Communication in Mathematical Modeling and Applications
http://ntmsci.com/cmma

Chebyshev collocation method for the two-dimensional heat equation

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Received: 23 December 2017, Accepted: 30 January 2018
Published online: 20 March 2018.

Abstract: The purpose of this study is to apply the Chebyshev collocation method to the two- dimensional heat equation. The method converts the two-dimensional heat equation to a matrix equation, which corresponds to a system of linear algebraic equations. Error analysis and illustrative example is included to demonstrate the validity and applicability of the technique.

Keywords: Heat equation, Chebyshev collocation method, Chebyshev polynomial solutions, error analysis of collocation methods.

1 Introduction

Laplace's equation is one of the most significant equations in physics. It is the solution to problems in a wide variety of fields including thermodynamics and electrodynamics. Today, the theory of complex variables is used to solve problems of heat flow, fluid mechanics, aerodynamics, electromagnetic theory and practically every other field of science and engineering. A broad class of steady-state physical problems can be reduced to finding the harmonic functions that satisfy certain boundary conditions. The Dirichlet problem for the Laplace equation is one of the above mentioned problems.

The Dirichlet problem is to find a function $U(z)$ that is harmonic in a bounded domain $D \subset R^2$, is continuous up to the boundary ∂D of D, assumes the specified values $U_0(z)$ on the boundary ∂D , where $U_0(z)$ is a continuous function on ∂D. Let D be a rectangular region and ∂D is the boundary of D and can be formulated as

$$\nabla^2 U = 0, \qquad z \in D, \qquad U|_{z\in\partial D} = U_0(z). \tag{1}$$

Here, for a point (x,y) in the plane R^2, one takes the complex notation $z = x+yi$, $U(z) = U(x,y)$ and $U_0(z) = U_0(x,y)$ are real functions and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator. Similarly the Dirichlet problem for the Poisson equation can be formulated as

$$\nabla^2 U = h(z), \qquad z \in D, \qquad U|_{z\in\partial D} = U_0(z). \tag{2}$$

The Green function of the Dirichlet problem for the Laplace differential equation in a triangle region was expressed in terms of elliptic functions and the solution of problem was based on the Green function, and therefore on elliptic functions by Kurt and Sezer [1,2]. Solution of the two-dimensional heat equation in a square region was given by Kurt

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[3]. Analytic solution of two-dimensional heat equation was given for some regions by Baykuş Savaşaneril et al. [4,5,6,7]. The Chebyshev tau technique for the solution of Laplace's equation [8] and Chebyshev tau matrix method for Poisson-type equations in irregular domain [9] were studied by Ahmadi et al. and Kong et al. Error analysis of the Chebyshev collocation method for linear second-order partial differential equations was expressed by Yuksel et al. [10, 11]. Gas Dynamics Equation arising in shock fronts [12] and solution of conformable fractional partial differential equations by reduced differential transform method [13] were studied by Tamsir et al. and O. Acan et al. n-dimensional differential transformation method for solving PDEs is studied by Kurnaz et al. [14].

In this study, we find an approximate solution of Eq. (2) using a truncated Chebyshev series, such that

$$U(x,y)=\sum_{r=0}^{N}\sum_{s=0}^{N}a_{r,s}T_{r,s}(x,y), \tag{3}$$

where $T_{r,s}(x,y)=T_r(x)T_s(y)$ and $a_{r,s}$'s are unknown constants to be determined. Here, $T_r(x)$ and $T_s(y)$ denote the Chebyshev polynomials of degree r and s, respectively, defined by $T_r(x)=\cos(r\arccos(x))$ and $T_s(y)=\cos(s\arccos(y))$. We choose the collocation points as the extremes of the Chebyshev polynomials $T_r(x)$ and $T_s(y)$ as

$$x_n=\cos\left(\frac{(N-n)}{N}\right)\pi \quad \text{and} \quad y_l=\cos\left(\frac{(N-n)}{N}\right)\pi; \quad n,l=0,1,\dots,N \tag{4}$$

2 Fundamental relations

To find the numerical solution of the two-dimensional heat equation with the Chebyshev collocation method, it is necessary to evaluate the Chebyshev coefficients of the approximate solution. For convenience, Eq. (3) can be written in the matrix form [10].

