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Projective Graphs Obtained from Projective Planes

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Abstract

In this paper, we introduced a new method to relate two areas, graph theory and projective geometry that have a long history and very deep theories. We investigated the combinatorial properties of the graphs which are obtained from finite projective planes by using this new method. Also, we examined the relations between these combinatorial properties and the order of the projective plane.

Keywords: Projective planes, Graphs, Degree sequences.

Projektif Düzlemlerden Elde Edilen Projektif Graflar

Özet

Bu makalede, uzun bir tarihe ve derin teorilere sahip olan iki alanı, graf teorisi ve projektif geometri arasında bir ilişki kurmak için yeni bir metot sunduk. Bu yeni metodu kullanarak sonlu projektif düzlemlerden elde edilen grafların kombinatoryal özelliklerini araştırdık. Aynı zamanda bu kombinatoryal özellikler ile projektif düzlemin mertebesi arasındaki ilişkileri inceledik.

Anahtar Kelimeler: Projektif düzlemler, Graflar, Köşe derece dizileri.

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1. Introduction

In this section, we start with some definitions and fundamental notions regarding projective planes from [8, 9].

Definition 1 *A projective plane π is an ordered pair (P, L) which we call the elements of P as points and the elements of L as lines where every line consists of some points, with the properties*

P1) *Any two distinct points lie on a unique line,*

P2) *Any two lines meet,*

P3) *There exists a set of four points no three of them are collinear.*

If $P \cup L$ is a finite set then the projective plane $\pi = (P, L)$ is called as finite projective plane. If $P \cup L$ is finite then we know that P is finite and L is finite, and also, we denote the number of points with $|P| = v$ and the number of lines with $|L| = b$.

From now on, we assume that π is a projective plane. If every line consists of k+1 points and every point is on k+1 line in a projective plane where $k \geq 2$ is an integer, then we call the number k as the order of the projective plane.

In a projective plane π of order k, we know that

1) π has $v = k^2 + k + 1$ points,

2) π has $b = k^2 + k + 1$ lines.

Example 2 (The Fano Plane) In 1892, the Italian mathematician Gino Fano discovered the smallest projective plane. It is the geometry with seven points and seven lines given by

$P=\{1,2,3,4,5,6,7\}$,

$L=\{l_1 =\{1,2,3\},l_2=\{3,4,5\},l_3=\{5,6,1\},l_4=\{1,7,4\},l_5=\{2,7,5\},l_6=\{3,7,6\},l_7=\{2,4,6\}\}$.

The Fano plane is represented in Figure 1. Here it may appear that the circular-like line l_7={2,4,6} meets with line l_5={2,7,5} at 2 and somewhere between 5 and 7. However the sets {2,4,6} and {2,7,5} intersects only at 2.

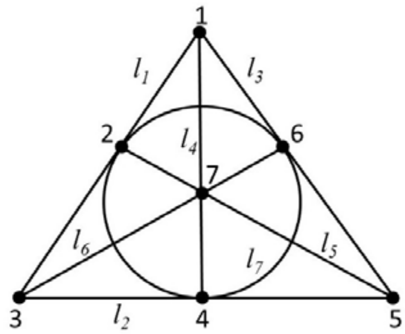

Figure 1. Projective plane of order 2

When a projective plane is given, there is a number k≥2 as the order. But for a given number k, it is not necessary to have a projective plane that has k as the order. It is a matter of ongoing studies that whether or not there are projective planes in which order.

Let us now recall some fundamental notions of graph theory from [3, 4, 10, 13, 14].

Definition 3 *A multigraph G = (V, E) consists of a non-empty finite set V = V(G) of vertices together with a finite set E = E(G) of edges such that*

1) Each edge consists of two vertices,

2) Any two distinct vertices are joined by a finite number (including zero) of edges.

Let G = (V, E) be a multigraph with order n which is the number of vertices and size m which is the number of edges. Let u and v be two vertices in graph G. If u and v are connected to each other by an edge e, this situation will be denoted by e=uv. The vertices u and v are called adjacent vertices and the edge e is said to be incident with u and v.

