An Application of Galerkin Method to Generalized Benjamin-Bona-Mahony-Burgers Equation

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Abstract

In this study, Galerkin finite element method is applied to generalized Benjamin-Bona-Mahony-Burgers (gBBM-B) equation. Quadratic B-spline functions are used as interpolation function. Stability analysis is investigated based on von Neumann theory. The performance of the proposed method is checked by two test problems with zero boundary conditions. As a result, it is observed that applied method is successful and efficient to show motions of some waves.

Keywords: Benjamin-Bona-Mahony-Burgers equation, Finite element, Galerkin method.

Genelleştirilmiş Benjamin-Bona-Mahony Denklemine Galerkin Metodunun Bir Uygulaması

Özet

Bu çalışmada, genelleştirilmiş Benjamin-Bona-Mahony (gBBM-B) denklemine Galerkin sonlu elemanlar yöntemi uygulanmıştır. İnterpolasyon fonksiyonu olarak kuadratik B-spline fonksiyonlar kullanılmıştır. Von Neumann teorisine bağlı olarak kararlılık analizi incelenmiştir. Sıfır sınır şartları ile iki test problemi yardımıyla önerilen yöntemin

performansı kontrol edilmiştir. Sonuç olarak, uygulanan yöntemin bazı dalga hareketlerini göstermek için başarılı ve etkili olduğu gözlenmiştir.

Anahtar Kelimeler: Benjamin-Bona-Mahony-Burgers denklemi, Sonlu eleman, Galerkin metodu.

Introduction

The dynamics of shallow water waves is a seriously growing research area in the area of fluid dynamics. There are various models to describe the shallow water waves along lakeshores and beaches. These are Korteweg-de Vries (KdV) equation [1-3], Kawahara equation [4], equal width (EW) wave equation [5, 6], regularized long wave (RLW) equation [7, 8], Burgers equation [9, 10] and others [11, 12]. This paper will focus on the generalized Benjamin-Bona-Mahony-Burgers (gBBM-B) equation with following form

$$u_t - \mu u_{xxt} - \alpha u_{xx} + \beta u_x + \sigma u^p u_x = 0. \tag{1}$$

where $u(x,t)$ represents fluid velocity in the horizontal direction x, u_t is evolution term, u_x is dispersion term, u_{xx} is dissipative term and $u^p u_x$ is nonlinear term [13]. Also, α, β, μ, σ are arbitrary constants, p is positive integer and α satisfying

$$\alpha = -\frac{p}{2}\left[-\frac{p^2(p+4)a_1\sqrt{-\dfrac{c^2(p+2)\mu\sigma\left(2\beta(p+1)(p+2)\mu c^2 + p^2\sigma a_1^2\right)}{p^4(p+1)^2}}}{\sqrt{2}c^3(p+2)^2\mu} \right]. \tag{2}$$

Many analytical and numerical methods including finite difference method [14], meshless method [15], tanh method [16], homotopy analysis method [17], exp-function method [18], tan-cot function method [19], collocation method [20] were applied to BBM-B equation in the past decades.

The organization of the paper is as follows: In Section 1, we describe quadratic B-spline interpolation function and its some properties. In Section 2, the gBBM-B equation is

considered and obtained its numerical solution using Galerkin finite element method based on quadratic B-splines. In Section 3, stability of the applied method is investigated based on von Neumann theory. In Section 4, numerical simulation is performed to check the performance of the proposed method. Finally, a brief conclusion is given in Section 5.

1. Quadratic B-splines and Properties

In this study, gBBM-B equation (1) is considered where α, β, μ and σ are arbitrary parameters, p is positive integer and the subscripts x and t denote the spatial and temporal differentiations, respectively.

Firstly, solution domain of the problem is restricted over an interval $a \leq x \leq b$. Space interval $[a,b]$ is separated into uniformly sized finite elements of length h by the knots x_m like that $a = x_0 < x_1 < ... < x_N = b$. Lengths of these finite elements are $h = \dfrac{b-a}{N} = (x_{m+1} - x_m)$ for $m = 1, 2, ..., N$.

