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APPROXIMATION BY TWO DIMENSIONAL GADJIEV-IBRAGIMOV TYPE OPERATORS

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Abstract

In the present paper we introduce linear positive operators which are defined in [6] by generalization of Gadjiev-Ibragimov operators and give some approximation properties of these operators in the space of continuous functions of two variables on a compact set. We find certain moments of this operator and estimate for approximation error of the operators in terms of modulus of continuity. Then, we give some approximation properties of these operators.

Keywords: Gadjiev-Ibragimov Operators; Linear Positive Operators; Volkov Theorem.

1. Introduction

Gadjiev and Ibragimov, defined a general sequence of positive operators and studied someapproximation properties of this operators. Several generalizations of this operator have been studied in the one dimensional case by different researchers[1,2,3,6]. We introduce a generalization of linear positive operators in two dimensions which given in [4, 5]. Then we give some approximation properties of two dimensional Gadjiev-Ibragimov operators.

We give the construction of operators in the next section. Then we present some auxiliary result and approximation with the help of modulus of continuity will be given.

2. Construction of Operators

Definition 2.1. Let (α_n), (β_n) and (γ_n) be sequences of real numbers sequences such as

$$\lim_{n\to\infty} \beta_n = \infty, \lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0 \quad \text{and} \ \lim_{n\to\infty} \frac{\alpha_n}{\beta_n} n = 1$$

and

$$\lim_{m\to\infty} \gamma_m = \infty, \lim_{m\to\infty} \frac{\alpha_m}{\gamma_m} = 0 \quad \text{and} \ \lim_{m\to\infty} \frac{\alpha_m}{\gamma_m} m = 1.$$

$K_{n,\vartheta}(x)$ and $K_{m,\mu}(y)$ get a function satisfies the following conditions;

1) Let $n, m \in \mathbb{N}$ and $\vartheta, \mu \in \mathbb{N}_0$. For every finite A and $(x, y) \in C([0, A] \times [0, A])$ such that

$$(-1)^{\vartheta} K_{n,\vartheta}(x) \geq 0 \text{ and } (-1)^{\mu} K_{m,\mu}(y) \geq 0.$$

2) For any $(x, y) \in [0, A]$,

$$\sum_{\vartheta=0}^{\infty} K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!} = 1 \text{ and} \sum_{\mu=0}^{\infty} K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!} = 1.$$

3) For any $(x, y) \in [0, A]$,

$$K_{n,\vartheta}(x) = -nxK_{n+k,\vartheta-1}(x) \text{ and } K_{m,\mu}(x) = -myK_{m+l,\mu-1}(y)$$

where $n + k, m + l \in \mathbb{N}_0$ and k,l are constants independent of ϑ, μ.

Taking these equations into account, let us define a two variable generalization of Gadjiev-Ibragimov's operator for $f \in C([0, A] \times [0, A])$

$$L_{n,m}(f, x, y) = \sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty} f(\frac{\vartheta}{\beta_n}, \frac{\mu}{\gamma_m})K_{n,\vartheta}(x)K_{m,\mu}(y)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}\frac{(-\alpha_m)^{\mu}}{\mu!} \quad (1)$$

Here we use

$$P_{n,m}(x, y) = K_{n,\vartheta}(x)K_{m,\mu}(y)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}\frac{(-\alpha_m)^{\mu}}{\mu!}.$$

Lemma 2.1 $L_{n,m}$ defined by Equation 1 is linear and positive operators.

Proposition 2.1. Let $f \in C([0, A] \times [0, A])$for the operator given by Equation 1 we have

$$i)L_{n,m}(1, x, y) = 1.$$

$$ii)\ L_{n,m}(t_1, x, y) = \frac{\alpha_n}{\beta_n}nx.$$

$$iii)L_{n,m}(t_2, x, y) = \frac{\alpha_m}{\gamma_m}my.$$

$$iv)L_{n,m}({t_1}^2 + {t_2}^2, x, y) = \left(\frac{\alpha_n}{\beta_n}\right)^2 n(n + k)x^2 + \frac{\alpha_n}{{\beta_n}^2}nx + \left(\frac{\alpha_m}{\gamma_m}\right)^2 m(m + l)y^2 + \frac{\alpha_m}{{\gamma_m}^2}my.$$

Proof. (i) In Definition 2.1. 2) we get

$$L_{n,m}(1, x, y) = \sum_{\vartheta=0}^{\infty} K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}\sum_{\mu=0}^{\infty} K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!} = 1.$$