Definition 4 *An edge with identical ends is called a loop, and an edge with distinct ends a link. Two or more links with the same pair of ends are said to be parallel edges.*

Definition 5 *A graph G is called simple if it has no loops nor parallel edges.*

In this work, we are going to deal with simple graphs.

Definition 6 *The degree of a vertex* $v \in V(G)$ *is denoted by* d_v *or* $d(v)$ *and defined as the number of edges in G incident to v.*

If the degree of v is one, then it is called a pendant vertex.

The smallest and biggest vertex degrees of G are denoted by δ and Δ, respectively

$$\delta(G) = \min\{d(v) \mid v \in V(G)\},$$

$$\Delta(G) = \max\{d(v) \mid v \in V(G)\}.$$

Definition 7 *A graph G is said to be regular if every vertex in G has the same degree. More precisely, G is said to be k-regular if* $d(v) = k$ *for each vertex v in G, where* $k \geq 0$.

Definition 8 *Assume that* $V(G)=\{u_1,u_2,u_3,\ldots,u_n\}$. *Call the sequence* $D=\{d(u_1),d(u_2),d(u_3),\ldots,d(u_n)\}$ *degree sequence of graph G where the vertices in G are renamed such that* $d(u_1)\geq d(u_2)\geq d(u_3)\geq\ldots\geq d(u_n)$.

Written with multiplicities, the degree sequence of a graph G can also be written as

$$DS(G) = \{1^{x_1}, 2^{x_2}, 3^{x_3}, \cdots, \Delta^{x_\Delta}\}$$

where x_i's are the numbers of the vertices which has i as the degree. Therefore x_i's are non-negative integers.

A graph G is called a realization of the set D if the degree sequence of G is equal to D.

For a realizable degree sequence, there may be more than one graph having this degree sequence. For example, two graphs in Figure 2 have the same degree sequence.

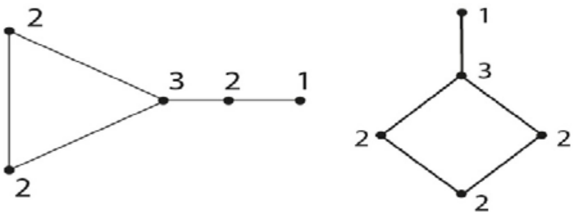

Figure 2. Graphs with the same DS

The most well-known result to determine realizability is the Havel-Hakimi process. For this and some other algorithms [1, 2, 5, 6, 7, 11, 12, 15].

Definition 9 *A walk in a graph G is an alternating sequence of vertices and edges beginning and ending at vertices,*

$$v_0 e_0 v_1 e_1 v_2 \ldots v_{k-1} e_{k-1} v_k$$

where $k \geq 1$ and e_i is incident with v_i and v_{i+1} for each $i=0,1,2,\ldots,k-1$.

The walk $v_0e_0v_1e_1v_2\ldots v_{k-1}e_{k-1}v_k$ is also called a v_0-v_k walk with its initial vertex v_0 and terminal vertex v_k. This representation can be interpreted as one can go from v_0 to v_1 with the e_0 edge and so on. The length of the walk $v_0e_0v_1e_1v_2\ldots v_{k-1}e_{k-1}v_k$ is defined as k, which is the number of occurrences of edges in the sequence.

Definition 10 *A walk is called a path if no vertex in it is visited more than once.*

A graph is connected when there is a path between every pair of vertices. A graph that is not connected is called disconnected. There are many results about the connectedness of a given graph.

Definition 11 *A u-v walk is closed if u = v, that is, its initial and terminal vertices are the same and open otherwise.*

Definition 12 *A closed walk of length at least 2 in which no edges are repeated is called a circuit in any multigraph.*

The length of a closed walk in a graph G must be at least 3.

Definition 13 *A circuit is called a cycle if no vertex is repeated (except the initial and terminal vertices). A cycle of length n is denoted by* C_n.

