

Codes over $\mathbb{Z}_p[u]/\langle u^r \rangle \times \mathbb{Z}_p[u]/\langle u^s \rangle$

Research Article

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Abstract: In this paper we generalize $\mathbb{Z}_2\mathbb{Z}_2[u]$ -linear codes to codes over $\mathbb{Z}_p[u]/\langle u^r \rangle \times \mathbb{Z}_p[u]/\langle u^s \rangle$ where p is a prime number and $u^r = 0 = u^s$. We will call these family of codes as $\mathbb{Z}_p[u^r, u^s]$ -linear codes which are actually special submodules. We determine the standard forms of the generator and parity-check matrices of these codes. Furthermore, for the special case $p = 2$, we define a Gray map to explore the binary images of $\mathbb{Z}_2[u^r, u^s]$ -linear codes. Finally, we study the structure of self-dual $\mathbb{Z}_2[u^2, u^3]$ -linear codes and present some examples.

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1. Introduction

Linear codes are the most important family of codes in coding theory. Because they have some advantages compared to the non-linear codes. For example, linear codes have an easier encoding and decoding process than non-linear arbitrary codes. In the beginning, studies on linear codes were mainly over fields, especially binary fields. Later, codes over extension fields and, in general, finite fields were studied by many researchers. In 1994, a remarkable paper written by Hammons et al.[8] brought a new direction to researches on coding theory. In this paper, they showed that some well known non-linear codes are actually binary images of some linear codes over the ring of integers modulo 4, i.e., \mathbb{Z}_4 . After this work a lot of research has been directed towards codes over rings.

The structure of binary linear codes and quaternary linear codes have been studied in details for the last sixty years. However, in 2010, Borges et al. have introduced a new class of error correcting codes, actually mixed alphabet codes, over $\mathbb{Z}_2 \times \mathbb{Z}_4$ called $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes in [7]. These family of codes generalizes the class of binary linear codes and the class of quaternary linear codes. For positive integers α, β , a $\mathbb{Z}_2\mathbb{Z}_4$ -additive code \mathcal{C} is defined as a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ with $\alpha + 2\beta = n$. Another important ring with four elements which is not isomorphic to \mathbb{Z}_4 is the ring $\mathcal{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 = \{0, 1, u, 1 + u\}$ where $u^2 = 0$. There are many studies related to codes over the ring $\mathcal{R} = \mathbb{Z}_2 + u\mathbb{Z}_2$ in the literature, the reader

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may see some of them in [1, 2, 6]. Recently, Aydogdu et al. have introduced $\mathbb{Z}_2\mathbb{Z}_2[u]$ -linear codes in [3] where $\mathbb{Z}_2[u] = \{0, 1, u, 1 + u\} = \mathcal{R}$. Even though the structures of these codes and the structures of $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes are similar $\mathbb{Z}_2\mathbb{Z}_2[u]$ -linear codes have some advantages compared to $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes. One of these advantages is that, the Gray (binary) images of $\mathbb{Z}_2\mathbb{Z}_2[u]$ -linear codes are always linear, but this is not always the case for codes over $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. Another advantage of working with such linear codes is that the factorization of polynomials in $\mathbb{Z}_2[x]$ is also valid since \mathbb{Z}_2 is a subring of \mathcal{R} (see [6]) and Hensel's lift is not necessary.

The generalizations of $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes are considered in the literature. For instance, Aydogdu and Siap introduced $\mathbb{Z}_2\mathbb{Z}_{2^s}$ and $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ additive codes in [4] and [5] respectively. This paper attempts to study a further generalization. An important reason motivating us to study these families of additive codes is that they can be mapped to codes over finite fields via Gray maps and moreover they offer some interesting algebraic structures.

In this paper, we generalize $\mathbb{Z}_2\mathbb{Z}_2[u]$ -linear codes to codes over $\mathbb{Z}_p[u]/\langle u^r \rangle \times \mathbb{Z}_p[u]/\langle u^s \rangle = \mathbb{Z}_p[u^r, u^s]$ where p is a prime number and $u^r = 0 = u^s$. We also determine the structure of these codes by giving the standard forms of generator and parity-check matrices. Finally, we present binary images of these codes for the special case $p = 2$ and illustrate some examples. Throughout the paper we assume that $r \leq s$.