(ii) Using Definition 2.1. conditional of 2) and 3)

$$L_{n,m}(t_1, x, y) = \sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{\vartheta}{\beta_n}P_{n,m}(x, y)$$

$$= \sum_{\vartheta=0}^{\infty}\frac{\vartheta}{\beta_n}K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}$$

$$= \frac{\alpha_n}{\beta_n}nx\sum_{\vartheta=1}^{\infty} K_{n+k,\vartheta-1}(x)\frac{(-\alpha_n)^{\vartheta-1}}{(\vartheta-1)!}$$

$$=\frac{\alpha_n}{\beta_n}nx \qquad\qquad (n+k)\in\mathbb{N}_0.$$

(iii) Definition of $L_{n,m}$ we have

$$L_{n,m}(t_2,x,y)=\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{\mu}{\gamma_m}K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}$$

$$=\frac{-\alpha_m}{\gamma_m}\sum_{\vartheta=0}^{\infty}K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}\sum_{\mu=1}^{\infty}-myK_{m+l,\mu-1}(y)\frac{(-\alpha_m)^{\mu-1}}{(\mu-1)!}$$

$$=\frac{\alpha_m}{\gamma_m}my\sum_{\vartheta=0}^{\infty}K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}\sum_{\mu=1}^{\infty}K_{m+l,\mu-1}(y)\frac{(-\alpha_m)^{\mu-1}}{(\mu-1)!}$$

$$=\frac{\alpha_m}{\gamma_m}my \qquad\qquad (m+l)\in\mathbb{N}_0.$$

(iv)For $(n+k)\in\mathbb{N}_0$

$$L_{n,m}({t_1}^2,x,y)=\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\left(\frac{\vartheta}{\beta_n}\right)^2P_{n,m}(x,y)$$

$$=\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}\frac{\vartheta(\vartheta-1)}{{\beta_n}^2}K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}$$

$$+\frac{1}{{\beta_n}^2}\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}\vartheta K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}$$

$$=\frac{{\alpha_n}^2}{{\beta_n}^2}n(n+k)x^2\sum_{\vartheta=2}^{\infty}K_{n+k,\vartheta-2}(x)\frac{(-\alpha_n)^{\vartheta-2}}{(\vartheta-2)!}+\frac{1}{\beta_n}\frac{\alpha_n}{\beta_n}nx$$

$$=\left(\frac{\alpha_n}{\beta_n}\right)^2n(n+k)x^2+\frac{\alpha_n}{{\beta_n}^2}nx. \tag{2}$$

Similarly for $(m+l)\in\mathbb{N}_0$ we get

$$L_{n,m}({t_2}^2,x,y)=\left(\frac{\alpha_m}{\gamma_m}\right)^2m(m+l)y^2+\frac{\alpha_m}{{\gamma_m}^2}my \tag{3}$$

and using Equation 2 and Equation 3 we have

$$L_{n,m}({t_1}^2+{t_2}^2,x,y)=\left(\frac{\alpha_n}{\beta_n}\right)^2n(n+k)x^2+\frac{\alpha_n}{{\beta_n}^2}nx+\left(\frac{\alpha_m}{\gamma_m}\right)^2m(m+l)y^2+\frac{\alpha_m}{{\gamma_m}^2}my.$$

Theorem 2.1. For every $f\in C([0,A]\times[0,A])$

$$\lim_{n\to\infty}\left\|L_{n,m}(f,x,y)-f(x,y)\right\|=0.$$

Proof. We show conditional of Volkov Theorem. Clearly we have

$$\lim_{n\to\infty}\left\|L_{n,m}(1,x,y)-1\right\|=0.$$

Using $\frac{\alpha_n}{\beta_n}n\to 1$ we write

$$\left\|\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{\vartheta}{\beta_n}P_{n,m}(x,y)-x\right\|=\left\|\frac{\alpha_n}{\beta_n}nx-x\right\|.$$