2. The Relation of Graphs with Finite Projective Planes

We are going to be investigating for the answer to the following question: What if someone perceives the lines of a geometric structure as an element of graph theory and what could be obtained from this? To do so, we are introducing a new concept, a new perspective to the field. In this study, we work only with finite projective planes.

2.1 The Method of Obtaining Graphs from Finite Projective Planes

In this part, a method will be given to obtain a graph from a given finite projective plane.

Definition 14 *Let* $\pi = (P, L)$ *be a finite projective plane of order k. If* l_i *is a line in* π*, then it must have k+1 points. We take this line as an ordered k+1-tuple* $l_i=(l_{i1},l_{i2},l_{i3},\ldots,l_{ik},l_{i(k+1)})$*. It is obvious that* $k\geq 2$.

We establish a new set "$s(l_i)$*" for any line* $l_i=(l_{i1},l_{i2},l_{i3},\ldots,l_{ik},l_{i(k+1)})$*, we define*

$$l_{i,j} := \begin{cases} \{l_{ij}, l_{i(j+1)}\}; & 1 \leq j \leq k+1 \\ \{l_{i(k+1)}, l_{i1}\}; & j = k+1 \end{cases}$$

$$s(l_i) := \{l_{i,j} \mid j=1,2,\ldots,k,k+1\}$$

$$:= \{l_{i,1},l_{i,2},l_{i,3},\ldots,l_{i,k-1},l_{i,k},l_{i,k+1}\}$$

$$:= \{\{l_{i1},l_{i2}\},\{l_{i2},l_{i3}\},\ldots,\{l_{i(k-1)},l_{ik}\},\{l_{ik},l_{i(k+1)}\},\{l_{i(k+1)},l_{i1}\}\}.$$

Throughout this paper $s(l_i)$ is called as ordered k+1-gon corresponding to line l_i.

If we try to obtain a graph, from a finite projective plane $\pi = (P, L)$, then we have to change our perception of lines. To do so, we use the method given in Definition 14.

Let's consider a line $l_i=(l_{i1},l_{i2},l_{i3},...,l_{ik},l_{i(k+1)})$. For $1\leq r \leq k+1$ any point l_{ir}, is incident with only $l_{i,r-1}$ and $l_{i,r}$.

For a line $l_i=\{l_{i1},l_{i2},l_{i3},...,l_{ik},l_{i(k+1)}\}$, $s(l_i)$ can be considered as a C_{k+1}.

Also, we know that a C_3 is often called a triangle, a C_4 a quadrilateral, a C_5 a pentagon, a C_6 a hexagon, and so on.

In the following theorem, we obtain a graph, from a finite projective plane π of any order, by considering every element of obtained $s(l_i)$ for a line l_i, as an edge. It is obvious that every element of an $s(l_i)$ obtained from an l_i is a set of exactly two points.

Theorem 15 (Principal Theorem) *If $\pi = (P, L)$ is a finite projective plane of order k then we have a graph $G = (V, E)$ such that*

$$V(G) = P,$$

$$E(G) = \bigcup_{l_i \in L} s(l_i).$$

Proof. For any line $l_i=\{l_{i1},l_{i2},l_{i3},...,l_{ik},l_{i(k+1)}\} \in L$ it is trivial that $| l_i |= k+1 \geq 3$ and every set $l_{i,j}$ consists of exactly 2 vertices.

Let $x, y \in V(G) = P$ be distinct elements. From P1) there exists only one line

$$l_i=\{l_{i1},l_{i2},l_{i3},...,l_{ik},l_{i(k+1)}\}$$

passing through the points x and y. Therefore for $1 \leq r,t \leq k+1$ there exists

$$x = l_{ir} , y = l_{it}$$

in l_i.

Since $r\neq t$ we take $r<t$ without loss of generality.

When $t = r+1$ there is an edge

$$l_{i,r} = \{l_{ir}, l_{i(r+1)}\} = \{l_{ir},l_{it}\} = \{x,y\}$$

incident with x and y. In other words when $t = r+1$ there is only one edge passing through x and y and that is $l_{i,r}$.