Eq. (1) is solved by taking

$$\begin{aligned} u(a,t) = 0, \ u(b,t) = 0, \\ u_x(a,t) = 0, u_x(b,t) = 0 \quad , \quad t > 0 \end{aligned} \tag{3}$$

homogeneous boundary conditions and

$$u(x,0) = f(x) \quad , \quad a \leq x \leq b. \tag{4}$$

the initial condition.

The cubic B-spline approximation functions $\phi_m(x)$ are defined as

$$\phi_m(x) = \frac{1}{h^2} \begin{cases} (x_{m+2} - x)^2 - 3(x_{m+1} - x)^2 + 3(x_m - x)^2, & [x_{m-1}, x_m], \\ (x_{m+2} - x)^2 - 3(x_{m+1} - x)^2, & [x_m, x_{m+1}], \\ (x_{m+2} - x)^2, & [x_{m+1}, x_{m+2}], \\ 0, & elsewhere. \end{cases} \tag{5}$$

at the knots x_m are defined over the interval $[a,b]$ for $m = -1(1)N$ [21]. All spline

functions apart from $\phi_{m-1}(x)$, $\phi_m(x)$ and $\phi_{m+1}(x)$ are zero over the element $[x_m, x_{m+1}]$. Each quadratic B-spline function covers three elements so that each element $[x_m, x_{m+1}]$ is covered by three spline functions. The values of $\phi_m(x)$ and its first derivative may be tabulated as in Table 1.

Table 1. Quadratic B-splines and its derivatives at nodes x_m

x	x_{m-1}	x_m	x_{m+1}	x_{m+2}
$\phi_m(x)$	0	1	1	0
$h\phi_m'(x)$	0	-2	2	0

The set of these approximation functions $\{\phi_{-1}(x), \phi_0(x), \ldots, \phi_N(x)\}$ forms a basis for approximate solution which will be defined over $[a, b]$. A global approximation is signified in terms of the quadratic B-spline approximation functions as

$$u_N(x, t) = \sum_{i=-1}^{N} \phi_i(x) \delta_i(t), \qquad (6)$$

where $\delta_i(t)$ are time dependent parameters to be determined from the boundary and weighted residual conditions. A typical finite interval $[x_m, x_{m+1}]$ can be converted to the interval $[0, 1]$ by a local coordinate transformation defined as

$$h\xi = x - x_m \quad , \quad 0 \leq \xi \leq 1 \qquad (7)$$

In this way, it can be studied more easily on the defined new interval $[0, 1]$. In this case, quadratic B-spline functions (5) can be defined in terms of ξ over the interval as the following.

$$\phi_{m-1}(\xi) = (1 - \xi)^2,$$
$$\phi_m(\xi) = 1 + 2\xi - 2\xi^2, \qquad (8)$$
$$\phi_{m+1}(\xi) = \xi^2.$$

Variation of the function $u(x, t)$ over element $[x_m, x_{m+1}]$ is approximated by

$$u_N(\xi,t) = \sum_{i=m-1}^{m+1} \phi_i(\xi)\delta_i(t), \qquad (9)$$

where δ_{m-1}, δ_m, δ_{m+1} act as element parameters and B-splines ϕ_{m-1}, ϕ_m, ϕ_{m+1} as element shape functions.

Using trial function (6) and quadratic splines (5), the values of u and u' at the knots are determined in terms of the element parameters δ_m by

$$u_N(x_m,t) = u_m = \delta_{m-1} + \delta_m,$$

$$u'_m = \frac{2}{h}(-\delta_{m-1} + \delta_{m+1}).$$

where the symbol "$'$" denotes first differentiation with respect to x. The splines $\phi_m(x)$ and its first derivative vanish outside the interval $[x_{m-1}, x_{m+2}]$.