The Chebyshev approximate solution of Eq. (2)

$$U(x,y)=\sum_{r=0}^{N}\sum_{s=0}^{N}a_{r,s}T_{r,s}(x,y)$$

can be written in a matrix from as

$$U(x,y)=\mathbf{T}(x)\mathbf{Q}(y)\overline{\mathbf{A}} \tag{5}$$

where

$$\mathbf{T}(x)=\begin{bmatrix}T_0(x) & T_1(x) & \cdots & T_N(x)\end{bmatrix}_{1\times(N+1)}$$

$$\mathbf{Q}(y)=\begin{bmatrix} T_0(y) & \cdots & T_N(y) & 0 & \cdots 0 \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & T_0(y) & 0 & 0 & & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots 0 \cdots & T_0(y) & \cdots & T_N(y) \end{bmatrix}_{(N+1)\times(N+1)^2}$$

and $\overline{\mathbf{A}}$ is the unknown Chebyshev coefficients matrix

$$\overline{\mathbf{A}}=\begin{bmatrix}a_{0,0} & a_{0,1} & \cdots & a_{0,N} & a_{1,0} & a_{1,1} & \cdots & a_{1,N} & \cdots & a_{N,0} & a_{N,1} & \cdots & a_{N,N}\end{bmatrix}_{(N+1)^2\times 1}$$

The $(i+j)$ th-ordered partial derivatives of $U(x,y)$ in Eq. (5) can be written as [10,11]

$$U(x,y)^{(i,j)}(x,y) = \mathbf{T}(x)(\mathbf{J}^T)^i\mathbf{Q}(y)(\overline{\mathbf{J}})^j\overline{\mathbf{A}} \tag{6}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2.2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 3 & 0 & 2.3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2.4 & 0 & 2.4 & 0 & \cdots & 0 & 0 \\ 5 & 0 & 2.5 & 0 & 2.5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 2N & 0 & 2N & 0 & \cdots & 2N & 0 \\ N & 0 & 2N & 0 & 2N & \cdots & 2N & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \quad \overline{\mathbf{J}} = \begin{bmatrix} J^T & 0 & \cdots & 0 \\ 0 & J^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J^T \end{bmatrix}_{(N+1)^2\times(N+1)^2}$$

The matrix form for the conditions is

$$\sum_{k=1}^{1}\sum_{i=0}^{1}\sum_{j=0}^{1} a_{i,j}^{k} U^{(i,j)}(\alpha_k,\beta_k) = \lambda_k. \tag{7}$$

3 Matrix solution of the problem

Each term in Eq. (1) can be given in the matrix equation by Eq. (6) [10,11]

$$\begin{aligned} &\mathbf{A}(x,y)\mathbf{T}(x)(\mathbf{J}^T)^2\mathbf{Q}(y)\overline{\mathbf{A}} + \mathbf{B}(x,y)\mathbf{T}(x)\mathbf{J}^T\mathbf{Q}(y)(\overline{\mathbf{J}})\overline{\mathbf{A}} + \mathbf{C}(x,y)\mathbf{T}(x)\mathbf{Q}(y)(\overline{\mathbf{J}})^2\overline{\mathbf{A}} \\ &+\mathbf{D}(x,y)\mathbf{T}(x)\mathbf{J}^T\mathbf{Q}(y)\overline{\mathbf{A}} + \mathbf{E}(x,y)\mathbf{T}(x)\mathbf{Q}(y)(\overline{\mathbf{J}})\overline{\mathbf{A}} + \mathbf{F}(x,y)\mathbf{T}(x)\mathbf{Q}(y)\overline{\mathbf{A}} = \mathbf{G}. \end{aligned} \tag{8}$$

