A new approach to the trial function for Homotopy Perturbation Method for nonlinear problems

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Abstract

In this paper, we give a new approach to the trial function for Homotopy Perturbation Method (HPM) which is powerful, easy-to-use and effective approximate analytical mathematical tool. By generalizing of the trial function, we obtain better approximate series solutions for the linear or nonlinear problems, and we minimize the computational work and the time by choosing trial function properly.

Keywords: Nonlinear differential equations, homotopy perturbation method, series solutions.

1. Introduction

When we want to obtain mathematical models of physical or biological phenomena, we generally get nonlinear partial differential equations. However, to find exact solutions of these problems is difficult. In recent years, we see that there are so many mathematical methods to find approximate solutions of nonlinear problems which come from various field of science and engineering. Some of them are Perturbation Method (PM), Adomian Decomposition Method (ADM) [5], HPM [2,4,7,8,9,10,11,12], HAM [3,13,14,15,12], Variational Iteration Method [16], G'/G Method [17], Homogeneous Balance Method [18], F-expansion Method [19], Symmetry Method [20], Exp-function Method [21].

PM is one of the well-known methods which based on the existence of small/large parameters in the equation. These parameters are called perturbation quantities. Because of this restriction, PM can only be applied for weakly non-linear problems.

In ADM, we need to calculate Adomian polynomials which are difficult to obtain.

In HPM, one can obtain approximate solution even if there is no restriction on small/large physical parameters. The method contains an embedding parameter p. The application of HPM in nonlinear problems has been presented by many researchers [2,4,7,8,9,10,11,12].

HAM, originally presented by Liao in 1992, has an embedding parameters p and non-zero auxiliary parameter h which provides us with a simple way to adjust and control the radius of convergence of series solution. Later, this parameter h is called convergence-control parameter.

The main purpose of all the above methods is to find series solutions which converge to the exact solutions of the problems. Many researchers try to find which method is better than the others. There are a lot of papers which include comparison of the methods. Generally, we choose problems which have an exact solutions for testing numerical and approximate methods. Even those special exact solutions that do not have a clear physical meaning can be used as "test problems" to verify the consistency and estimate errors of various numerical, asymptotic and approximate analytical methods. In this paper, we generalize trial function to get better approximate solutions of the linear or nonlinear problems. Not only do we obtain better approximate solutions, but also we get less computational work.

2. The Homotopy Perturbation Method

The basic idea of this method is the following. We consider the nonlinear differential equation

$$A(u) - f(\mathbf{r}) = 0, \qquad \mathbf{r} \in \Omega \tag{1}$$

with boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0, \qquad \mathbf{r} \in \Gamma$$

where A is a general differential operator, B is a boundary operator, $f(\mathbf{r})$ is a known analytic function. Γ is the boundary of the domain Ω. In general, A can be written in two parts which are linear operator L, and nonlinear operator N. Therefore, Eq.(1) can be written as follows

$$L(u) + N(u) - f(\mathbf{r}) = 0.$$

According to the homotopy technique, we can construct a homotopy $\Phi(\mathbf{r}, t; p) : \Omega \times [0, 1] \to \mathbf{R}$ which satisfies

$$H(\Phi, p) = (1 - p)[L(\Phi) - L(u_0)] + p[A(\Phi) - f(\mathbf{r})] = 0$$

where $p \in [0, 1]$ is an embedding parameter, $\mathbf{r} \in \Omega$, u_0 is an initial condition of Eq.(1), $\Phi(\mathbf{r}, t; p)$ is an unknown function. We can write this homotopic equation as follows

$$H(\Phi, p) = L(\Phi) - L(u_0) + pL(u_0) + p[N(\Phi) - f(\mathbf{r})] = 0$$

It is easy to see that this homotopy satisfies following initial conditions,

$$\begin{aligned} H(\Phi, 0) &= L(\Phi) - L(u_0) = 0 \\ H(\Phi, 1) &= A(\Phi) - f(\mathbf{r}) = 0. \end{aligned}$$