Then we have

$$\lim_{n\to\infty}\left\|L_{n,m}(t_1,x,y)-x\right\|=0.$$

Similarly for $\frac{\alpha_m}{\gamma_m}m\to 1$ we get

$$\lim_{n\to\infty}\left\|L_{n,m}(t_2,x,y)-y\right\|=0.$$

Also by Proposition 2.1 iv)

$$\lim_{n\to\infty}\left\|L_{n,m}({t_1}^2+{t_2}^2,x,y)-x^2-y^2\right\|=0.$$

Example 2.1. The convergence of $L_{n,m}(f,x,y)$ to $f(x,y)=e^{1+2x}+y$ for $\alpha_n=\alpha_m=1,\beta_n=n$, $\gamma_m=m$ is illustrated in Figure1. $n=m=1(\text{brown}), n=m=3(\text{green}), n=m=10(\text{magenta})$

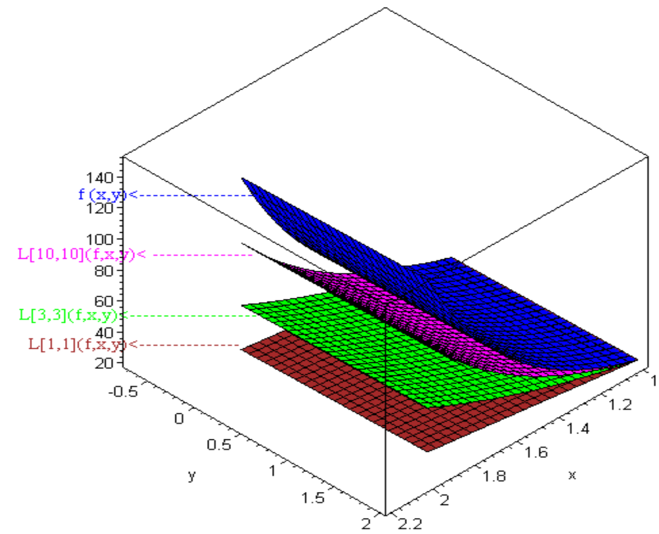

Figure 1:Approximation of $L_{n,m}(f,x,y)$

The first three moments of operators are given next Lemma.

Lemma 2.2. Let $(x,y)\in[0,A]\times[0,A]$ and for $n,m\in\mathbb{N}$ the following equalities hold.

$i) L_{n,m}(1,x,y)=1.$

$ii) L_{n,m}(t_1-x,x,y)=x\left(\frac{\alpha_n}{\beta_n}-1\right).$

$iii) L_{n,m}((t_1-x)^2,x,y)=\left[\left(\frac{\alpha_n}{\beta_n}\right)^2 n(n+k)-\frac{2\alpha_n}{\beta_n}n+1\right]x^2+\frac{\alpha_n}{{\beta_n}^2}nx.$

Proof.

(i) Clearly $L_{n,m}(1,x,y)=1.$

(ii)

$$L_{n,m}(t_1-x,x,y)=\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\left(\frac{\vartheta}{\beta_n}-x\right)P_{n,m}(x,y)=x\left(\frac{\alpha_n}{\beta_n}n-1\right)$$

(iii)

$$L_{n,m}((t_1-x)^2,x,y)=\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}(t_1-x)^2 P_{n,m}(x,y)$$

$$=\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\left(\frac{\vartheta}{\beta_n}\right)^2 P_{n,m}(x,y)-2x\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{\vartheta}{\beta_n}P_{n,m}(x,y)+x^2\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}P_{n,m}(x,y)$$

$$=\left(\frac{\alpha_n}{\beta_n}\right)^2 n(n+k)x^2+\frac{\alpha_n}{{\beta_n}^2}nx-2x^2\left(\frac{\alpha_n}{\beta_n}n\right)+x^2$$

$$=x^2\left[\left(\frac{\alpha_n}{\beta_n}\right)^2 n(n+k)-\frac{2\alpha_n}{\beta_n}n+1\right]+\frac{\alpha_n}{{\beta_n}^2}nx.$$

Remark 2.1 Similar equality is provided for $L_{n,m}(t_2-y,x,y)$ and $L_{n,m}((t_2-y)^2,x,y)$.

Now we estimate modulus of continuity of operators definition 2.1 in $C([0,A]\times[0,A])$.