2. Implementation of The Method

By applying the Galerkin method to the Eq. (1) with weight function $w(x)$, the weak form of Eq. (1) is obtained as

$$\int_a^b w(u_t - \mu u_{xxt} - \alpha u_{xx} + \beta u_x + \sigma u^p u_x)dx = 0. \qquad (10)$$

For a single element $[x_m, x_{m+1}]$, the following equation is obtained by using transformation (7) into the equation (10)

$$\int_0^1 w\left[u_t - \left(\frac{1}{h^2}\right)\mu u_{\xi\xi t} - \left(\frac{1}{h^2}\right)\alpha u_{\xi\xi} + \left(\frac{1}{h}\right)\beta u_\xi + \left(\frac{1}{h}\right)\sigma u^p u_\xi\right]d\xi = 0. \qquad (11)$$

Also, Eq. (11) can be written as

$$\int_0^1 w\left[u_t - a_1 u_{\xi\xi t} - a_2 u_{\xi\xi} + a_3 u_\xi + a_4 u^p u_\xi\right]d\xi = 0. \qquad (12)$$

where $a_1 = \dfrac{\mu}{h^2}$, $a_2 = \dfrac{\alpha}{h^2}$, $a_3 = \dfrac{\beta}{h}$, $a_4 = \dfrac{\sigma}{h}$ and $z_m = u^p$. Integrating Eq. (12) by parts and using Eq. (1) lead to

$$\int_0^1 [wu_t + a_1 w_\xi u_{\xi t} + a_2 w_\xi u_\xi + a_3 wu_\xi + a_4 z_m wu_\xi] d\xi - a_1 wu_{\xi t} \mid_0^1 - a_2 wu_\xi \mid_0^1 = 0. \quad (13)$$

Taking the weight function as quadratic B-spline shape functions given by Eq. (5) and substituting approximation (9) in integral equation (13) with some manipulation, the element contributions are obtained in the form

$$\sum_{i,j=m-1}^{m+1} \left[\int_0^1 \phi_i \phi_j d\xi + a_1 \int_0^1 \phi_i' \phi_j' d\xi - a_1 \phi_i \phi_j' \mid_0^1 \right] \dot{\delta}_j^e$$

$$(14)$$

$$+ \sum_{i,j=m-1}^{m+1} \left[a_2 \int_0^1 \phi_i' \phi_j' d\xi + a_3 \int_0^1 \phi_i \phi_j' d\xi + a_4 z_m \int_0^1 \phi_i \phi_j' d\xi - a_2 \phi_i \phi_j' \mid_0^1 \right] \delta_j^e = 0.$$

In matrix notation, this equation becomes

$$A^e + a_1 \left(B^e - C^e \right)] \dot{\delta}^e + \left[a_2 B^e + \left(a_3 + a_4 z_m \right) D^e - a_2 C^e \right] \delta^e = 0, \quad (15)$$

where $\delta^e = (\delta_{m-1}, \; \delta_m, \; \delta_{m+1})^T$ are the element parameters and the dot denotes differentiation with respect to t. The element matrices A^e, B^e, C^e and D^e are given by the following integrals:

$$A_{ij}^e = \int_0^1 \phi_i \phi_j d\xi = \frac{1}{30} \begin{bmatrix} 6 & 13 & 1 \\ 13 & 54 & 13 \\ 1 & 13 & 6 \end{bmatrix}$$

$$B_{ij}^e = \int_0^1 \phi_i' \phi_j' d\xi = \frac{2}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$C_{ij}^e = \phi_i \phi_j' \mid_0^1 = 2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$D_{ij}^e = \int_0^1 \phi_i \phi_j' d\xi = \frac{1}{6} \begin{bmatrix} -3 & 2 & 1 \\ -8 & 0 & 8 \\ -1 & -2 & 3 \end{bmatrix}$$

where the suffices i, j take only the values $m-1$, m, $m+1$ for the typical element $[x_m, x_{m+1}]$. A lumped value for ω and ϖ are found from $(u_m + u_{m+1})^p / 2^p$ as

$$\lambda_m = \frac{1}{2^p}(\delta_{m-1} + 2\delta_m + \delta_{m+1})^p. \tag{16}$$

