ISSN 2587-1943

DIFFERENCE SCHEMES METHODS FOR THE FRACTIONAL ORDER DIFFERENTIAL EQUATION SENSE OF CAPUTO DERIVATIVE

Mahmut Modanli 1

Original scientific paper

This study gives numerical solution of the fractional order partial differential equation defined by Caputo fractional derivative. Laplace transform method is used for the exact solution of this equation depend on intial-boundary value problem. The difference schemes are constructed for this equation. The stability of this difference schemes is proved. Error analysis is performed by comparing the exact solution with the approximate solution. The effectiveness of the method is shown from the error analysis table.

Keywords: *Caputo derivative, Difference scheme method ,Fractional order differential equation, Laplace method, Stability estimates*.

1 Introduction

Partial differential equation is very important for the many fields. The numerical and the exact solutions of this differential equations have many various methods. In [7,10], the authors studied high-order linear complex differential equations. Fractional order differential equation has many applications in engineering, physic, finance, physics and seismology [1-3]. The Atangana-Baleanu (AB) derivative was applied successfully in modelling of various real phenomena such as [4-6]. Some methods were applied the fractional and partial differential equation in [13,14]. Modanli applied two different method for fractional telegraph differential equation depend on nonlocal initial conditions [12].

In [8], the authors worked to implicit difference approximation for the time fractional heat equation with the nonlocal condition. Finally, Sarboland gave numerical solution of time fractional partial differential equations using multiquadric quasi-interpolation [9].

In this article, we consider fractional order differential equation defined by Caputo derivative to obtain numerical results. Then, the stability inequality is showed by the given the initial conditions. We examine the following fractional order differential equation

$$\begin{cases} {}_0^C D_t^\alpha u(t,x) + \frac{\partial u(t,x)}{\partial x} + u(t,x), \\ -\frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \\ 0 < x < L,\ \ 0 < t < T, \\ u(0,x) = r_1(x),\ \ 0 \le t \le T, \\ u(t,X_L) = u(t,X_R) = 0,\ X_L \le x \le X_R, \\ 0 < \alpha \le 1. \end{cases} \tag{1}$$

Now, we shall recall some basic definitions and properties of fractional calculus theory for fractional order differential equation.

Defination 1.1 The Caputo fractional derivative $D_t^\alpha u(t,x)$ of order α with respect to time is defined as: as:

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = D_t^\alpha u(t,x) =$$

$$\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-p)^{\alpha-n+1}} \frac{\partial^\alpha u(p,x)}{\partial p^\alpha} dp, \tag{2}$$

$$(n-1 < \alpha < n),$$

and for $\alpha = n \in N$ defined as:

$$D_t^\alpha u(t,x) = \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^n u(t,x)}{\partial t^n}.$$

2 Constructed difference schemes and its stability

We consider a rectangular domain $w^h \times w^\tau$ for the difference scheme method. For this method, suppose that $h = \frac{L}{M}$ for x-axis and $\tau = \frac{T}{N}$ for t −axis as grid mess, then we get

$$x_n = X_L + nh,\ n = 1,2,\dots,M,\ \ t_k = k\tau, k = 1,2,\dots,N..$$

We write the original differential equation (1) at the point $(x_n,\ t_k) \in w^h \times w^\tau$ as

$${}_0^C D_t^\alpha u(\,t_k, x_n) + \frac{\partial u(\,t_k, x_n)}{\partial x} + u(\,t_k, x_n),$$

$$-\frac{\partial^2 u(\,t_k, x_n)}{\partial x^2} = f(\,t_k, x_n),$$

Now, we shall give the following definition for construct the difference scheme method.

Definition 2.1 First-order approach method for the calculation of the problem (2) given by the formula:

$$D_t^\alpha u(t_k, x_n) = D_t^\alpha u_n^k$$

$$\cong g_{\alpha,\tau} \sum_{j=1}^{k} w_j^{(\alpha)} (u_n^{k-j+1} - u_n^{k-j}), \quad (3)$$

where $g_{\alpha,\tau} = \frac{1}{\Gamma(2-\alpha)\tau^\alpha}$ and $w_j^{(\alpha)} = (j+1)^{1-\alpha} - j^{1-\alpha}$. Using the last values, one has the following approximation

$$\frac{\partial^\alpha u(t_k, x_n)}{\partial x^\alpha} = g_{\alpha,\tau}$$

$$\times \left[w_1 u_n^k - w_k u_n^0 + \sum_{j=1}^{k-1} (w_{k-j+1} - w_{k-j}) u_n^j\right]. \quad (4)$$