Definition 2.2. Let $D\subset\mathbb{R}^2$ and $f:D\to\mathbb{R}$ bounded function. $K\subset D$ compact domain and let $(x=(x_1,x_2),\ y=(y_1,y_2))$ using partial modulus of continuity

$$\omega_1 f(f,\delta)=sup\{|f(x_1,y)-f(x_2,y)|:(x_1,y),(x_2,y)\in K,|x_1-x_2|\le\delta\}$$

$$\omega_2 f(f,\delta)=sup\{|f(x,y_1)-f(x,y_2)|:(x,y_1),(x,y_2)\in K,|y_1-y_2|\le\delta\}.$$

Theorem 2.1. Every $f\in C([0,A]\times[0,A])$ and let sequences of $(\alpha_n),(\beta_n),(\gamma_m)$ defined as in definition 2.1. Then for sufficiently large *n*,*m*

$$\left\|L_{n,m}(f,x,y)-f(x,y)\right\|_{C[0,A]}\le K_1 w_2(f,\delta_m)+K_2 w_1(f,\delta_n)$$

where K is a constant independent of *n*,*m* for $\delta_n=\sqrt{\left(n\frac{\alpha_n}{\beta_n}-1\right)^2+\frac{\alpha_n}{\beta_n}+\frac{1}{A\beta_n}}$ and

$\delta_m=\sqrt{\left(m\frac{\alpha_m}{\gamma_m}-1\right)^2+\frac{\alpha_m}{\gamma_m}+\frac{1}{A\gamma_m}}.$

Proof. Clearly using Definition 2.2 and Cauchy-Schwarz inequality we get

$$N_1 = \sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\left|f\left(\frac{\vartheta}{\beta_n},\frac{\mu}{\gamma_m}\right)-f\left(\frac{\vartheta}{\beta_n},y\right)\right|P_{n,m}(x,y)$$

$$\leq \sum_{\mu=0}^{\infty}\omega_2(f,\delta_m)\left[1+\frac{\left|\frac{\mu}{\gamma_m}-y\right|}{\delta_m}\right]K_{m,\mu}(y)\frac{(-\alpha_m)^\mu}{\mu!}$$

$$\leq \omega_2(f,\delta_m)\left\{1+\frac{1}{\delta_m}\sum_{\mu=0}^{\infty}\left|\frac{\mu}{\gamma_m}-y\right|\sqrt{K_{m,\mu}(y)\frac{(-\alpha_m)^\mu}{\mu!}}\sqrt{K_{m,\mu}(y)\frac{(-\alpha_m)^\mu}{\mu!}}\right\}$$

$$\leq \omega_2(f,\delta_m)\left\{1+\frac{1}{\delta_m}\sqrt{\sum_{\mu=0}^{\infty}\left|\frac{\mu}{\gamma_m}-y\right|^2 K_{m,\mu}(y)\frac{(-\alpha_m)^\mu}{\mu!}}\right\}.$$

Using Proposition 2.1

$$\sum_{\mu=0}^{\infty}\left|\frac{\mu}{\gamma_m}-y\right|^2 K_{m,\mu}(y)\frac{(-\alpha_m)^\mu}{\mu!} = \sum_{\mu=0}^{\infty}\left[\left(\frac{\mu}{\gamma_m}\right)^2-2y\frac{\mu}{\gamma_m}+y^2\right]K_{m,\mu}(y)\frac{(-\alpha_m)^\mu}{\mu!}$$

$$=\left(\frac{\alpha_m}{\gamma_m}\right)^2 m(m+l)y^2+\frac{\alpha_m}{{\gamma_m}^2}my-2y\frac{\mu}{\gamma_m}+y^2.$$

from $\left(\frac{\mu}{\gamma_m}-y\right)^2=\left(\frac{\mu}{\gamma_m}\right)^2-2y\frac{\mu}{\gamma_m}+y^2$

$$\left|L_{n,m}(f,x,y)-f(x,y)\right|\leq \omega_2(f,\delta_m)\left\{1+\frac{1}{\delta_m}\left(\sum_{\mu=0}^{\infty}\left(\frac{\mu}{\gamma_m}\right)^2 K_{m,\mu}(y)\frac{(-\alpha_m)^\mu}{\mu!}\right.\right.$$

$$\left.\left.-2y\sum_{\mu=0}^{\infty}\frac{\mu}{\gamma_m}K_{m,\mu}(y)\frac{(-\alpha_m)^\mu}{\mu!}+y^2\sum_{\mu=0}^{\infty}K_{m,\mu}(y)\frac{(-\alpha_m)^\mu}{\mu!}\right)^{1/2}\right\}$$

$$=\omega_2(f,\delta_m)\left\{1+\frac{1}{\delta_m}\left(L_{n,m}({t_2}^2,x,y)-2yL_{n,m}(t_2,x,y)+y^2L_{n,m}(1,x,y)\right)^{1/2}\right\}.$$