By assembling all contributions from all elements, Eq. (15) leads to the following matrix equation;

$$A + a_1(B-C)]\dot{\delta} + \left[a_2 B + (a_3 + a_4 \lambda_m) D - a_2 C \right] \delta = 0, \tag{17}$$

where $\delta = (\delta_{-1}, \delta_0, ..., \delta_{N-1}, \delta_N)^T$ are global element parameters. The rows of matrices A, B, C and $(a_3 + a_4 \lambda_m) D$ have the following form:

$$A = \frac{1}{30}(1, 26, 66, 26, 1),$$

$$B = \frac{2}{3}(-1, -2, 6, -2, -1)$$

$$C = (0, 0, 0, 0, 0), \tag{18}$$

$$D = \frac{1}{6} \begin{bmatrix} -(a_3 + a_4 \lambda_1), -2(a_3 + a_4 \lambda_1) - 8(a_3 + a_4 \lambda_2), \\ 3(a_3 + a_4 \lambda_1) - 3(a_3 + a_4 \lambda_3), \\ 8(a_3 + a_4 \lambda_2) + 2(a_3 + a_4 \lambda_3), (a_3 + a_4 \lambda_3) \end{bmatrix}$$

where

$$\lambda_1 = \frac{1}{2^p}(\delta_{m-1} + 2\delta_m + \delta_{m+1})^p,$$

$$\lambda_2 = \frac{1}{2^p}(\delta_m + 2\delta_{m+1} + \delta_{m+2})^p, \tag{19}$$

$$\lambda_3 = \frac{1}{2^p}(\delta_{m+1} + 2\delta_{m+2} + \delta_{m+3})^p.$$

Replacing the time derivative of the parameter δ by Crank-Nicolson formulation and parameter $\dot{\delta}$ by usual forward finite difference approximation

$$\delta = \frac{1}{2}\left(\delta^{n+1} + \delta^n\right) \quad , \quad \dot{\delta} = \frac{1}{\Delta t}\left(\delta^{n+1} - \delta^n\right) \tag{20}$$

into Eq. (17), it gives the $(N+2) \times (N+2)$ matrix system

$$\left\{\left[A + a_1\left(B - C\right)\right] + \left[a_2 B + \left(a_3 + a_4\lambda_m\right)D - a_2 C\right]\frac{\Delta t}{2}\right\}\delta^{n+1}$$

$$\tag{21}$$

$$= \left\{\left[A + a_1\left(B - C\right)\right] - \left[a_2 B + \left(a_3 + a_4\lambda_m\right)D - a_2 C\right]\frac{\Delta t}{2}\right\}\delta^n,$$

where Δt is time step. Applying the boundary conditions (3) to the system (22), a $N \times N$ matrix system is obtained. This system is efficiently solved with a variant of the Thomas algorithm. A typical member of the matrix system (22) may be written in terms of the nodal parameters δ^n and δ^{n+1} as

$$\gamma_1\delta_{m-2}^{n+1} + \gamma_2\delta_{m-1}^{n+1} + \gamma_3\delta_m^{n+1} + \gamma_4\delta_{m+1}^{n+1} + \gamma_5\delta_{m+2}^{n+1} = \gamma_5\delta_{m-2}^n + \gamma_4\delta_{m-1}^n + \gamma_3\delta_m^n + \gamma_2\delta_{m+1}^n + \gamma_1\delta_{m+2}^n \tag{22}$$

where

$$\gamma_1 = \frac{1}{30} - \frac{2a_1}{3} - \frac{a_2\Delta t}{3} - \frac{(a_3 + a_4\lambda_m)\Delta t}{12},$$