Applying Taylor expansion with respect to x, the first and second order difference schemes are obtained as the following form

$$u_x(t_k, x_n) \cong \frac{u_{n+1}^k - u_{n-1}^k}{2h}, \quad (5.a)$$

$$u_{xx}(t_k, x_n) \cong \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2}. \quad (5.b)$$

Using the formula (3), (5.a) and (5.b) the difference schemes formula for the equation (1), we obtain

$$\begin{cases} g_{\alpha,\tau}(u_n^{k+1} - u_n^k) \\ +g_{\alpha,\tau} \sum_{j=1}^{k} w_j^{(\alpha)} (u_n^{k-j+1} - u_n^{k-j}) \\ +u_n^k + \frac{u_{n+1}^k - u_{n-1}^k}{2h} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} \\ \qquad = f(t_k, x_n) = f_n^k, \\ 1 \le k \le N-1, \ 1 \le n \le M-1, \\ u_n^0 = r_1(x_n), \ 0 \le n \le M, \\ u_0^k = u_M^k = 0, \ 0 \le k \le N. \end{cases} \quad (6)$$

Here $g_{\alpha,\tau} = \frac{1}{\Gamma(2-\alpha)\tau^\alpha}$ and $w_j^{(\alpha)} = (j+1)^{1-\alpha} - j^{1-\alpha}$.

We can rewrite the formula (6) as the following form

$$\begin{cases} \left(-\frac{1}{h^2} + \frac{1}{2h}\right) u_{n+1}^k + \left(-\frac{1}{h^2} + \frac{1}{2h}\right) u_{n-1}^k \\ +g_{\alpha,\tau} \sum_{j=1}^{k} w_j^{(\alpha)} (u_n^{k-j+1} - u_n^{k-j}) \\ + \left(-g_{\alpha,\tau} + \frac{2}{h^2} + 1\right) u_n^k + g_{\alpha,\tau} u_n^{k+1} = f_n^k, \\ 1 \le k \le N-1, \ 1 \le n \le M-1, \\ u_n^0 = r_1(x_n), \ 0 \le n \le M, \\ u_0^k = u_M^k = 0, \ 0 \le k \le N. \end{cases} \quad (7)$$

The formula (7) can be written in matrix form,

$$\begin{cases} Au_{n+1} + Bu_n + Cu_{n-1} = D\varphi_n, \\ u_0 = u_M = \vec{0}, \ 1 \le n \le M-1, \end{cases} \quad (8)$$

where $\varphi_n = f_n^k$, A, B, C and D are $(N+1) \times (N+1)$ matrices.

Using the modified Gauss-Elimination method, the formula (8) can be convert into the difference scheme as the following form

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \ n = M-1, \ldots, 2, 1, 0. \quad (9)$$

From the formulas (8) and (9), the following fair formulas can be obtain easily

$$\begin{cases} \alpha_{n+1} = -(B + A\alpha_n)^{-1} A, \\ \beta_{n+1} = (B + A\alpha_n)^{-1} (D\varphi_n - A\beta_n), \\ 1 \le n \le M-1, \end{cases} \quad (10)$$

Now, we shall the stability estimates theorem for the formula (8)

Theorem 2. 1 If $-g_{\alpha,\tau} + \frac{2}{h^2} + 1 + g_{\alpha,\tau}(2^{1-\alpha} - 1) > 0$, for $0 < \alpha \le 1$, then, the stability estimates are satisfied for the formula (8)

Proof. Using the formulas (9), (10) and the methods [8,11], the proof theorem is clear.

3 Numerical results

Example 3.1 We consider the following fractional order differential equation defined by the Caputo derivative

$$\begin{cases} {}_0^C D_t^{\alpha} u(t,x) + \frac{\partial u(t,x)}{\partial x} + u(t,x) \\ -\frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \\ f(t,x) = x(1-x)(\cos\left(x+t+\frac{\alpha\pi}{2}\right) \\ +(3-2x^2)\cos(x+t) \\ -(x^2-x)\sin(x+t) \\ 0 < x < 1,\ \ 0 < t < 1, \\ u(0,x) = x(1-x)cosx,\ \ 0 \le x \le 1 \\ u(t,0) = u(t,1) = 0,\ \ 0 \le t \le 1, \\ 0 < \alpha \le 1. \end{cases} \tag{11}$$

The exact solution of the formula (11) is obtained $u(t,x) = x(1-x)cos(x+t)$ by using Laplace transform method. We have used a procedure of modified Gauss elimination method for difference equation (6). We calculate the maximum norm for the error analysis using by

$$\in = \max_{\substack{n=0,1,\ldots,M \\ k=0,1,2,\ldots,N}} |u(t,x) - u(t_k,x_n)|,$$

where $u(t_k,x_n)$ is the approximate solution and $u(t,x)$ is the exact solution. Thus, the Table 1. gives error analysis for the difference scheme method.

