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RESEARCH ARTICLE / ARAŞTIRMA MAKALESİ

Beta Kesirli Türevli Kesirli Mertebeden Sınır Değer Problemlerinin Yaklaşık Çözümleri

Approximate Solutions of Fractional Boundary Value Problems Based on Beta Fractional Derivative

Sebahat Ebru DAS[1]

[1] *Yıldız Teknik Üniversitesi, Fen-Edebiyat Fakültesi, Matematik Bölümü, 34220, İstanbul, Türkiye*

Öz

Bu çalışmada, farklı sınır koşulları için kesirli mertebeden bir diferansiyel denklem sınıfını ele aldık. Bu kesirli mertebeden sınır değer problemlerinin(KSDP) yaklaşık çözümlerini elde etmek için Sinc Sıralama Yöntemi (SSY) uygulanmıştır. Kesirli türevler için Beta türevi alınmıştır. Ayrıca, bir takım test örnekleri sayısal simulasyonlarla birlikte verilmiştir. Yakınsaklık analizi SSY 'nin tutarlı ve etkin bir yöntem olduğunu göstermiştir.

Anahtar Kelimeler: Sinc Sıralama Yöntemi, Beta Türevi, Kesirli Sınır Değer Problemleri

Abstract

In this work, we deal with a class of fractional differential equations with different boundary conditions. Sinc-collocation method (SCM) has been employed to obtain the approximate solution for these fractional boundary value problems (FBVPs). Beta-derivative is taken for the fractional derivatives. Intercalarity, several test samples with numerical simulations are handled. Convergence analysis shows that SCM is a consistent and effective method.

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Keywords: Sinc-collocation method, Beta-derivative, Fractional boundary value problems.

I. INTRODUCTION

Fractional differential equations (FDEs) have been quite popular and attractive for many researchers owing to their practical applications in miscellaneous areas of engineering, science, etc. The reason is that mathematical modelling constructed on fractional derivatives either in compliance with time or space or both are more reliable and efficiently describe diversity of natural phenomena. For detailed information on fractional calculus, we refer to reader the monographs [1-6].

After the identification of the FDEs, various numerical methods are developed to work on the approximate solutions of FDEs by many authors. Some of these known numerical methods are Finite Difference Method[7], Variational Iteration Method[8-9], Adomian Decomposition Method[10-11], Homotopy Perturbation Method[12], Homotopy Analysis Method[13], Wavelets[14-15], Spectral Methods[16] and Sinc-Collocation Method[17-20]. If we take a brief look at the applications of these methods: In [7], Li and Zeng investigated the stability and convergence of some FDEs using the fractional Euler, fractional Adams and high order methods which are related to Finite Difference Method. In [8], Wu and Lee handled the fractional variational iteration method considering the modified Riemann-Liouville derivative to find the approximate solution of some FDEs. In [9], Singh and Kumar dealt with time fractional partial differential equations. By the use of the fractional variational iteration method, three test samples are taken to see the efficiency and accuracy of the solutions. Caputo sense is

Sorumlu yazar/Corresponding Author: Sebahat Ebru DAS, Tel: 050.542.06650, e-posta: eyeni@yildiz.edu.tr
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considered in this work. In the work of Gejji and Jafari [10], for converting a multi-order FDE to a system of FDE containing Caputo fractional derivatives, a new algorithm is enhanced. Moreover, some illustrative examples are presented. In [11], Wakil, Elhanbaly and Abdou worked on three different models with fractional-time derivative. Adomian Decomposition Method is applied to these models and then behaviours of the solutions are investigated. In [12], A new algorithm is developed for the solution of some second-order boundary value problems with two-point boundary conditions. In the algorithm, firstly an ordinary differential equation is transformed to an integral equation which has already satisfied the boundary conditions. Then Homotopy Analysis Method is considered to obtain the solution of the equation. In the work of Ghazanfari and Veisi [13], an extended Homotopy Analysis Method is considered to investigate the nonlinear fractional wave equation. Results approve the accurancy and efficiency of the method. In [14], A few kinds of FBVPs's solutions are tried to determine using by the Haar wavelet operational matrices of integration. By this method, FDE is transformed to a system which consist of algebraic equations. Numerical results are given by tables and graphs. In the paper [15], Haar wavelet collocation method is applied to the multi-term FDEs. Comparisons between other numerical methods are made and these results are illustrated with tables and graphs. Baleanu, Bhrawy and Taha [16] dealed with modified generalized Laguerre spectral tau and collocation methods for the solution of linear and nonlinear multi-term FDEs. A new algorithm is derived for expressing the modified generalized Laguerre polynomials. An effective technique is developed for solving the linear multi-term FDEs using a modified generalized Laguerre tau method.