$$\gamma_2 = \frac{26}{30} - \frac{4a_1}{3} - \frac{2a_2\Delta t}{3} - \frac{10(a_3 + a_4\lambda_m)\Delta t}{12},$$

$$\gamma_3 = \frac{66}{30} + 4a_1 + 2a_2\Delta t, \tag{23}$$

$$\gamma_4 = \frac{26}{30} - \frac{4a_1}{3} - \frac{2a_2\Delta t}{3} + \frac{10(a_3 + a_4\lambda_m)\Delta t}{12},$$

$$\gamma_5 = \frac{1}{30} - \frac{2a_1}{3} - \frac{a_2\Delta t}{3} + \frac{(a_3 + a_4\lambda_m)\Delta t}{12},$$

which all depend on δ^n. The initial vector of parameters $\delta^0 = (\delta_{-1}^0, \ldots, \delta_N^0)$ must be determined to iterate the system (22). To do this, the approximation is rewritten over the interval $[a,b]$ at time $t = 0$ as follows:

$$u_N(x,0) = \sum_{m=-1}^{N} \phi_m(x)\delta_m^0 \tag{24}$$

where the parameters δ_m^0 will be determined. $u_N(x,0)$ are required to satisfy the following relations at the mesh points x_m for $m = 0,1,\ldots,N$,

$$u_N(x_m,0) = u(x_m,0),$$

$$u_N'(x_0,0) = u'(x_N,0) = 0. \tag{25}$$

The above conditions lead to a matrix system of the form

$$\begin{bmatrix} 1 & -1 & & & \\ 1 & 1 & & & \\ & & \ddots & & \\ & & & 1 & 1 \end{bmatrix} \begin{bmatrix} \delta_{-1}^0 \\ \delta_0^0 \\ \vdots \\ \delta_{N-1}^0 \\ \delta_N^0 \end{bmatrix} = \begin{bmatrix} u_N'(x_0,0) \\ u_N(x_0,0) \\ \vdots \\ u_N(x_{N-1},0) \\ u_N(x_N,0) \end{bmatrix}$$

which can be solved using a variant of the Thomas algorithm.

3. Stability of the Scheme

The stability analysis is based on the von Neumann theory. The growth factor ξ of the error in a typical mode of amplitude

$$\delta_m^n = \xi^n e^{imkh}, \tag{26}$$

where h the element size and k is the mode number, is determined from a linearization of the numerical scheme. Substituting the Fourier mode (26) into (27) gives the following equality

$$
\begin{aligned}
&\varepsilon_1 \xi^{n+1} e^{i(m-2)kh} + \varepsilon_2 \xi^{n+1} e^{i(m-1)kh} + \varepsilon_3 \xi^{n+1} e^{i(m)kh} + \varepsilon_4 \xi^{n+1} e^{i(m+1)kh} + \varepsilon_5 \xi^{n+1} e^{i(m+2)kh} \\
&= \eta_5 \xi^n e^{i(m-2)kh} + \eta_4 \xi^n e^{i(m-1)kh} + \eta_3 \xi^n e^{i(m)kh} + \eta_2 \xi^n e^{i(m+1)kh} + \eta_1 \xi^n e^{i(m+2)kh}.
\end{aligned} \tag{27}
$$

where

$$\varepsilon_1 = 1 - E - M - K(\beta + \sigma z_m), \qquad \eta_1 = 1 - E + M - K(\beta + \sigma z_m),$$

$$\varepsilon_2 = 26 - 2E - 2M - 10K(\beta + \sigma z_m), \eta_2 = 26 - 2E + 2M - 10K(\beta + \sigma z_m),$$

$$\varepsilon_3 = 66 + 6E + 6M, \qquad \eta_3 = 66 + 6E - 6M, \tag{28}$$

$$\varepsilon_4 = 26 - 2E - 2M + 10K(\beta + \sigma z_m), \eta_4 = 26 - 2E + 2M + 10K(\beta + \sigma z_m),$$

$$\varepsilon_5 = 1 - E - M + K(\beta + \sigma z_m), \qquad \eta_5 = 1 - E + M + K(\beta + \sigma z_m),$$

and

$$E = \frac{20\mu}{h^2}, \quad M = \frac{10\alpha}{h^2}, \quad K = \frac{5\Delta t}{2h}. \tag{29}$$