Table 1. Error Analysis

$\tau = \frac{1}{N}, h = \frac{1}{M}$			
The difference scheme (11)			
$N = 5, M = 25$	$N = 10, M = 100$	$N = 20, M = 400$	$N = 25, M = 625$
0.0531	0.0405	0.0370	0.0366

Tablo 1. Error analysis are calculated for the variable values $0 < t < 1, 0 < x < 1$ and $\alpha = 0.5$ of the approximation solution by helping the difference formula (6).

Remark 3.1 From Table 1., we see that the numerical results are consistent and stabile for $\tau = h^2$ which is the condition of Theorem 3.1.

4 Conclusion

In this paper,the first order difference schemes for the equation (1) are constructed. Stability inequalities are proved for given difference schemes. Approximate solutions for numerical experiment are found by the difference-method. Error analysis is performed by comparing the exact solution with the approximate solution. The effectiveness of the method was seen from the error analysis table. MATLAB program is used for all numerical calculations.

References

[1] Celik, C. and Duman,M. Crank-Nicholson method for the fractional equation with the Riezs fractional derivative. Journal of computational physics, 231:1743-1750, 2012.

[2] Gorial, I.I. , Numerical methods for fractional reaction-dispersion equation with Riesz space fractional derivative. Eng. and Tech. Journal, 29:709-715, 2011.

[3] Jafari, H. and Gejii, V.D., Solving linear and nonlinear fractional diffusion and wave equations by adomian decomposition. Appl. Math. and Comput., 180:488-497, 2006.

[4] Atangana, A. and Koca I. Chaos in a simple nonlinear system with Atangana--Baleanu derivatives with fractional order. Chaos, Solitons & Fractals, 89, 447-454.

[5] Alkahtani, R.T., Chua's circuit model with Atangana--Baleanu derivative with fractional order. Chaos, Solitons & Fractals, 89, 547-551, 2016.

[6] Owolabi,K.M and Atangana, A., Chaotic behaviour in system of noninteger-order ordinary differential equations. Chaos, Solitons & Fractals, 115, 362-370, 2018.

[7] Dusunceli, F. and Celik, E., Fibonacci matrix Polynomial Method For LinearComplex Differential Equations, Asian Journal of Mathematics and Computer Research, 15(3): 229-238, 2017.

[8] Karatay, İ., Bayramoğlu, Ş. R., and Şahin, A., Implicit difference approximation for the time fractional heat equation with the nonlocal condition. Applied Numerical Mathematics, 61(12), 1281-1288, 2011.

[9] Sarboland, M., Numerical solution of time fractional partial differential equations using multiquadric quasi-interpolation scheme. European Journal of Computational Mechanics, 1-20, (2018).

[10] Dusunceli, F. and Celik, E., An Effective Tool: Numerical Solutions by Legendre Polynomials for High-Order Linear Complex Differential Equations, British Journal of Applied Science & Technology, . 8(4): 348-355, 2015.

[11] Modanli, M, ve Akgül, A., Numerical solution of fractional telegraph differential equations by theta-method: The European Physical Journal Special Topics, 226.16-18: 3693-3703, 2017.

[12] Modanli, M., Two numerical methods for fractional partial differential equation with nonlocal boundary value problem: Advances in Difference Equations, 2018(1), 333, 2018.

[13] Dusunceli, F., Solutions for the Drinfeld-Sokolov Equation Using an IBSEFM Method, MSU Journal of Science, 6(1), 505-510. DOI : 10.18586/msufbd.403217, 2018.

[14] Dusunceli, F. and Celik, E., Numerical Solution For High-Order LinearComplex Differential Equations with Variable Coefficients, Numerical Methods for Partial Differential Equations, DOI: 10.1002/num.22222, 2017.

Authors' addresses

Mahmut Modanli, PhD
Harran University
Faculty of Arts and Sciences
Depart of the mathematics
Sanliurfa, Turkey
mmodanli@harran.edu.tr