In our work, we consider the following FBVP

$$\mu_2(x)y''(x)+\mu_1(x)y'(x)+\mu_\beta(x)y^{(\beta)}(x)+\mu_0(x)y(x)=f(x) \tag{1.1}$$

with the boundary conditions

$$y(a)=0\ ,\ y(b)=0 \tag{1.2}$$

where $y^{(\beta)}$ is the Beta-derivative for $0<\beta\leq 1$. Beta derivative is a relatively new defined fractional derivative [28] which has applications especially in biology and medicine [29-30] . For instance, A. Atangana and B.S.T Alkahtani constructed a model for supporting the spread of the Rubella virus with the help of beta derivative in [31]. After that, they made the stability and uniqueness analysis of the model. In [32] A. Atangana observed the spread of the Ebola virus in pregnant women to set up a mathematical model.

We tackle the SCM for the solution of the equation. Method is extensively applied in the examination of the models encountered in physics and engineering applications. For instance, SCM is used to obtain numerical solutions to BVPs for second-order Fredholm integro-differential equations in [17]. In [18], SCM is used to solve the Lane-Emden equation which is a nonlinear ordinary differential equation. Blasius equation which comes from boundary layer equations is solved via the SCM in [19]. In [20], the solution of Multi-Point BVPs is presented by means of SCM. For more work for SCM, see [21]-[27].

This paper is consisted of four main sections. In section 2, definition of Beta-derivative, important definitions and theorems related to the Sinc-collocation method are introduced. In section 3, the SCM is implemented to the equation (1.1) for the boundary conditions (1.2). In section 4, the convergence analysis of the sinc approximation is presented. In section 5, the comparison of the solutions for certain test examples are shown in both tables and in graphs. In section 6, the obtained results are interpreted.

II. PRELIMINARIES

In this section, Beta-derivative of a function is defined. Then, significant definitions and theorems which are used in SCM are given. For more information see [28]-[37].

Definition 1. *[38]* Let $f(x)$ be a function which is given as

$$f(x):[a,\infty)\to\mathbb{R}$$

$$D_x^\beta\big(f(x)\big)=\lim_{\varepsilon\to 0}\frac{f\left(x+\varepsilon\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-f(x)}{\varepsilon}\quad,\quad \forall x\geq a,\beta\epsilon(0,1] \tag{2.1}$$

provided the limit exists.

Theorem 1. Assume that Let $f:[a,\infty)\rightarrow\mathbb{R}$ be a function both differentiable and beta-differentiable; g be a differentiable function defined on $\mathbb{R}$. Then,

$$D_x^{\beta}\left((g\circ f)(x)\right)=\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}f'(x)g'(f(x)) \tag{2.2}$$

[38].

Definition 2. *[36]* The function

$$sinc(x)=\begin{cases}\frac{\sin\pi x}{\pi x}, & x\neq 0\\ 1, & x=0\end{cases}. \tag{2.3}$$

is called the *Sinc function*.

Definition 3. *[36]* The translated sinc function with space points are defined by

$$S(k,h)(x)=sinc\left(\frac{x-kh}{h}\right)=\begin{cases}\frac{\sin\pi\left(\frac{x-kh}{h}\right)}{\pi\left(\frac{x-kh}{h}\right)}, & x\neq kh\\ 1, & x=kh\end{cases}. \tag{2.4}$$

where $h>0$ and $k=0,\pm1,\pm2,\ldots$

To establish the approximation on the interval (a,b), the conformal map is defined as

$$\phi(z)=In\left(\frac{z-a}{b-z}\right) \tag{2.5}$$

$$\tilde{z}=\phi^{-1}(w)=\frac{a+be^{w}}{1+e^{w}}. \tag{2.6}$$

is the inverse map of $w=\phi(z)$. Here the basis functions are attained using the composite translated Sinc functions given as

$$S_k(z)=S(k,h)(z)\circ\phi(z)=sinc\left(\frac{\phi(z)-kh}{h}\right) \tag{2.7}$$

The sinc grid points $z_k\epsilon(a,b)$ in D_E are real numbers, so that they can denoted by x_k. For the evenly spaced points $\{kh\}_{k=-\infty}^{\infty}$, the image corresponding to these points is defined by

$$x_k=\phi^{-1}(kh)=\frac{a+be^{kh}}{1+e^{kh}},\ k=0,\pm1,\pm2,\ldots. \tag{2.8}$$