Now, if Euler's formula

$$e^{ikh} = \cos(kh) + i\sin(kh), \tag{30}$$

is used in Eq. (27) and this equation is simplified to the following growth factor:

$$\xi = \frac{\omega_1 - i\varpi}{\omega_2 + i\varpi}, \tag{31}$$

in which

$$\omega_1 = (66 + 6E - 6M) + (52 - 4E + 4M)\cos(kh) + (2 - 2E + 2M)\cos(2kh),$$
$$\omega_2 = (66 + 6E + 6M) + (52 - 4E - 4M)\cos(kh) + (2 - 2E - 2M)\cos(2kh), \qquad (32)$$
$$\varpi = 20K(\beta + \sigma z_m)\sin(kh) + 2K(\beta + \sigma z_m)\sin(2kh).$$

The von Neumann criterion for stability $|\xi| \leq 1$ will be satisfied when

$$\mu\{(66h^2 + 60\alpha\Delta t) + (52h^2 - 40\alpha\Delta t)\cos(kh) + (h^2 - 20\alpha\Delta t)\cos(2kh)\} > 0. \qquad (33)$$

4. Results and Discussion

Numerical results of the gBBM-B equation are obtained for two problems with zero boundary conditions.

Example 1. Consider gBBM-B equation (1) with boundary condition

$$u(-1,t) = u(1,t) = 0 \ , \ t \in [0,3] \qquad (34)$$

and initial condition
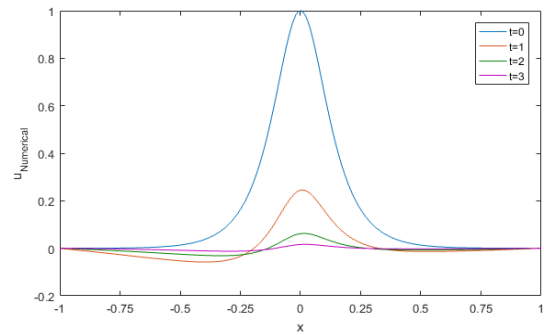
$$u(x,0) = sech[k(x - x_0)] \ , \ x \in [-1,1]. \qquad (35)$$

The parameters are taken as $\beta = -1$, $\mu = -1$, $\sigma = 1$, $a_1 = 1$, $c = 1$, $h = 0.01$ and $\Delta t = 0.01$. Also, coefficient of dissipative term is chosen as $\alpha = -0.61$, -1.16, -1.49, -1.75 for $p = 1, 3, 5, 7$, respectively. Amplitudes of the initial solitary waves are $amp = 1$. Numerical solution of Eq. (1) is given in Table 2 for some values of x in selected times. It can be observed from Figure 1, the height of the solution waves to u is more and more low with time elapsing according to the effect of dissipative term u_{xx} and its coefficient. Also, the wave is faster damping as the nonlinear effect increases. Because, coefficient of dissipative term α is based on degree of the nonlinearity from Eq. (2).