By substituting the collocation points (4) into Eq. (8), we obtain the linear algebraic equation:

$$\begin{aligned} &\mathbf{A}(x_n,y_l)\mathbf{T}(x_n)(\mathbf{J}^T)^2\mathbf{Q}(y_l)\overline{\mathbf{A}} + \mathbf{B}(x_n,y_l)\mathbf{T}(x_n)\mathbf{J}^T\mathbf{Q}(y_l)(\overline{\mathbf{J}})\overline{\mathbf{A}} \\ &+\mathbf{C}(x_n,y_l)\mathbf{T}(x_n)\mathbf{Q}(y_l)(\overline{\mathbf{J}})^2\overline{\mathbf{A}} + \mathbf{D}(x_n,y_l)\mathbf{T}(x_n)\mathbf{J}^T\mathbf{Q}(y_l)\overline{\mathbf{A}} \\ &+\mathbf{E}(x_n,y_l)\mathbf{T}(x_n)\mathbf{Q}(y_l)(\overline{\mathbf{J}})\overline{\mathbf{A}} + \mathbf{F}(x_n,y_l)\mathbf{T}(x_n)\mathbf{Q}(y_l)\overline{\mathbf{A}} = \mathbf{G}(x_n,y_l). \end{aligned} \tag{9}$$

The fundamental matrix equation in Eq. (9) is as follows:

$$(\mathbf{AT}(\mathbf{J}^T)^2\mathbf{Q}\overline{\mathbf{A}} + \mathbf{BTJ}^T\mathbf{Q}(\overline{\mathbf{J}})\overline{\mathbf{A}} + \mathbf{CTQ}(\overline{\mathbf{J}})^2\overline{\mathbf{A}} + \mathbf{DTJ}^T\mathbf{Q}\overline{\mathbf{A}} + \mathbf{ETQ}(\overline{\mathbf{J}})\overline{\mathbf{A}} + \mathbf{FTQ}\overline{\mathbf{A}}) = \mathbf{G}. \tag{10}$$

where, Eq. (10) corresponds to a system of $(N+1)^2$ linear algebraic equations with unknown Chebyshev coefficients $a_{0,0}, a_{0,1}, \ldots, a_{0,N}, a_{1,0}, a_{1,1}, \ldots, a_{1,N}, \ldots, a_{N,0}, a_{N,1}, \ldots, a_{N,N}$. Also we can write Eq. (10) such that

$$\underbrace{(\mathbf{AT}(\mathbf{J}^T)^2\mathbf{Q} + \mathbf{BTJ}^T\mathbf{Q}(\overline{\mathbf{J}}) + \mathbf{CTQ}(\overline{\mathbf{J}})^2 + \mathbf{DTJ}^T\mathbf{Q} + \mathbf{ETQ}(\overline{\mathbf{J}}) + \mathbf{FTQ})}_{\mathbf{W}}\overline{\mathbf{A}} = \mathbf{G}. \tag{11}$$

which can be written simply as

$$\mathbf{W}\overline{\mathbf{A}} = \mathbf{G}. \tag{12}$$

Similarly, by substituting the collocation points (4) into Eq. (7) under the complicated conditions, we obtain, respectively,

$$(\sum_{k=1}^{t}\sum_{i=0}^{1}\sum_{j=0}^{1} a_{i,j}^{k}\mathbf{T}(a_k)(\mathbf{J}^T)^i\mathbf{Q}(\beta_k)(\bar{\mathbf{J}})^j)\overline{\mathbf{A}} = \lambda_k,$$

or shortly

$$\mathbf{V}\,\bar{\mathbf{A}} = \lambda_k. \tag{13}$$

To obtain the Chebyshev series solution of Eq. (2) under conditions (7), the augmented matrix form of Equations (12) and (13) is as follows

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] = \begin{bmatrix} \mathbf{V} \; ; \; \lambda_k \\ \mathbf{W} \; ; \; \mathbf{G} \end{bmatrix}. \tag{14}$$

Therefore, the unknown Chebyshev coefficients are obtained as

$$\bar{\mathbf{A}} = \left(\tilde{\tilde{\mathbf{W}}}\right)^{-1}\tilde{\tilde{\mathbf{G}}}, \tag{15}$$

where $\left[\tilde{\tilde{\mathbf{W}}};\tilde{\tilde{\mathbf{G}}}\right]$ is generated by using the Gauss elimination method and then by removing zero rows of the augmented matrix $\left[\tilde{\tilde{\mathbf{W}}};\tilde{\tilde{\mathbf{G}}}\right]$. The reason for using the Gauss elimination for this direct solution is because of the non-invertible case of the matrix $\tilde{\tilde{\mathbf{W}}}$. When the conditions are added to the linear algebraic system, some rows can be the same because of the symmetry of Chebyshev collocation points. These terms can be eliminated by the Gauss elimination method.