Lemma 1. If ϕ is the conformal 1-1 mapping which simply connects the domain D_E onto D_S. Then

$$\delta_{jk}^{(0)}=[S(j,h)\circ\phi(x)]|_{x=x_k}=\begin{cases}1, & j=k\\ 0, & j\neq k\end{cases}$$

$$\delta_{jk}^{(1)}=h\frac{d}{d\phi}[S(j,h)\circ\phi(x)]\Big|_{x=x_k}=\begin{cases}0, & j=k\\ \frac{(-1)^{k-j}}{k-j}, & j\neq k\end{cases}$$

$$\delta_{jk}^{(2)}=h^2\frac{d^2}{d\phi^2}[S(j,h)\circ\phi(x)]\Big|_{x=x_k}=\begin{cases}-\frac{\pi^2}{3}, & j=k\\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j\neq k\end{cases} \tag{2.9}$$

[40].

III. THE SINC-COLLOCATION METHOD

Let

$$y_n(x) = \sum_{k=-M}^{N} c_k S_k(x) \quad , \quad n = M + N + 1 \tag{3.1}$$

be the solution form of (1.1). $S_k(x)$ is the composite function of $S(k,h)$ and $\phi(x)$. The unknown coefficients $c_k c_k$ in (3.1) are determined with the help of the SCM.

Theorem 2. The first and the second derivatives of (3.1) are given by

$$\frac{d}{dx} y_n(x) = \sum_{k=-M}^{N} c_k \phi'(x) \frac{d}{d\phi} S_k(x) \tag{3.2}$$

$$\frac{d^2}{dx^2} y_n(x) = \sum_{k=-M}^{N} c_k \left(\phi''(x) \frac{d}{d\phi} S_k(x) + (\phi'(x))^2 \frac{d^2}{d\phi^2} S_k(x)\right) \tag{3.3}$$

Theorem 3. The Beta derivative of (3.1) for $0 < \beta \leq 10 < \beta \leq 1$ is presented by

$$y_n^{(\beta)}(x) = \sum_{k=-M}^{N} c_k \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \phi'(x) \frac{d}{d\phi} S_k(x) \tag{3.4}$$

Proof. The Beta derivative of (3.1) is written as

$$y_n(x) = \sum_{k=-M}^{N} c_k S_k^{(\beta)}(x) \quad . \tag{3.5}$$

with the help of Theorem 1, we can write

$$S_k^{(\beta)}(x) = \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} S_k'(x) \tag{3.6}$$

When we write (3.6) in (3.5), we obtain (3.4).

□

By replacing each term of (1.1) with the approach defined in (3.1)-(3.4) and the producting with $\left(\frac{1}{\phi'}\right)^2$, we determine

$$\sum_{k=-M}^{N} \left[c_k \left\{\sum_{i=0}^{2} g_i(x) \frac{d^i}{d\phi^i} S_k\right\}\right] = \left(f(x) \left(\frac{1}{\phi'(x)}\right)^2\right) \tag{3.7}$$

where $\quad g_0(x) = \mu_0(x) \left(\frac{1}{\phi'(x)}\right)^2$

$$g_1(x) = \left[\left(\mu_1(x) + \mu_\beta(x) \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{1}{\phi'(x)} - \mu_2(x) \left(\frac{1}{\phi'(x)}\right)'\right)\right]$$

$$g_2(x) = \mu_2(x) \tag{3.8}$$

From [37], it is known that

$$\delta_{jk}^{(0)} = \delta_{kj}^{(0)} \quad , \quad \delta_{jk}^{(1)} = -\delta_{kj}^{(1)} \quad , \quad \delta_{jk}^{(2)} = \delta_{kj}^{(2)} \tag{3.9}$$

