j\in[1,4]; p\in[1,2] \\ \sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4} x_{ijk}^{p}-\Phi_{\tilde{c}_k}^{-1}(1-\delta_k)\le 0, k\in[1,2] \\ x_{ijk}^{p}\ge 0, i\in[1,3]; j\in[1,4]; k\in[1,2]; p\in[1,2] \end{cases} \end{cases} \tag{7}$$

Suppose that the confidence levels are $\gamma_i^p=0.9;\beta_j^p=0.9;\delta_k=0.9$ and $i=1,2.3, j=1,2,3,4, k=1,2, p=1,2$, respectively.

Reconstructing model(7) as a single objective programming problem under the system constraints neclecting the different objective, we obtain

$$\begin{cases} \min\sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p}e_{ijk}^{p} \\ s.t.\begin{cases} \sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p}-\Phi_{\tilde{a}_i^p}^{-1}(1-\alpha_i^p)\le 0, i\in[1,3]; p\in[1,2] \\ \Phi_{\tilde{b}_j^p}^{-1}\left(\beta_j^p\right)-\sum_{k=1}^{3}\sum_{k=1}^{2} x_{ijk}^{p}\le 0, j\in[1,4]; p\in[1,2] \\ \sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4} x_{ijk}^{p}-\Phi_{\tilde{c}_k}^{-1}(1-\delta_k)\le 0, k\in[1,2] \\ x_{ijk}^{p}\ge 0, i\in[1,3]; j\in[1,4]; k\in[1,2]; p\in[1,2] \end{cases} \end{cases} \tag{8}$$

$$\begin{cases} \max \sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p}\left(\frac{\pi\sigma_{ijk}^{p}}{\sqrt{3}}\right) \\ s.t. \begin{cases} \sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p} - \Phi_{\tilde{a}_i^p}^{-1}(1-\alpha_i^p) \le 0, i\in[1,3]; p\in[1,2] \\ \Phi_{\tilde{b}_j^p}^{-1}\left(\beta_j^p\right) - \sum_{k=1}^{3}\sum_{k=1}^{2} x_{ijk}^{p} \le 0, j\in[1,4]; p\in[1,2] \\ \sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4} x_{ijk}^{p} - \Phi_{\tilde{c}_k}^{-1}(1-\delta_k) \le 0, k\in[1,2] \\ x_{ijk}^{p} \ge 0, i\in[1,3]; j\in[1,4]; k\in[1,2]; p\in[1,2] \end{cases} \end{cases} \quad (9)$$

Solving problem (8), and (9),respectively, we obtain the following individual solutions.

$$\min \mathrm{E}[f_1] = 1235.5 \text{ and } \max \mathrm{E}[f_1] = 6869.500$$
$$\min \mathrm{E}[f_2] = 694.322 \text{ and } \max \mathrm{E}[f_2] = 2125.142$$

Applying the distance minimization method (6) to model (7), model (7) is reduced to the following single objective programmng problem.

$$\begin{cases} \min \sqrt{\left(\sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p}e_{ijk}^{p} - \mathrm{E}[f_1]\right)^2 + \left(\sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p}\frac{\pi\sigma_{ijk}^{p}}{\sqrt{3}} - \mathrm{E}[f_2]\right)^2} \\ s.t. \begin{cases} \sum_{j=1}^{4}\sum_{k=1}^{2} x_{ijk}^{p} - \Phi_{\tilde{a}_i^p}^{-1}(1-\alpha_i^p) \le 0, i\in[1,3]; p\in[1,2] \\ \Phi_{\tilde{b}_j^p}^{-1}\left(\beta_j^p\right) - \sum_{k=1}^{3}\sum_{k=1}^{2} x_{ijk}^{p} \le 0, j\in[1,4]; p\in[1,2] \\ \sum_{p=1}^{2}\sum_{i=1}^{3}\sum_{j=1}^{4} x_{ijk}^{p} - \Phi_{\tilde{c}_k}^{-1}(1-\delta_k) \le 0, k\in[1,2] \\ x_{ijk}^{p} \ge 0, i\in[1,3]; j\in[1,4]; k\in[1,2]; p\in[1,2] \end{cases} \end{cases} \quad (10)$$

where the confidence levels are $\gamma_i^p = 0.9; \beta_j^p = 0.9; \delta_k = 0.9$ and $i = 1,2.3$, $j = 1,2,3,4, k = 1,2, p = 1,2$, respectively.

Taking $\mathrm{E}[f_1] = 1235.5$ and $\mathrm{E}[f_2] = 2125.142$ in problem (10), it is solved to obtain the optimal solutions by using Maple 2017 optimization toolbox, and the the solutions obtained are as follows:

$$x_{122}^1 = 34.500,\ x_{141}^1 = 2, x_{142}^1 = 0.500,\ x_{232}^1 = 17,\ x_{312}^1 = 80,\ x_{342}^1 = 25,\ x_{131}^2 = 105,\ x_{222}^2 = 64,\ x_{232}^2 = 73.500,$$
$$x_{312}^2 = 59.500,\ x_{342}^2 = 11.$$

By putting these parameters into Model (7), the minimum cost obtained is $\mathrm{E}[f_1] = 1712$.

6 Conclusions

This paper presented the model of entropy based multi-item solid transportation problem with uncertain variables. Applying the expected value and chance-constrained programming to the given MISTP with uncertain variables, a deterministic model for the MISTP is first obtained. By taking the entropy of the objective function, the model is transformed into a multiobjective MISTP. Thanks to entropy are reached to all roots with minimum cost. In order to show its optimal solution, the distance minimization method is applied to the multiobjective model. Finally, a numerical experiment is given to demonstrate the applications of the models. The entropy function can be applied to different uncertain transportation models such as step fixed-charge, multiobjective and multilevel solid transportation and so on.

Compliance with ethical standards

The author declares that he has no conflict of interest. This article does not contain any studies with human participants performed by the author.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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