4 Accuracy of the solution and error analysis

We can easily check the accuracy of the method. Since the truncated Chebyshev series given in Eq. (3) is an approximate solution of Eq. (2), when the function $U(x,y)$ and its derivatives are substituted in Eq.(2),the resulting equation must be satisfied approximately; that is, for $(x,y) = (x_q,y_q) \in \{-a \le x_q \le a, \quad -b \le y_q \le b\}$ $q = 0,1,2,\ldots$ $E(x_q,y_q) = \left|D(x_q,y_q) - \lambda I(x_q,y_q)\right| \cong 0$ and $E(x_q,y_q) \le 10^{-k_q}$ (k_q positive integer) If max $10^{-k_q} = 10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_q,y_q)$ at each of the points becomes smaller than the prescribed 10^{-k}. On the other hand,the error can be estimated by the function

$$E_N = \sum_{r=0}^{N}\sum_{s=0}^{N} a_{r,s}T_{r,s}(x,y), -g(x,y) - I(x,y). \tag{16}$$

If $E_N(x,y) \to 0$ when N is sufficiently large enough, then the error decreases.

5 Numerical example

In this section, the efficiency of the method is demonstrated with the numerical result of example [7] and it has been solved by a computer code written in Maple.

The boundary of a rectangular sheet of metal is kept at constant temperature 50^0C on the upper edge, 20^0C on the bottom edge, and 0^0C on the other two edges. After a sufficient period of time, the temperature inside the plate reaches an equilibrium distribution. This steady-state temperature distribution $U(x,y)$ is determined in this application. Since no heat sources are present in the plate, the steady-state temperature U must satisfy

$$U_{xx}(x,y)+U_{yy}(x,y)=0; \qquad \left(-\frac{K}{2\lambda}\le x\le\frac{K}{2\lambda}, \quad -\frac{K'}{2\lambda}\le y\le\frac{K'}{2\lambda}\right) \tag{17}$$

The boundary conditions are

$$\begin{aligned} &U\left(x,-\tfrac{K'}{2\lambda}\right)=20^0C, \quad U\left(x,\tfrac{K'}{2\lambda}\right)=50^0C; \quad -\tfrac{K}{2\lambda}\le x\le\tfrac{K}{2\lambda} \\ &U\left(-\tfrac{K}{2\lambda},y\right)=U\left(\tfrac{K}{2\lambda},y\right)=0^0C; \quad -\tfrac{K'}{2\lambda}\le y\le\tfrac{K'}{2\lambda} \end{aligned} \tag{18}$$

We obtain the approximate solutions for $N=5,7,9$ which are based on the truncated double Chebyshev series $U(x,y)=\sum_{r=0}^{N}\sum_{s=0}^{N}a_{r,s}T_{r,s}(x,y)$ on rectangular domain$-\frac{K}{2\lambda}\le x\le\frac{K}{2\lambda}, \quad -\frac{K'}{2\lambda}\le y\le\frac{K'}{2\lambda}$.

$$(\mathbf{T}(x_i)(\mathbf{J}^T)^2\mathbf{Q}(y_l)+\mathbf{T}(x_i)\mathbf{Q}(y_l)(\overline{\mathbf{J}})^2)\overline{\mathbf{A}}=0, \quad i,l=0,...,N.$$

where

$$\mathbf{T}(\mathbf{J}^T)^2\mathbf{Q}+\mathbf{T}\mathbf{Q}(\overline{\mathbf{J}})^2=\mathbf{W}, \mathbf{W}\overline{\mathbf{A}}=0$$