The changing process of p from zero to unity is just that of $\Phi(\mathbf{r}, t; p)$ from $u_0(\mathbf{r}, t)$ to $u(\mathbf{r}, t)$. Expanding $\Phi(\mathbf{r}, t; p)$ in taylor series with respect to the p, we can write

$$\Phi(\mathbf{r}, t; p) = u_0(\mathbf{r}, t) + pu_1(\mathbf{r}, t) + p^2 u_2(\mathbf{r}, t) + \cdots, \tag{2}$$

when $p = 1$ this infinite series gives us the series solution of the problem.

$$u(\mathbf{r}, t) = \lim_{p \to 1} \Phi(\mathbf{r}, t; p) = u_0(\mathbf{r}, t) + u_1(\mathbf{r}, t) + u_2(\mathbf{r}, t) + \cdots, \tag{3}$$

Let trial function be $u_0(x, t) = P_0(t)g(x)$ which satisfies the following condition,

$$u_0(x, 0) = u(x, 0) = P_0(0)g(x)$$

where $g(x)$ is the initial condition and $P_0(0) = 1$ when $t = 0$.

3. Applications

Example 1 : Let's consider the special case of evolution equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 2\frac{\partial^3 u}{\partial x^2 \partial t}, \quad -\infty < x < \infty, \quad t > 0$$

with initial condition

$$u(x, 0) = \exp(-x).$$

We construct a homotopic equation by using He's method

$$(1 - p)[L(\Phi) - L(u_0)] + p[A(\Phi) - f(\mathbf{r})] = 0 \tag{4}$$

where L is a linear operator,

$$L(\Phi) = \frac{\partial \Phi}{\partial t} \tag{5}$$

and A is a whole linear operator,

$$A(\Phi) = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x} - 2\frac{\partial^3 \Phi}{\partial x^2 \partial t} \tag{6}$$

and u_0 is a trial function such that

$$u_0(x, t) = P_0(t)\exp(-x), \quad P_0(0) = 1. \tag{7}$$

If we substitute Eq.(5), Eq.(6) and Eq.(7) into Eq.(4), then we obtain the following homotopic equation

$$\frac{\partial \Phi}{\partial t} - P_0'(t)\exp(-x) = p\left(2\frac{\partial^3 \Phi}{\partial x^2 \partial t} - \frac{\partial \Phi}{\partial x} - P_0'(t)\exp(-x)\right). \tag{8}$$

If $p = 0$, then above equation satisfies trial function. If $p = 1$, then the above equation is just as original differential equation. Substituting Eq.(2) into Eq.(8) and equating coefficients of p, we obtain the following differential equations,

$$\begin{aligned}
p^0 &: \quad \{\frac{\partial u_0}{\partial t} - P_0'(t)\exp(-x) = 0, \qquad u_0(x,0) = \exp(-x) \\
p^1 &: \quad \{\frac{\partial u_1}{\partial t} = 2\frac{\partial^3 u_0}{\partial x^2 \partial t} - \frac{\partial u_0}{\partial x} - P_0'(t)\exp(-x), \qquad u_1(x,0) = 0 \\
p^2 &: \quad \{\frac{\partial u_2}{\partial t} = 2\frac{\partial^3 u_1}{\partial x^2 \partial t} - \frac{\partial u_1}{\partial x}, \qquad u_2(x,0) = 0 \\
&\vdots \\
p^n &: \quad \{\frac{\partial u_n}{\partial t} = 2\frac{\partial^3 u_{n-1}}{\partial x^2 \partial t} - \frac{\partial u_{n-1}}{\partial x}, \qquad u_n(x,0) = 0
\end{aligned}$$