For $y\in[0,A]$ we write $L_{n,m}({t_2}^2,x,y), L_{n,m}(t_2,x,y)$ and $L_{n,m}(1,x,y)$ using $\lim_{m\to\infty}\gamma_m=\infty$, $\lim_{m\to\infty}\frac{\alpha_m}{\gamma_m}=0$ and $\lim_{m\to\infty}\frac{\alpha_m}{\gamma_m}m=1$ equation for a large *m* and using the equalites $\frac{\alpha_m}{\gamma_m}\leq 1$ and $\frac{\alpha_m}{\gamma_m}m\leq 2$

$$\left|L_{n,m}(f,x,y)-f(x,y)\right|\leq \omega_2(f,\delta_m)\left\{1+\frac{1}{\delta_m}\left(\frac{m(m+l)}{{\gamma_m}^2}{\alpha_m}^2A^2+\frac{1}{\gamma_m}\frac{\alpha_m}{\gamma_m}mA\right.\right.$$

$$\left.\left.-2A\frac{\alpha_m}{\gamma_m}m+A^2\right)\right\}^{1/2}$$

$$\leq \omega_2(f,\delta_m)\left\{1+\frac{A}{\delta_m}\left(A^2\left[\left(\frac{\alpha_m}{\gamma_m}\right)^2m^2-2\frac{\alpha_m}{\gamma_m}m+1\right]+A^2\left[\left(\frac{\alpha_m}{\gamma_m}\right)^2ml+\frac{1}{A}\frac{1}{\gamma_m}\frac{\alpha_m}{\gamma_m}m+1\right]\right)^{1/2}\right\}$$

Then if we choose $\delta_m = \sqrt{\left(m\frac{\alpha_m}{\gamma_m}-1\right)^2+\frac{\alpha_m}{\gamma_m}+\frac{1}{A\gamma_m}}$ we have the following inequality constant K_1 independent on m

$$\left\|L_{n,m}(f,x,y)-f(x,y)\right\|_{C[0,A]} \le K_1 w_2\left(f, \sqrt{\left(m\frac{\alpha_m}{\gamma_m}-1\right)^2+\frac{\alpha_m}{\gamma_m}+\frac{1}{A\gamma_m}}\right).$$

Similarly for N_2 using Cauchy-Schwarz inequality and Proposition 2.1

$$N_2 = \sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\left|f\left(\frac{\vartheta}{\beta_n},y\right)-f(x,y)\right| P_{n,m}(x,y)$$

$$\le \omega_1(f,\delta_n)\left\{1+\frac{1}{\delta_n}\sqrt{\sum_{\vartheta=0}^{\infty}\left|\frac{\vartheta}{\beta_n}-x\right|^2 K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}}\right\}$$

So we get

$$\sum_{\vartheta=0}^{\infty}\left|\frac{\vartheta}{\beta_n}-x\right|^2 K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!} = \left(\frac{\alpha_n}{\beta_n}\right)^2 n(n+k)x^2+\frac{\alpha_n}{{\beta_n}^2}nx-2x\frac{\alpha_n}{\beta_n}n+x^2.$$

Using $\left(\frac{\vartheta}{\beta_n}-x\right)^2 = \left(\frac{\vartheta}{\beta_n}\right)^2-2x\frac{\vartheta}{\beta_n}+x^2$ and there for a large n

$$\left|L_{n,m}(f,x,y)-f(x,y)\right| \le \omega_1(f,\delta_n)\left\{1+\frac{1}{\delta_n}\left(L_{n,m}({t_1}^2,x,y)-2xL_{n,m}(t_1,x,y)\right.\right.$$

$$\left.\left.+x^2L_{n,m}(1,x,y)\right)^{1/2}\right\}.$$

For $x\in[0,A]$ we write $L_{n,m}({t_1}^2,x,y), L_{n,m}(t_1,x,y)$ and $L_{n,m}(1,x,y)$ using $\lim_{m\to\infty}\beta_n=\infty$, $\lim_{m\to\infty}\frac{\alpha_n}{\beta_n}=0$ and $\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}n=1$ equation for a large n and using the equalites $\frac{\alpha_n}{\beta_n}\le 1$ and $\frac{\alpha_n}{\beta_n}n\le 2$

$$\left|L_{n,m}(f,x,y)-f(x,y)\right| \le \omega_1(f,\delta_n)\left\{1+\frac{1}{\delta_n}\left(\frac{n(n+k)}{{\delta_n}^2}{\alpha_n}^2A^2+\frac{1}{\beta_n}\frac{\alpha_n}{\beta_n}nA-2A\frac{\alpha_n}{\beta_n}n+A^2\right)^{1/2}\right\}$$

$$\le \omega_1(f,\delta_n)\left\{1+\frac{2kA}{\delta_n}\left[\left(n\frac{\alpha_n}{\beta_n}-1\right)^2+\frac{\alpha_n}{\beta_n}+\frac{1}{A\beta_n}\right]^{1/2}\right\}.$$