Theorem 4. Let us consider the BVP (1.1)-(1.2). Then the Discrete Sinc-Collocation system for determining the unknown coefficients $\{c_k\}_{k=-M}^{N}$ of the approximate solution is given as

$$\sum_{k=-M}^{N} \left[c_k \left\{\sum_{i=0}^{2} \frac{g_i(x_j)(-1)^i}{h^i} \delta_{jk}^{(i)}\right\}\right] = \left(f(x_j) \left(\frac{1}{\phi'(x_j)}\right)^2\right) \quad , \quad j = -M, \dots, N \tag{3.10}$$

Some notations are defined to rewrite (3.10) in the matrix form. Let $\mathbf{D}(y)$ and $\mathbf{I}^{(i)}$ be $n \times n$ matrices as follows

$$D(y) = \begin{bmatrix} y(x_{-M}) & 0 & \dots & 0 \\ 0 & y(x_{-M+1}) & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & 0 & \dots & y(x_{-N}) \end{bmatrix} \qquad (3.11)$$

$$\mathbf{I}^{(i)} = \left[\delta_{jk}^{(i)}\right] \quad , \quad i,k = 0,1,2 \qquad (3.12)$$

If we write (3.11) and (3.12) in (3.10), we can represent it as

$$\boldsymbol{Ac} = \boldsymbol{B} \qquad (3.13)$$

where

$$\mathbf{A} = \sum_{i=0}^{2} \frac{1}{h^i} \mathbf{D}(g_i)\mathbf{I}^{(i)}$$

$$\mathbf{B} = \mathbf{D}\left(\frac{f}{\phi'}\right)\mathbf{I}^{(i)}$$

$$\mathbf{c} = (c_{-M}, c_{-M+1}, \dots, c_N)^T \qquad (3.14)$$

Finally, we can reach the approximate solution of (3.1) after finding the unknown coefficients c_k in the system (3.10).

IV. CONVERGENCE ANALYSIS

Let $H^2(D_E)$ be a class of analytic functions f in D_E and satisfies the following condition:

$$\int_{\phi^{-1}(p+T)} |f(z)|dz \to 0 \, , x \to \pm\infty \qquad (4.1)$$

where $T = \left\{iq: |q| < d \leq \frac{\pi}{2}\right\}$. Also f satisfies the following equation on the boundary of D_E :

$$N(f) = \int_{\partial D} |f(z)dz| < \infty \qquad (4.2)$$

Theorem 5. Assume that $\phi' \epsilon H^2(D_E)$; then, for all $z\epsilon(0,1)$,

$$E(f,h)(z) = \left|f(z) - \sum_{k=-\infty}^{\infty} f(kh)S(k,h)o\phi(z)\right| \leq \frac{N(f\phi')}{2d\pi\sinh(\pi d/h)} \leq 2\frac{N(f\phi')}{\pi d} e^{-\pi d/h} \qquad (4.3)$$

Additionally, let us take $h = \sqrt{\pi d/\alpha N} \leq 2\pi d/In2$.

If

$$|f(z)| \leq Ce^{-\alpha|\phi(z)|} \quad , \quad z\epsilon\Gamma \qquad (4.4)$$

for some positive constants C and α , then

$$\left|f(z) - \sum_{k=-N}^{N} f(kh)S(k,h)o\phi(z)\right| \leq K\sqrt{N}e^{-\sqrt{\pi d\alpha N}} \qquad (4.5)$$

Here K only depends on f, d and α. As a result, the convergence rate of Sinc approximation is exponential [39].

V. NUMERICAL RESULTS

In this section, SCM is applied to various test examples. In each example, we take that $d = \frac{\pi}{2}$ and *M*=*N*. For the application of the method we use *Mathematica10*. It is seen that the method is an effective method when the numerical results are examined.