When t>r+1 if t=k and r=1 then there is an edge

$$l_{i,k} = \{l_{ik},l_{i1}\} = \{l_{it},l_{ir}\} = \{y,x\}$$

incident with y and x. In other words when t>r+1 if t = k+1 and r = 1 there is only one edge passing through y and x and that is $l_{i,k+1}$.

In other cases, it is obvious from the construction of $s(l_i)$ that there is no edge incident with x and y. Thus we show that G = (V, E) is a graph.

Throughout this work, we are going to use the method introduced by Theorem 15 to obtain a graph from a finite projective plane unless otherwise noted.

Definition 16 *A graph obtained from a projective plane π of order k is called as "projective graph of order k corresponding to π" or if there will not be any confusion "projective graph of order k".*

Now we will examine how projective graphs are obtained when the projective plane of order 2 and the projective plane of order 3 is taken as π = (P, L).

Example 17 We are going to investigate the graph obtained from the smallest projective plane which given in Example 2.

P={1,2,3,4,5,6,7},

$L=\{l_1=\{1,2,3\},l_2=\{3,4,5\},l_3=\{5,6,1\},l_4=\{1,7,4\},l_5=\{2,7,5\},l_6=\{3,7,6\},l_7=\{2,4,6\}\}$.

Since each line of the Fano plane has 3 points, k+1 = 3. In other words, every line of this projective plane considered as an ordered 3-gon or as a C_3 in graph theory.

The vertex and edge sets of the graph that is obtained from the projective plane of order 2 are as follows:

$$V(G) = \{1,2,3,4,5,6,7\},$$

$E(G) = \{l_{1,1}=\{1,2\}, l_{1,2}=\{2,3\}, l_{1,3}=\{3,1\}, l_{2,1}=\{3,4\}, l_{2,2}=\{4,5\}, l_{2,3}=\{5,3\}, l_{3,1}=\{1,6\}, l_{3,2}=\{6,5\}, l_{3,3}=\{5,1\}, l_{4,1}=\{1,7\}, l_{4,2}=\{7,4\}, l_{4,3}=\{4,1\}, l_{5,1}=\{2,7\}, l_{5,2}=\{7,5\}, l_{5,3}=\{5,2\}, l_{6,1}=\{3,7\}, l_{6,2}=\{7,6\}, l_{6,3}=\{6,3\}, l_{7,1}=\{2,4\}, l_{7,2}=\{4,6\}, l_{7,3}=\{6,2\}\}$.

Now we determine the vertex degrees for each vertex:

$$d(1)=6,\ d(2)=6,\ d(3)=6,\ d(4)=6,\ d(5)=6,\ d(6)=6,\ d(7)=6.$$

As a result of the degree sequence which is given down below, G is regular, see Figure 3.

$$\{6,6,6,6,6,6,6\} = \{6^{(7)}\}$$

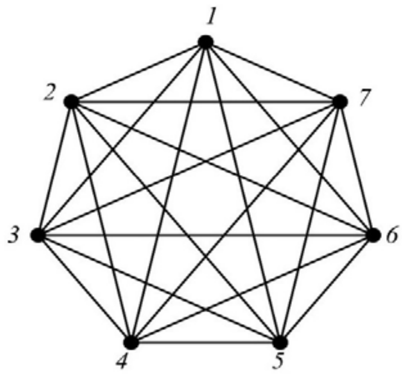

Figure 3. The regular graph obtained from the projective plane of order 2

Example 18 Let us take the projective plane of order 3 with points and lines given, respectively

$P'=\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$,

$L'=\{l_1'=\{1,2,3,11\},\ l_2'=\{4,5,6,11\},\ l_3'=\{7,8,9,11\},\ l_4'=\{1,4,7,13\},\ l_5'=\{2,5,8,13\},$

$l_6'=\{3,6,9,13\},\ l_7'=\{1,5,9,12\},\ l_8'=\{2,6,12,7\},\ l_9'=\{4,8,12,3\},\ l_{10}'=\{7,5,3,10\},$

$l_{11}'=\{1,10,6,8\},\ l_{12}'=\{4,2,10,9\},\ l_{13}'=\{10,11,12,13\}\}$

and illustrated by Figure 4.