Matrix forms of the conditions can be written as

$$U\left(x_i,-\tfrac{K'}{2\lambda}\right)=\mathbf{T}(x_i)\mathbf{Q}\left(-\tfrac{K'}{2\lambda}\right)\overline{\mathbf{A}}=20$$

$$U\left(x_i,\tfrac{K'}{2\lambda}\right)=\mathbf{T}(x_i)\mathbf{Q}\left(\tfrac{K'}{2\lambda}\right)\overline{\mathbf{A}}=50$$

and

$$U\left(\pm\frac{K}{2\lambda},y_l\right)=\mathbf{T}(\pm\frac{K}{2\lambda})\,\mathbf{Q}\,(y_l)\,\overline{\mathbf{A}}=0$$

and fundamental matrix equations of the conditions can be written as

$$TQ\left(-\tfrac{K'}{2\lambda}\right)\overline{A}=20; K_1\overline{A}=20$$

$$TQ\left(\tfrac{K'}{2\lambda}\right)\overline{A}=50; K_2\overline{A}=50$$

$$T\left(-\tfrac{K}{2\lambda}\right)Q\overline{A}=0; K_3\overline{A}=0$$

$$T\left(\tfrac{K}{2\lambda}\right)Q\overline{A}=0; K_4\overline{A}=0$$

To obtain the solution of Eq. (17) under conditions (18), the augmented matrix is formed as follows

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]=\begin{bmatrix}\mathbf{K}_1\,; & 20\\ \mathbf{K}_2\,; & 50\\ \mathbf{K}_3\,; & 0\\ \mathbf{K}_4\,; & 0\\ \mathbf{W}\,; & 0\end{bmatrix}$$

Therefore, the unknown Chebyshev coefficients are obtained as

$$\bar{\mathbf{A}} = \left(\tilde{\tilde{\mathbf{W}}}\right)^{-1} \tilde{\tilde{\mathbf{G}}}.$$

for $N = 5$;

$$T = \begin{bmatrix} 1 & x & 2x^2-1 & 4x^3-3x & 8x^4-8x^2+1 & 16x^5-20x^3+5x \end{bmatrix}_{1\times 6}$$

$$Q = \begin{bmatrix} T & 0 & 0 & 0 & 0 & 0 \\ 0 & T & 0 & 0 & 0 & 0 \\ 0 & 0 & T & 0 & 0 & 0 \\ 0 & 0 & 0 & T & 0 & 0 \\ 0 & 0 & 0 & 0 & T & 0 \\ 0 & 0 & 0 & 0 & 0 & T \end{bmatrix}_{6\times 36}, \quad J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 3 & 0 & 6 & 0 & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & 0 \\ 5 & 0 & 10 & 0 & 10 & 0 \end{bmatrix}_{6\times 6}, \quad \bar{J} = \begin{bmatrix} J^T & 0 & 0 & 0 & 0 & 0 \\ 0 & J^T & 0 & 0 & 0 & 0 \\ 0 & 0 & J^T & 0 & 0 & 0 \\ 0 & 0 & 0 & J^T & 0 & 0 \\ 0 & 0 & 0 & 0 & J^T & 0 \\ 0 & 0 & 0 & 0 & 0 & J^T \end{bmatrix}_{36\times 36}$$

Error analysis for the equation in (17) can be seen in Fig. 1.