Then if we choose $\delta_n = \sqrt{\left(n\frac{\alpha_n}{\beta_n}-1\right)^2+\frac{\alpha_n}{\beta_n}+\frac{1}{A\beta_n}}$ we have the following inequality constant K_2 independent on n

$$\left\|L_{n,m}(f,x,y)-f(x,y)\right\|_{C[0,A]} \le K_2 w_1\left(f, \sqrt{\left(n\frac{\alpha_n}{\beta_n}-1\right)^2+\frac{\alpha_n}{\beta_n}+\frac{1}{A\beta_n}}\right).$$

Then proof is completed.

Now we want to find the rate of convergence of the sequence of operators $L_{n,m}(f,x,y)$.

Example 2.2. The error bound of the function $f(x,y) = \frac{x^2+y^2}{10}$, $\alpha_n = 1$, $\beta_n = n$.

n,m	Error bound for modulus of continuity of $f(x,y)$
10	0.8755417528
10^2	0.2422741700
10^3	0.0731541753
10^4	0.0227874170
10^5	0.0071714175
10^6	0.0022643417
10^7	0.0007157018
10^8	0.0002262902
10^9	0.0000715558

Table 1:The error bound of $f(x,y) = \frac{x^2+y^2}{10}$.

3. Approximation Properties in C_ρ^k

Definition 3.1. For $(x,y) \in (0,\infty) \times (0,\infty)$ and let $f \in C_\rho^k$. Then two dimensional generalized Gadjiev-Ibragimov operators defined by

$$L_{n,m}(f,x,y) = \sum_{\vartheta=0}^{\infty} \sum_{\mu=0}^{\infty} f\left(\frac{\vartheta}{\beta_n}, \frac{\mu}{\gamma_m}\right) K_{n,\vartheta}(x) K_{m,\mu}(y) \frac{(-\alpha_n)^\vartheta}{\vartheta!} \frac{(-\alpha_m)^\mu}{\mu!}. \tag{4}$$

Lemma3.1. The following equalities hold for Equation 4

$i)L_{n,m}(1,x,y) = 1$

$ii)\ L_{n,m}(t_1,x,y) = \frac{\alpha_n}{\beta_n} nx$

$iii)L_{n,m}(t_2,x,y) = \frac{\alpha_m}{\beta_m} my$

$iv)L_{n,m}({t_1}^2 + {t_2}^2,x,y) = \left(\frac{\alpha_n}{\beta_n}\right)^2 n(n+k)x^2 + \frac{\alpha_n}{{\beta_n}^2} nx + \left(\frac{\alpha_m}{\gamma_m}\right)^2 m(m+l)y^2 + \frac{\alpha_m}{{\gamma_m}^2} my.$

Theorem 3.1. Let $\rho(x,y) = 1 + x^2 + y^2$ and $L_{n,m}: C_\rho \to B_\rho$ sequences of linear positive operators defined by Equation 4 then every $f \in C_\rho^k$

$$\lim_{n,m\to\infty} \left\| L_{n,m}(f,x,y) - f(x,y) \right\|_\rho = 0.$$

Proof. Using Volkov Theorem clearly

$$\lim_{n,m\to\infty} \left\| L_{n,m}(1,x,y) - 1 \right\|_\rho = 0.$$

We have $L_{n,m}(t_1,x,y)=\frac{\alpha_n}{\beta_n}nx$ then definition of norm in C_ρ

$$\left\|L_{n,m}(t_1,x,y)-x)\right\|_\rho=\sup_{(x,y)\in(0,\infty)\times(0,\infty)}\left|\frac{x}{1+x^2+y^2}\right|\left|\frac{\alpha_n}{\beta_n}n-1\right|$$
$$\leq\left|\frac{\alpha_n}{\beta_n}n-1\right|.$$