If we solve above equations for unknowns u_n's, we obtain

$$\begin{aligned}
u_0(x,t) &= P_0(t)\exp(-x) \\
u_1(x,t) &= [P_0(t) - 1 + \int P_0(t)dt]\exp(-x) = P_1(t)\exp(-x) \\
u_2(x,t) &= [2P_1(t) + \int P_1(t)dt]\exp(-x) = P_2(t)\exp(-x) \\
u_3(x,t) &= [2P_2(t) + \int P_2(t)dt]\exp(-x) = P_3(t)\exp(-x) \\
&\vdots \\
u_n(x,t) &= [2P_{n-1}(t) + \int P_{n-1}(t)dt]\exp(-x) = P_n(t)\exp(-x), \qquad n > 1.
\end{aligned}$$

Therefore series solution can be written in the following

$$u(x,t) = [P_0(t) + P_1(t) + P_2(t) + P_3(t) + ...]\exp(-x).$$

As we see that once we compute u_1, we can get the other terms of the series solution without computing equations. This gives us less computational work and time. Let's give some examples for $P_0(t)$. If $P_0(t) = 1$, then terms of the series solution of the problem is the following

$$\begin{aligned}
u_0(x,t) &= \exp(-x) \\
u_1(x,t) &= t\exp(-x) \\
u_2(x,t) &= (\frac{t^2}{2} + 2t)\exp(-x) \\
u_3(x,t) &= (\frac{t^3}{3!} + 2t^2 + 4t)\exp(-x) \\
u_4(x,t) &= (\frac{t^4}{24} + t^3 + 6t^2 + 8t)\exp(-x) \\
u_5(x,t) &= (\frac{t^5}{120} + \frac{t^4}{3} + 4t^3 + 16t^2 + 16t)\exp(-x) \\
&\vdots
\end{aligned}$$

Having u_n for $n = 0, 1, 2, 3, 4, 5$, the solution $u(x,t)$ is

$$u(x,t) = \sum_{n=0}^{5} u_n = (\frac{t^5}{120} + \frac{t^4}{3} + 5t^3 + \frac{49t^2}{2} + 31t + 1)\exp(-x). \tag{9}$$

If $P_0(t) = t + 1$, then we obtain the terms of the series solution of the problem

$$\begin{aligned}
u_0(x,t) &= (t+1)\exp(-x) \\
u_1(x,t) &= (\frac{t^2}{2} + 2t)\exp(-x) \\
u_2(x,t) &= (\frac{t^3}{3!} + 2t^2 + 4t)\exp(-x) \\
u_3(x,t) &= (\frac{t^4}{24} + t^3 + 6t^2 + 8t)\exp(-x) \\
u_4(x,t) &= (\frac{t^5}{120} + \frac{t^4}{3} + 4t^3 + 16t^2 + 16t)\exp(-x). \\
&\vdots
\end{aligned}$$

Computing u_n for $n = 0, 1, 2, 3, 4$, the solution $u(x,t)$ is same as Eq.(9). If $P_0(t) = \frac{t^2}{2} + 3t + 1$, then terms of the series solution of the problem is the following

$$\begin{aligned}
u_0(x,t) &= (\frac{t^2}{2} + 3t + 1)\exp(-x) \\
u_1(x,t) &= (\frac{t^3}{3!} + 2t^2 + 4t)\exp(-x) \\
u_2(x,t) &= (\frac{t^4}{24} + t^3 + 6t^2 + 8t)\exp(-x) \\
u_3(x,t) &= (\frac{t^5}{120} + \frac{t^4}{3} + 4t^3 + 16t^2 + 16t)\exp(-x). \\
&\vdots
\end{aligned}$$

Computing u_n for $n = 0, 1, 2, 3$, the solution $u(x,t)$ is same as Eq.(9). As we see that if we choose $P_0(t)$ properly, we obtain the series solution of the problem in three iteration.