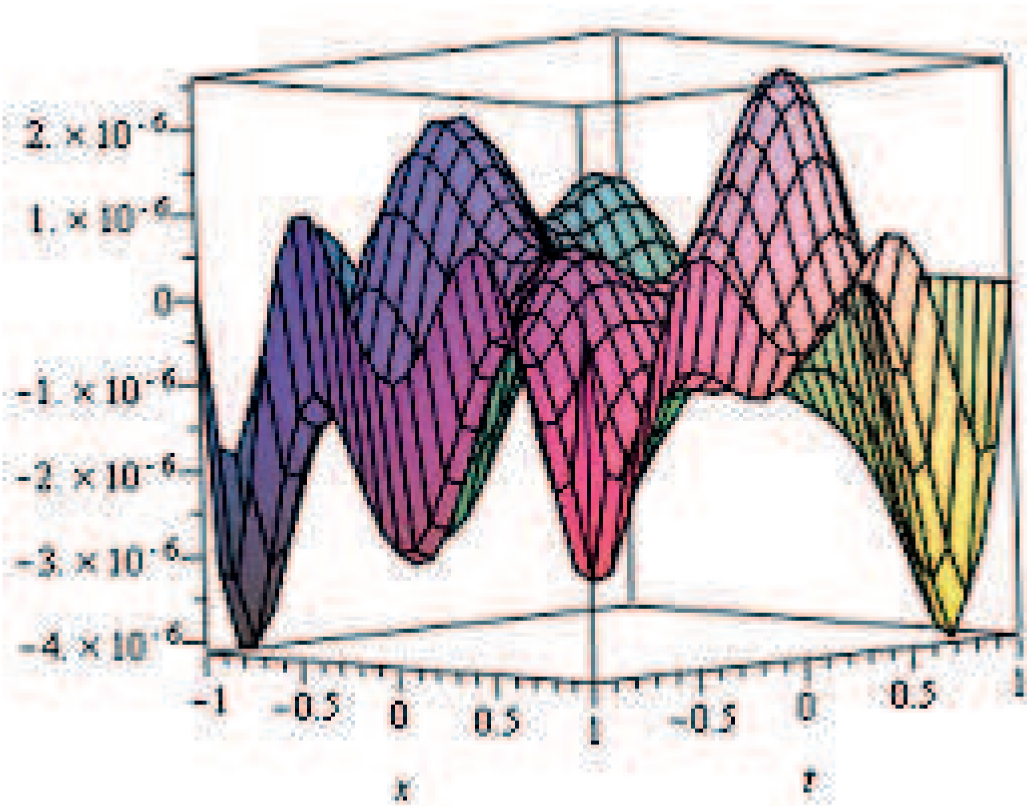

Fig. 1: Error analysis for $N = 9$.

From Table 1, it is obvious that the results get better as N increase.

Table 1: Comparison of the error analysis on ∂D that is the boundary of D for N=5,7,9.

x	y	N=5	N=7	N=9
0	-1	-3.6060×10^{-5}	-8.3312×10^{-7}	2.6254×10^{-6}
0	-0.8	-1.2322×10^{-4}	-2.6451×10^{-7}	9.4651×10^{-7}
0	-0.6	-6.1317×10^{-5}	-2.4327×10^{-7}	2.6016×10^{-7}
0	-0.4	-2.2821×10^{-4}	8.7531×10^{-7}	3.7960×10^{-8}
0	-0.2	1.0496×10^{-4}	2.6179×10^{-7}	1.3644×10^{-11}
0	0	8.6320×10^{-5}	-6.9187×10^{-7}	1.5891×10^{-9}
0	0.2	-2.3897×10^{-5}	-8.7441×10^{-7}	-6.1521×10^{-10}
0	0.4	-1.9570×10^{-4}	-4.1512×10^{-7}	1.0216×10^{-8}
0	0.6	-3.5767×10^{-4}	-4.0661×10^{-7}	3.6733×10^{-8}
0	0.8	8.6320×10^{-5}	-1.3148×10^{-6}	-1.1373×10^{-8}
0	1	-1.4303×10^{-4}	-4.67×10^{-10}	-3.2615×10^{-7}

The some calculating values of the error functions give in Table 2 that is clearly shown when N values increase, error function values rapidly decrease for $N = 5,7$ and 9.

Table 2: Comparison of the error and the solution for of N=5,7,9.