**Table 2**. Numerical results of Example 1 with different values of p

|	x/t	0	1	2	3
$p=1$	-0.5	.013475	-.067070	-.068412	-.047553
	0	1.00000	.464556	.207699	.089697
	0.5	.013475	-.005753	-.011119	-.010613
	x/t	0	1	2	3
$p=3$	-0.5	.013475	-.052916	-.024984	-.008422
	0	1.00000	.244697	.061510	.015861
	0.5	.013475	-.013665	-.007000	-.002474
	x/t	0	1	2	3
$p=5$	-0.5	.013475	-.040678	-.012503	-.002813
	0	1.00000	.168478	.030604	.005815
	0.5	.013475	-.013936	-.004229	-.000930
	x/t	0	1	2	3
$p=7$	-0.5	.013475	-.032560	-.007266	-.001208
	0	1.00000	.127011	.018016	.002692
	0.5	.013475	-.012840	-.002717	.002692
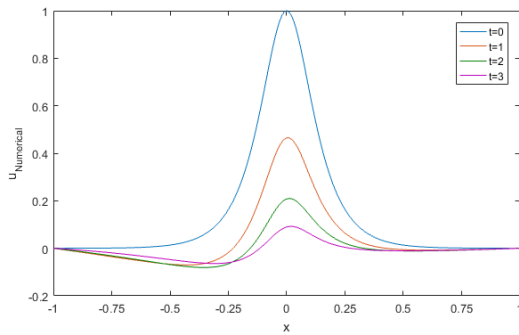
(a) $p = 1$

(b) $p = 3$
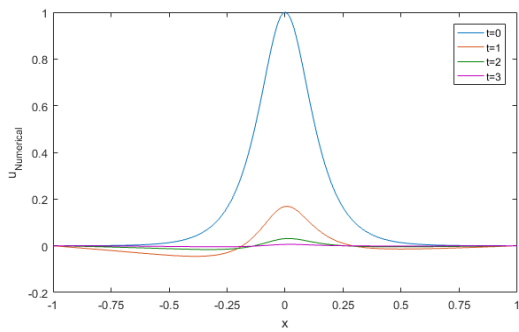
(c) $p = 5$
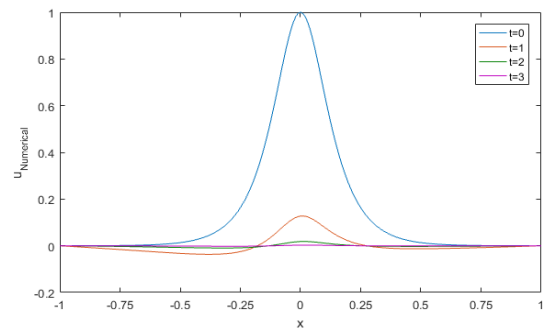
(d) $p = 7$

Figure 1. Numerical solution of $u(x,t)$ with different values of p for Example 1

Example 2. Consider gBBM-B equation (1) with boundary condition

$$u(-4,t) = u(4,t) = 0 \ , \ t \in [0,0.5] \tag{36}$$

and initial condition

$$u(x,0) = -\sin(\pi x) \ , \ x \in [-4,4]. \tag{37}$$

The parameters are chosen as $\beta = -1$, $\mu = -1$, $\sigma = 1$, $a_1 = 1$, $c = 1$, $h = 0.01$ and $\Delta t = 0.01$. Also, coefficient of dissipative term is taken as $\alpha = -0.61$, -1.34, -1.75, -2.07 for $p = 1, 4, 7, 10$, respectively. Numerical solution of Eq. (1) is given in Table 3 for some values of x in selected times. Figure 2 shows that the height of the numerical approximation to u decreases as time progresses. It can be seen from Eq. (2), the effect of the dissipative term u_{xx} is based on value of p. So, the height of the solution waves to u decreases faster as nonlinearity of Eq. (1) increases.