Example 2 : We consider the following problem

$$\frac{\partial u}{\partial t} + 2\frac{\partial^4 u}{\partial x^4} = \frac{\partial^3 u}{\partial x^2 \partial t}, \qquad -\infty < x < \infty, \qquad t > 0$$

with initial condition

$$u(x,0) = \sin x.$$

By same manipulating as we have done above, we obtain the following homotopic equation. Let trial function be $u_0(x,t) = P_0(t)\sin x$ where $P_0(0) = 1$.

$$\frac{\partial \Phi}{\partial t} - P_0'(t)\sin x = p\left(\frac{\partial^3 \Phi}{\partial x^2 \partial t} - 2\frac{\partial^4 \Phi}{\partial x^4} - P_0'(t)\sin x\right). \tag{10}$$

Substituting Eq.(2) into Eq.(10) and equating coefficients of p,

$$
\begin{aligned}
p^0 &: \quad \{\frac{\partial u_0}{\partial t} - P_0'(t)\sin x = 0, \qquad u_0(x,0) = \sin x,\\
p^1 &: \quad \{\frac{\partial u_1}{\partial t} = \frac{\partial^3 u_0}{\partial x^2 \partial t} - 2\frac{\partial^4 u_0}{\partial x^4} - P_0'(t)\sin x, \qquad u_1(x,0) = 0,\\
p^2 &: \quad \{\frac{\partial u_2}{\partial t} = \frac{\partial^3 u_1}{\partial x^2 \partial t} - 2\frac{\partial^4 u_1}{\partial x^4}, \qquad u_2(x,0) = 0,\\
&\vdots\\
p^n &: \quad \{\frac{\partial u_n}{\partial t} = \frac{\partial^3 u_{n-1}}{\partial x^2 \partial t} - 2\frac{\partial^4 u_{n-1}}{\partial x^4}, \qquad u_n(x,0) = 0.
\end{aligned}
$$

If we solve above equations for unknowns u_n's, we obtain the terms of the series solution

$$
\begin{aligned}
u_0(x,t) &= P_0(t)\sin x\\
u_1(x,t) &= -2[P_0(t) - 1 + \int P_0(t)dt]\sin x = P_1(t)\sin x\\
u_2(x,t) &= -[P_1(t) + 2\int P_1(t)dt]\sin x = P_2(t)\sin x\\
u_3(x,t) &= -[P_2(t) + 2\int P_2(t)dt]\sin x = P_3(t)\sin x\\
&\vdots\\
u_n(x,t) &= -[P_{n-1}(t) + 2\int P_{n-1}(t)dt]\sin x = P_n(t)\sin x, \qquad n > 1.
\end{aligned}
$$

Therefore series solution can be written in the following

$$u(x,t) = [P_0(t) + P_1(t) + P_2(t) + P_3(t) + ...]\sin x.$$

As we mentioned above, once we compute u_1, we can find the other terms of the series solution by using u_2, u_3, u_4,... directly. Let's give some examples for $P_0(t)$.

If $P_0(t) = 1$, then terms of the series solution of the problem is the following

$$
\begin{aligned}
u_0(x,t) &= \sin x\\
u_1(x,t) &= -2t\sin x\\
u_2(x,t) &= (2t^2 + 2t)\sin x\\
u_3(x,t) &= -(\frac{4t^3}{3} + 4t^2 + 2t)\sin x\\
u_4(x,t) &= (\frac{2t^4}{3} + 4t^3 + 6t^2 + 2t)\sin x\\
u_5(x,t) &= -(\frac{4t^5}{15} + \frac{8t^4}{3} + 8t^3 + 8t^2 + 2t)\sin x\\
u_6(x,t) &= (\frac{4t^6}{45} + \frac{4t^5}{3} + \frac{20t^4}{3} + \frac{40t^3}{3} + 10t^2 + 2t)\sin x.\\
&\vdots
\end{aligned}
$$

Having u_n for $n = 0, 1, 2, 3, 4, 5, 6$, the solution $u(x,t)$ is

$$u(x,t) = \sum_{n=0}^{6} u_n = (\frac{4t^6}{45} + \frac{4t^5}{3} + \frac{14t^4}{3} + 8t^3 + 6t^2)\sin x. \tag{11}$$