Example 1. Let us take the following FBVP

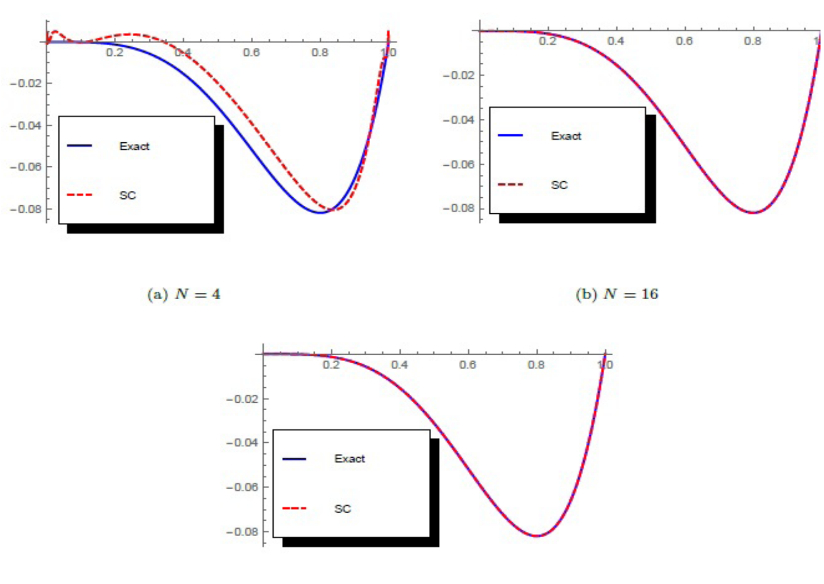
$$y''(x) + 0.5y^{(0.3)}(x) + y(x) = f(x) \quad , \quad y(0) = 0 \quad , \quad y(1) = 0$$

where $(x) = x^2(x^3 - x^2 + 20x - 12) + 0.5\left(x + \frac{1}{\Gamma(0.3)}\right)^{0.7} x^3(5x - 4)$.

This FBVP has an exact solution in the form of $y(x) = x^4(x-1)$. The approximate solution obtained with the aid of SCM of this problem is shown in Table 1. In addition to, the comparisons of the solutions of the example are shown graphically for different *N* values in Figure 1.

TABLE 1. Errors between the solutions for Example 1

x	*N*=4	*N*=8	*N*=16	*N*=32	*N*=64
0.1	1.516×10^{-4}	3.804×10^{-4}	3.806×10^{-5}	1.860×10^{-8}	2.409×10^{-9}
0.2	4.294×10^{-3}	3.142×10^{-4}	2.542×10^{-5}	1.334×10^{-6}	2.850×10^{-9}
0.3	8.097×10^{-3}	6.276×10^{-4}	1.535×10^{-5}	1.143×10^{-6}	1.625×10^{-9}
0.4	1.000×10^{-2}	1.027×10^{-3}	6.105×10^{-5}	8.423×10^{-7}	3.733×10^{-10}
0.5	1.148×10^{-2}	9.558×10^{-4}	3.122×10^{-5}	2.510×10^{-7}	2.422×10^{-10}
0.6	1.256×10^{-2}	8.046×10^{-4}	7.081×10^{-6}	5.791×10^{-7}	1.179×10^{-10}
0.7	1.098×10^{-2}	2.850×10^{-4}	1.294×10^{-5}	5.201×10^{-7}	5.853×10^{-10}
0.8	3.517×10^{-3}	7.066×10^{-4}	6.782×10^{-6}	2.145×10^{-8}	1.187×10^{-9}
0.9	5.140×10^{-3}	4.722×10^{-4}	2381×10^{-5}	1.759×10^{-7}	4.693×10^{-10}



(a) $N = 4$ (b) $N = 16$

(c) $N = 64$

FIGURE 1. The comparison of the solutions for Example 1

Example 2. Let us assume the following FBVP

$$y''(x) - xy'(x) + x^2 y^{(0.5)}(x) = f(x) \quad , \quad y(0) = 0 \quad , \quad y(1) = 0$$

where $f(x) = (x^2 - x - 2)cosx + (2x-1)sinx + x^2\left((1-x)cosx - sinx\right)\left(x + \frac{1}{\Gamma(0.5)}\right)^{0.5}$

The exact solution of this problem is $y(x) = (1-x)sinxy(x) = (1-x)sinx$. The numerical solutions determined by SCM of the problem are presented in Table 2. Furthermore, the comparisons of the solutions are given graphically in Figure 2.