so

$$\left\|L_{n,m}(t_1,x,y)-x\right\|_\rho=0.$$

Similarly using $L_{n,m}(t_2,x,y)=\frac{\alpha_m}{\gamma_m}my$ we get

$$\left\|L_{n,m}(t_2,x,y)-y\right\|_\rho=\sup_{(x,y)\in(0,\infty)\times(0,\infty)}\left|\frac{x}{1+x^2+y^2}\right|\left|\frac{\alpha_m}{\gamma_m}m-1\right|$$
$$\leq\left|\frac{\alpha_m}{\gamma_m}m-1\right|$$

then

$$\left\|L_{n,m}(t_2,x,y)-y\right\|_\rho=0.$$

We have $L_{n,m}({t_1}^2,x,y)=\left(\frac{\alpha_n}{\beta_n}\right)^2n(n+k)x^2+\frac{\alpha_n}{{\beta_n}^2}nx$ then

$$\left\|L_{n,m}({t_1}^2,x,y)-x^2\right\|_\rho=\sup_{(x,y)\in(0,\infty)\times(0,\infty)}\frac{\left|L_{n,m}({t_1}^2,x,y)-x^2\right|}{1+x^2+y^2}$$
$$\leq\left|\left(\frac{\alpha_n}{\beta_n}\right)^2n(n+k)-1\right|+\left|\frac{\alpha_n}{{\beta_n}^2}n\right|.$$

So we write

$$\lim_{n,m\to\infty}\left\|L_{n,m}({t_1}^2,x,y)-x^2\right\|_\rho\leq\lim_{n,m\to\infty}\left[\left|\left(\frac{\alpha_n}{\beta_n}\right)^2n(n+k)-1\right|+\left|\frac{\alpha_n n}{{\beta_n}^2}\right|\right].$$

Using $\lim_{n\to\infty}\left(\frac{\alpha_n}{\beta_n}\right)^2n(n+k)=1$ and $\lim_{n\to\infty}\frac{\alpha_n}{{\beta_n}^2}n=0$ we write

$$\lim_{n,m\to\infty}\left\|L_{n,m}({t_1}^2,x,y)-x^2\right\|_\rho=0.$$

Similarly for $L_{n,m}({t_2}^2,x,y)=\left(\frac{\alpha_m}{\gamma_m}\right)^2m(m+l)y^2+\frac{\alpha_m}{{\gamma_m}^2}my$

$$\lim_{n,m\to\infty}\left\|L_{n,m}({t_2}^2,x,y)-y^2\right\|_\rho\leq\lim_{n,m\to\infty}\left[\left|\left(\frac{\alpha_m}{\gamma_m}\right)^2m(m+l)-1\right|+\left|\frac{\alpha_m}{{\gamma_m}^2}m\right|\right]$$

so we get

$$\lim_{n,m\to\infty}\left\|L_{n,m}({t_2}^2,x,y)-y^2\right\|_\rho=0.$$

Consequently

$$\lim_{n,m\to\infty}\left\|L_{n,m}({t_1}^2+{t_2}^2,x,y)-(x^2+y^2)\right\|_\rho=0.$$

It means that for every $f \in C_\rho^k$

$$\lim_{n,m\to\infty} \left\| L_{n,m}(f,x,y) - f(x,y) \right\|_\rho = 0.$$

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References

[1] Aral, A. (2003) Approximation by Ibragimov-Gadjiev operators in polynomial weighted space, Proc. Inst. Math.Mech. Natl. Acad. Sci. Azerb., 19 35–44.

[2] Coskun, T. (2012) On the order of weighted approximation by positive linear operators, Turkish J. Math., 36 (1), pp. 113-120.

[3] Ghorbanalizadeh, A. M. (2014) On the order of weighted approximation of unbounded functions by some generalizationsof Gadjiev-Ibragimov operators, J. Appl. Anal.,20(1):87-92.

[4] Gonul, N., Coskun, E.(2012) Weighted approximation by positive linear operators, Proceedings of IMM of NAS of Azerbaijan, XXXVI (XLIV): 41-50.

[5] Gonul, N., Coskun, E. (2013) Approximation with modified Gadjiev–Ibragimov operators in C[0,A], J. Comput. Anal.Appl., 15 (5): 868-879.

[6] Ibragimov, I. I., Gadjiev, A. D. (1970) On a certain Family of linear positive operators, Soviet Math. Dokl., EnglishTrans., 11, 1092–1095.