If $P_0(t) = 1 - 2t$, then we obtain

$$\begin{aligned}
u_0(x,t) &= (1-2t)\sin x\\
u_1(x,t) &= (2t^2+2t)\sin x\\
u_2(x,t) &= -(\frac{4t^3}{3}+4t^2+2t)\sin x\\
u_3(x,t) &= (\frac{2t^4}{3}+4t^3+6t^2+2t)\sin x\\
u_4(x,t) &= -(\frac{4t^5}{15}+\frac{8t^4}{3}+8t^3+8t^2+2t)\sin x\\
u_5(x,t) &= (\frac{4t^6}{45}+\frac{4t^5}{3}+\frac{20t^4}{3}+\frac{40t^3}{3}+10t^2+2t)\sin x.\\
&\vdots
\end{aligned}$$

Computing u_n for $n = 0, 1, 2, 3, 4, 5$, the solution $u(x,t)$ is same as Eq.(11).

If $P_0(t) = 1 + 2t^2$, then the series solution of the problem is the following

$$\begin{aligned}
u_0(x,t) &= (1+2t^2)\sin x\\
u_1(x,t) &= -(\frac{4t^3}{3}+4t^2+2t)\sin x\\
u_2(x,t) &= (\frac{2t^4}{3}+4t^3+6t^2+2t)\sin x\\
u_3(x,t) &= -(\frac{4t^5}{15}+\frac{8t^4}{3}+8t^3+8t^2+2t)\sin x\\
u_4(x,t) &= (\frac{4t^6}{45}+\frac{4t^5}{3}+\frac{20t^4}{3}+\frac{40t^3}{3}+10t^2+2t)\sin x.\\
&\vdots
\end{aligned}$$

Having u_n for $n = 0, 1, 2, 3, 4$, the solution $u(x,t)$ is same as Eq.(11).

If $P_0(t) = (\frac{-4t^3}{3} - 2t^2 - 2t + 1)$, then the series solution of the problem is the following

$$\begin{aligned}
u_0(x,t) &= (\frac{-4t^3}{3}-2t^2-2t+1)\sin x\\
u_1(x,t) &= (\frac{2t^4}{3}+4t^3+6t^2+2t)\sin x
\end{aligned}$$

$$
\begin{aligned}
u_2(x,t) &= -(\frac{4t^5}{15} + \frac{8t^4}{3} + 8t^3 + 8t^2 + 2t)\sin x \\
u_3(x,t) &= (\frac{4t^6}{45} + \frac{4t^5}{3} + \frac{20t^4}{3} + \frac{40t^3}{3} + 10t^2 + 2t)\sin x. \\
&\vdots
\end{aligned}
$$

Having u_n for $n = 0, 1, 2, 3$, the solution $u(x,t)$ is same as Eq.(11). As we see that if we choose $P_0(t)$ properly, we obtain the series solution of the problem in three iteration.

Example 3 : We consider the cauchy problem for porous medium equation with a source term, which is a simple model for a nonlinear heat propagation in reactive medium

$$
\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right) + bu \tag{12}
$$

with the initial condition $u(x,0) = x^{-1}$. Exact solution of Eq.(12) is given as $u(x,t) = x^{-1}\exp(bt)$ [6]. Let trial function be $u_0(x,t) = P_0(t)x^{-1}$ where $P_0(0) = 1$. By using He's method, the homotopic equation is following

$$
\frac{\partial \Phi}{\partial t} - P_0'(t)x^{-1} = p\left[a\frac{\partial}{\partial x}\left(\Phi^{-2}\frac{\partial \Phi}{\partial x}\right) + b\Phi - P_0'(t)x^{-1}\right]. \tag{13}
$$