x	*N*=4	*N*=8	*N*=16	*N*=32	*N*=64
0.1	2.285×10^{-3}	1.919×10^{-4}	4.847×10^{-6}	1.642×10^{-8}	1.024×10^{-10}
0.2	5.516×10^{-4}	2.133×10^{-4}	1.221×10^{-7}	5.819×10^{-8}	4.910×10^{-11}
0.3	3.158×10^{-3}	3.412×10^{-6}	4.745×10^{-6}	2.505×10^{-9}	2.693×10^{-11}
0.4	4.322×10^{-3}	1.519×10^{-4}	2.026×10^{-6}	5.520×10^{-8}	2.503×10^{-11}
0.5	5.283×10^{-3}	3.163×10^{-4}	7.191×10^{-6}	4.342×10^{-8}	3.852×10^{-11}
0.6	6.144×10^{-3}	4.545×10^{-4}	1.379×10^{-5}	8.204×10^{-8}	5.076×10^{-11}
0.7	5.616×10^{-3}	2.435×10^{-4}	7.034×10^{-6}	1.605×10^{-7}	1.461×10^{-10}
0.8	1.834×10^{-3}	3.217×10^{-4}	7.769×10^{-6}	2.010×10^{-7}	2.663×10^{-10}
0.9	2.764×10^{-3}	2.023×10^{-4}	1.026×10^{-5}	5.843×10^{-9}	2.610×10^{-10}

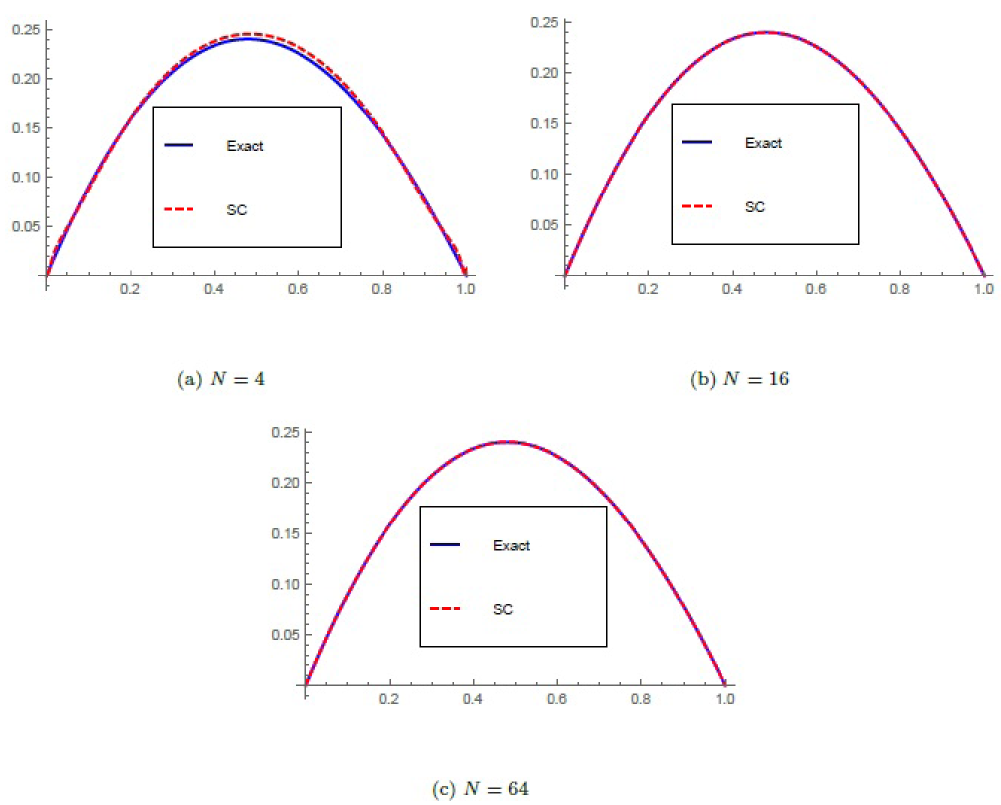


FIGURE 2. The comparison of the solutions for Example 2

VI. CONCLUSION

In this work we deal with the approximate solution of a class of FBVP which is given by (1.1)-(1.2). Beta derivative is considered as the fractional derivative. For the numerical results, SCM is used. This method gives accurate results for the (1.1) as presented in the previous section. As the result, we can say that Sinc Collocation algorithm is an effective tool for the determination of the approximate solution of (1.1).

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