Table 3. Numerical results of Example 2 with different values of p

	t/x	−2.5	−0.5	1.5	3.5
$p = 1$	0	.999995	1.00000	.999998	.999990
	0.25	.792202	.940395	.827620	.724142
	0.5	.689313	.885406	.621824	.474348
	t/x	−2.5	−0.5	1.5	3.5
$p = 4$	0	.999995	1.00000	.999998	.999990
	0.25	.663783	.750396	.680732	.611696
	0.5	.501421	.567684	.412100	.350838
	t/x	−2.5	−0.5	1.5	3.5
$p = 7$	0	.999995	1.00000	.999998	.999990
	0.25	.602231	.671759	.602740	.542607
	0.5	.432622	.463117	.300890	.268881
	t/x	−2.5	−0.5	1.5	3.5
$p = 10$	0	.999995	1.00000	.999998	.999990
	0.25	.556227	.613212	.552894	.500271
	0.5	.380033	.393360	.241680	.224114

(a) $p = 1$

(b) $p = 4$

(c) $p = 7$

(d) $p = 10$
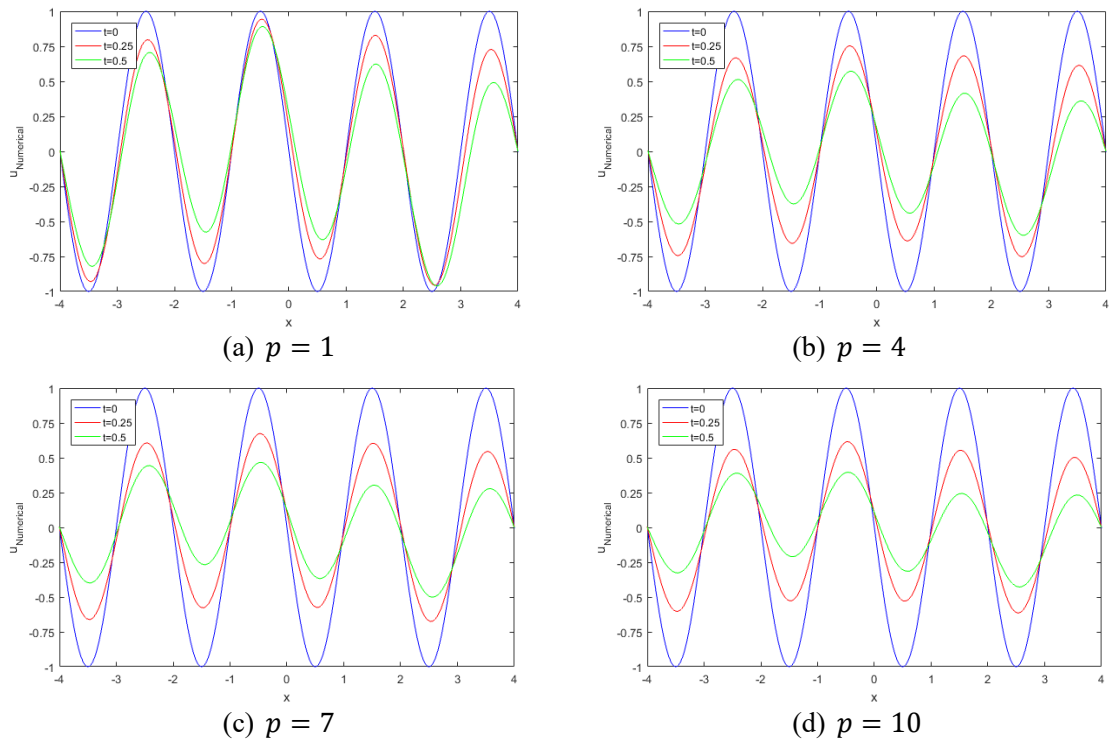
Figure 2. Numerical solution of $u(x,t)$ with different values of p for Example 2

5. Conclusion

In this paper, Galerkin finite element method based on quadratic B-spline interpolation function is applied to gBBM-B equation. Stability analysis is investigated based on von Neumann theory. Then, it is showed that the applied method is conditionally stable. Also, two test problems with zero boundary conditions is successfully studied to prove performance of the method. Consequently, it can be said that this method is a reliable method for calculating the numerical solutions of similar type non-linear equations and representing similar type waves.

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