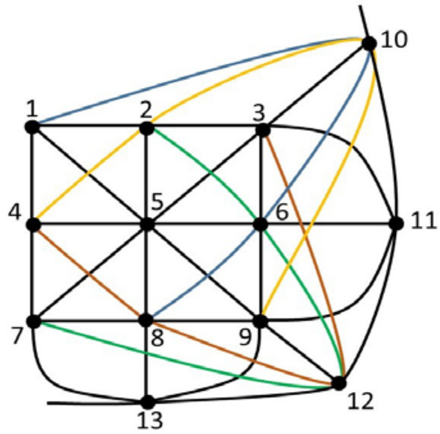

Figure 4. Projective plane of order 3

We know that the vertex set of the graph which is obtained from the projective plane of order 3 is equal to the set of points of the projective plane of order 3.

Every given line in this projective plane of order 3 consist of 4 elements, so k+1 = 4. In other words, every line of this projective plane considered as an ordered 4-gon or as a quadrilateral.

The vertex and edge sets of the graph G′ that is obtained from the projective plane of order 3 are as follows:

$$V(G') = P' = \{1,2,3,4,5,6,7,8,9,10,11,12,13\},$$

$$\begin{aligned}
E(G') = \{ & l_{1,1}'=\{1,2\}, l_{1,2}'=\{2,3\}, l_{1,3}'=\{3,11\}, l_{1,4}'=\{11,1\}, \\
& l_{2,1}'=\{4,5\}, l_{2,2}'=\{5,6\}, l_{2,3}'=\{6,11\}, l_{2,4}'=\{11,4\}, \\
& l_{3,1}'=\{7,8\}, l_{3,2}'=\{8,9\}, l_{3,3}'=\{9,11\}, l_{3,4}'=\{11,7\}, \\
& l_{4,1}'=\{1,4\}, l_{4,2}'=\{4,7\}, l_{4,3}'=\{7,13\}, l_{4,4}'=\{13,1\}, \\
& l_{5,1}'=\{2,5\}, l_{5,2}'=\{5,8\}, l_{5,3}'=\{8,13\}, l_{5,4}'=\{13,2\}, \\
& l_{6,1}'=\{3,6\}, l_{6,2}'=\{6,9\}, l_{6,3}'=\{9,13\}, l_{6,4}'=\{13,3\}, \\
& l_{7,1}'=\{1,5\}, l_{7,2}'=\{5,9\}, l_{7,3}'=\{9,12\}, l_{7,4}'=\{12,1\}, \\
& l_{8,1}'=\{2,6\}, l_{8,2}'=\{6,12\}, l_{8,3}'=\{12,7\}, l_{8,4}'=\{7,2\}, \\
& l_{9,1}'=\{4,8\}, l_{9,2}'=\{8,12\}, l_{9,3}'=\{12,3\}, l_{9,4}'=\{3,4\}, \\
& l_{10,1}'=\{7,5\}, l_{10,2}'=\{5,3\}, l_{10,3}'=\{3,10\}, l_{10,4}'=\{10,7\}, \\
& l_{11,1}'=\{1,10\}, l_{11,2}'=\{10,6\}, l_{11,3}'=\{6,8\}, l_{11,4}'=\{8,1\}, \\
& l_{12,1}'=\{4,2\}, l_{12,2}'=\{2,10\}, l_{12,3}'=\{10,9\}, l_{12,4}'=\{9,4\}, \\
& l_{13,1}'=\{10,11\}, l_{13,2}'=\{11,12\}, l_{13,3}'=\{12,13\}, l_{13,4}'=\{13,10\}\}.
\end{aligned}$$

We have 13 vertex and 52 edges for this graph G′.