Substituting Eq.(2) into Eq.(13) and equating coefficients of p, we get the following

$$
\begin{aligned}
p^0 &: \{\frac{\partial u_0}{\partial t} - P_0'(t)x^{-1} = 0, \qquad u_0(x,0) = x^{-1} \\
p^1 &: \{\frac{\partial u_1}{\partial t} = a\frac{\partial}{\partial x}\left(u_0^{-2}\frac{\partial u_0}{\partial x}\right) + bu_0 - P_0'(t)x^{-1}, \qquad u_1(x,0) = 0 \\
p^2 &: \{\frac{\partial u_2}{\partial t} = a\frac{\partial}{\partial x}\left(-2u_1u_0^{-3}\frac{\partial u_0}{\partial x} + u_0^{-2}\frac{\partial u_1}{\partial x}\right) + bu_1, \qquad u_2(x,0) = 0 \\
p^3 &: \{\frac{\partial u_3}{\partial t} = a\frac{\partial}{\partial x}\left(-2u_0^{-3}u_2\frac{\partial u_0}{\partial x} - 2u_0^{-3}u_1\frac{\partial u_1}{\partial x} + u_0^{-2}\frac{\partial u_2}{\partial x} + 3u_0^{-4}u_1^2\frac{\partial u_0}{\partial x}\right) + bu_2 \\
&\vdots
\end{aligned}
$$

If we solve above equations for unknowns u_n's, we obtain

$$
\begin{aligned}
u_0(x,t) &= x^{-1}P_0(t) \\
u_1(x,t) &= x^{-1}[-P_0(t) + 1 + b\int P_0(t)dt] = x^{-1}P_1(t) \\
u_2(x,t) &= x^{-1}\int P_1(t)dt = x^{-1}P_2(t) \\
u_3(x,t) &= x^{-1}\int P_2(t)dt = x^{-1}P_3(t)
\end{aligned}
$$

$$
\vdots
$$

$$
u_n(x,t) = x^{-1}\int P_{n-1}(t)dt = x^{-1}P_n(t)
$$

Therefore series solution can be written in the following

$$
u(x,t) = [P_0(t) + P_1(t) + P_2(t) + P_3(t) + ...]\, x^{-1}.
$$

As we mentioned above, once we compute u_1, we can find the other terms of the series solution by using u_2, u_3, u_4,... directly. Let's give some examples for $P_0(t)$.

If $P_0(t) = 1$, then terms of the series solution of the problem is $u_0(x,t) = x^{-1}$, $u_1(x,t) = btx^{-1}$ $u_2(x,t) = \frac{bt^2}{2}x^{-1}$, $u_3(x,t) = \frac{bt^3}{6}x^{-1}$, $u_4(x,t) = \frac{bt^4}{24}x^{-1}$, $u_5(x,t) = \frac{bt^5}{120}x^{-1}$...

If $P_0(t) = 1 + bt$, then terms of the series solution of the problem is $u_0(x,t) = (1+bt)x^{-1}$, $u_1(x,t) = \frac{bt^2}{2}x^{-1}$, $u_2(x,t) = \frac{bt^3}{6}x^{-1}$, $u_3(x,t) = \frac{bt^4}{24}x^{-1}$, $u_4(x,t) = \frac{bt^5}{120}x^{-1}$...

If $P_0(t) = 1 + bt + \frac{bt^2}{2}$, then terms of the series solution of the problem is $u_0(x,t) = (1 + bt + \frac{bt^2}{2})x^{-1}$, $u_1(x,t) = \frac{bt^3}{6}x^{-1}$, $u_2(x,t) = \frac{bt^4}{24}x^{-1}$, $u_3(x,t) = \frac{bt^5}{120}x^{-1}$... As we see that if we choose $P_0(t)$ properly, we obtain the series solution of the problem in three iteration. On the other hand, we obtain exact solution of the nonlinear problem, which is $u(x,t) = u_0 + u_1 + u_2 + ... = x^{-1}\exp(bt)$

Conclusion

As we see from above examples, we obtain better approximation by choosing $P_0(t)$ properly. In this way, we minimize the computational work and the time. Once we compute u_1, it is easy to find the other terms of the series solution of the problem without computing to the series equations. In addition to, we obtain better approximate solution of the problem no matter they are linear or nonlinear.

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