		N=5		N=7		N=9	
x	y	$E(x_q,y_q)$	$U(x,y)$	$E(x_q,y_q)$	$U(x,y)$	$E(x_q,y_q)$	$U(x,y)$
1	1	3.11×10^{-10}	1.0361581	-4.09×10^{-6}	0.6914707	2.35×10^{-5}	0.6897134
0.8	0.9	1.42×10^{-4}	1.0721373	-3.73×10^{-6}	0.7140524	-8.48×10^{-6}	0.6950974
0.6	0.7	4.69×10^{-5}	1.0418738	-1.22×10^{-6}	0.6865901	1.67×10^{-6}	0.6413776
0.5	0.5	4.37×10^{-5}	0.9844465	1.99×10^{-6}	0.6394523	-3.88×10^{-7}	0.5732205
0.4	0.3	6.78×10^{-5}	0.9363271	1.80×10^{-6}	0.5979623	-5.37×10^{-8}	0.5113068
0	0	8.6320×10^{-5}	0.8939214	-6.91×10^{-7}	0.5540799	1.58×10^{-9}	0.4361256
-0.1	-0.2	1.54×10^{-4}	0.8611805	7.20×10^{-7}	0.5217932	5.93×10^{-10}	0.3868576
-0.5	-0.3	3.79×10^{-4}	0.8089201	1.54×10^{-6}	0.4833937	1.78×10^{-7}	0.3521935
-0.8	-0.5	1.99×10^{-4}	0.7337253	-1.20×10^{-6}	0.4264064	7.51×10^{-7}	0.2979922
-1	-1	7.37×10^{-10}	0.7066744	-4.88×10^{-10}	0.3834902	5.71×10^{-6}	0.2412991

6 Conclusion

In this study, a technique has been developed for solving Laplace's equation with Dirichlet boundary condition. The method is based upon Chebyshev collocation method. The Chebyshev polynomials are utilized to solve the problem effectively. The method leads to solving a system of linear algebraic equations. The numerical result show that the accuracy improves with increasing the N. Tables and figure indicate that as N increases the errors decrease more rapidly; hence for better results, using large number N is recommended.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] N. Kurt, M. Sezer, A. Çelik, Solution of Dirichlet problem for a rectangular region in terms of elliptic functions, J. Comput. Math., 81, (2004), 1417-1426.

[2] N. Kurt, M. Sezer, Solution of Dirichlet problem for a triangle region in terms of elliptic functions, Appl.Math. Comput., 182, (2006), 73-78.

[3] N.Kurt, Solution of the two-dimensional heat equation for a square in terms of elliptic functions, Journal of the Franklin Institute, 345(3), (2007), 303-317.

[4] N. Baykuş Savaşaneril, H. Delibas, Analytic solution for two-dimensional heat equation for an ellipse region. NTMSCI 4(1) (2016) 65-70

[5] N. Baykuş Savaşaneril, H. Delibas, Analytic Solution for The Dirichlet Problem in 2-D Journal of Computational and Theoretical Nanoscience, ACCEPTED.

[6] Z. Hacıoglu, N. Baykuş Savaşaneril, H. Köse, Solution of Dirichlet problem for a square region in terms of elliptic functions, NTMSCI, 3(4), (2015), 98-103.

[7] E. Kurul, N. Baykuş Savaşaneril, Solution of the two-dimensional heat equation for a rectangular plate, NTMSCI, 3(4), (2015), 76-82.

[8] M.R. Ahmadi, H. Adibi, The Chebyshev tau technique for the solution of Laplace's equation, Applied Mathematics and Computation, 184(2), (2007), 895-900.

[9] W. Kong, X. Wu, Chebyshev tau matrix method for Poisson-type equations in irregular domain, Journal of Computational and Applied Mathematics, 228(1), (2009), 158-167.

[10] G. Yüksel, O.R. Işık, M. Sezer, Error analysis of the Chebyshev collocation method for linear second-order partial differential equations, International Journal of Computer Mathematics, (2014), http://dx.doi.org/10.1080/00207160.2014.966099

[11] G. Yüksel, Chebyshev polynomials solutions of second order linear partial differential equations, Ph.D. Thesis, Mugla University, Mugla, (2011).

[12] M. Tamsir, O. Acan, J. Kumar, A. Singh, Numerical Study of Gas Dynamics Equation arising in Shock Fronts, Asia Pacific J. Eng. Sci. Technol., 2 (2016), 17-25

[13] O. Acan, O. Firat, Y. Keskin, G. Oturanç, Solution of Conformable Fractional Partial Differential Equations by Reduced Differential Transform Method, Selcuk J. Appl. Math., (2016) (In press).

[14] A. Kurnaz, G. Oturanç, M.E. Kiris, n-dimensional differential transformation method for solving PDEs, International Journal of Computer Mathematics, 82(3), (2005), 369-380.