Now we determine the vertex degrees for each vertex in this G′:

$$d(1)=8,\ d(2)=8,\ d(3)=8,\ d(4)=8,\ d(5)=8,\ d(6)=8,\ d(7)=8,$$

$$d(8)=8,\ d(9)=8,\ d(10)=8,\ d(11)=8,\ d(12)=8,\ d(13)=8.$$

G′ is regular since the degree sequence obtained as down below, see Figure 5.

$$\{8,8,8,8,8,8,8,8,8,8,8,8,8\} = \{8^{(13)}\}$$

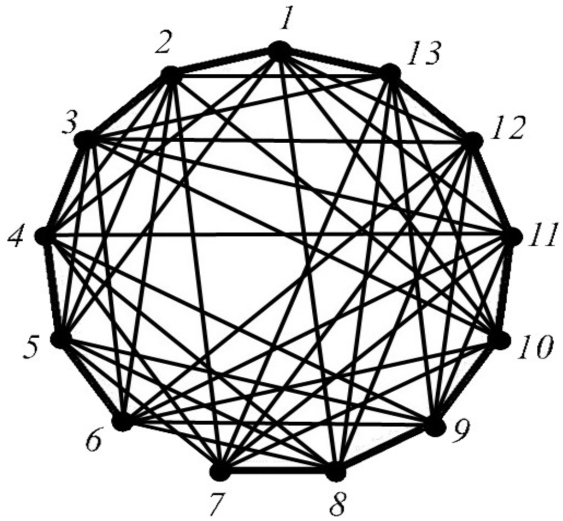

Figure 5. The regular graph obtained from the projective plane of order 3

Corollary 19 *Projective graphs of order k consist of* k^2+k+1 *vertices and* $(k^2+k+1)(k+1) = k^3+ 2k^2+2k+1$ *edges.*

Now we give a theorem for the characterization of the graphs that are obtained from projective planes.

Theorem 20 *Projective graphs of order k for* $k\geq2$ *are regular and has the degree sequence in the following form*

$$\{2(k+1),2(k+1),\ldots,2(k+1)\}= \{(2(k+1))^{(k^2+k+1)}\}.$$

Proof. Let $\pi = (P, L)$ be a projective plane of order k for $k\geq2$, and the graph $G = (V, E)$ is obtained from π. We know that in projective planes of order k, all lines have k+1 points and every point is exactly on k+1 distinct lines. As mentioned just before Theorem 15 for a point l_{ir} on a line $l_i=\{l_{i1},l_{i2},l_{i3},\ldots,l_{ik},l_{i(k+1)}\}$ there are exactly two edges $l_{i,r-1}$ and $l_{i,r}$ for that l_i.

The point l_{ir} is exactly on k+1 distinct lines so if it is incident with two edges for every line then we have 2(k+1) edges incident with the vertex l_{ir}. Therefore every vertex in G has the degree 2(k+1).

There are k^2+k+1 vertices in G since V(G) = P and $|P|=k^2+k+1$. Thus we have the degree sequence as below:

$$\{2(k+1),2(k+1),...,2(k+1)\}= \{(2(k+1))^{(k^2+k+1)}\}.$$

3. Open Questions

In this study, we start to give some relations between the disciplines of geometry and graph theory.

Although the relations between them have not been examined in detailed yet it is easily seen that there are some differences and similarities between the projective graphs of order k and the projective planes of order k.

There are also some open problems on this subject that we are constantly working on and which we think we are going to be able to examine and answer in the future. Some of them are in the following.

It is possible to obtain affine graphs as the geometric structure shift to affine planes.

What is the relation between projective and affine graphs which have the same order? For example, how does the existence and absence of the parallel axiom in projective and affine planes affect the graphs that are obtained from these planes?

We know that an affine plane is obtained when a line, with the points, is thrown away from a projective plane. Under which conditions we can find an affine graph of order k by deleting a C_{k+1}, with the vertices, from a projective graph of order k?

We know that affine planes of order k can be embedded into projective planes of order k. Can affine graphs of order k be embedded in projective graphs of order k?

How can we determine whether a given graph is a projective graph?

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