Let $\mathcal{R}_r = \mathbb{Z}_p + u\mathbb{Z}_p + u^2\mathbb{Z}_p + \dots + u^{r-1}\mathbb{Z}_p = \mathbb{Z}_p[u]/\langle u^r \rangle$ and $\mathcal{R}_s = \mathbb{Z}_p + u\mathbb{Z}_p + u^2\mathbb{Z}_p + \dots + u^{s-1}\mathbb{Z}_p = \mathbb{Z}_p[u]/\langle u^s \rangle$ be the finite rings with $u^r = 0 = u^s$. Let us define the $\mathbb{Z}_p[u]$ -module

$$\mathbb{Z}_p[u^r, u^s] = \{(a, b) \mid a \in \mathcal{R}_r \text{ and } b \in \mathcal{R}_s\}.$$

Being inspired by the definition of $\mathbb{Z}_2\mathbb{Z}_2[u]$ -linear codes, we give the following definition.

Definition 1.1. Let \mathcal{C} be the non-empty subset of $\mathbb{Z}_p^\alpha[u]/\langle u^r \rangle \times \mathbb{Z}_p^\beta[u]/\langle u^s \rangle$. If \mathcal{C} is a \mathcal{R}_r -submodule of $\mathbb{Z}_p^\alpha[u]/\langle u^r \rangle \times \mathbb{Z}_p^\beta[u]/\langle u^s \rangle$ then \mathcal{C} is called a $\mathbb{Z}_p[u^r, u^s]$ -linear code.

We understand from the definition of $\mathbb{Z}_p[u^r, u^s]$ -linear codes that the first α coordinates of the $\mathbb{Z}_p[u^r, u^s]$ -linear code \mathcal{C} are elements from \mathcal{R}_r and the remaining β coordinates are elements from \mathcal{R}_s . Also, it can be easily concluded that this code is isomorphic to an abelian group $\mathbb{Z}_p^{k_{r-1}} \times \mathbb{Z}_p^{2k_{r-2}} \times \dots \times \mathbb{Z}_p^{k_0} \times \mathbb{Z}_p^{s l_0} \times \mathbb{Z}_p^{(s-1)l_1} \times \dots \times \mathbb{Z}_p^{l_{s-1}}$. Considering all these parameters we will say \mathcal{C} is of type $(\alpha, \beta; k_0, k_1, \dots, k_{r-1}; l_0, l_1, \dots, l_{s-1})$. Further, the number of the codewords of the \mathcal{C} is $|\mathcal{C}| = p^{rk_0} p^{(r-1)k_1} \dots p^{k_{r-1}} \dots p^{sl_0} p^{(s-1)l_1} \dots p^{l_{s-1}}$.

Definition 1.2. Let \mathcal{C} be a $\mathbb{Z}_p[u^r, u^s]$ -linear code. Let us define \mathcal{C}_α (respectively \mathcal{C}_β) as the punctured code of \mathcal{C} by deleting the coordinates outside α (respectively β). If $\mathcal{C} = \mathcal{C}_\alpha \times \mathcal{C}_\beta$ then \mathcal{C} is called separable.

We also note that for a $\mathbb{Z}_2\mathbb{Z}_2[u]$ -linear code \mathcal{C} of type $(\alpha, \beta; k_0; k_1, k_2)$, the standard forms of the generator and the parity-check matrices are given by

$$G = \left[\begin{array}{cc|cc} I_{k_0} & A_1 & 0 & 0 & uT \\ 0 & S & I_{k_1} & A & B_1 + uB_2 \\ 0 & 0 & 0 & uI_{k_2} & uD \end{array} \right], \tag{1}$$

$$H = \left[\begin{array}{cc|ccc} -A_1^t & I_{\alpha-k_0} & -uS^t & 0 & 0 \\ -T^t & 0 & -(B_1 + uB_2)^t + D^t A^t & -D^t & I_{\beta-k_1-k_2} \\ 0 & 0 & -uA^t & uI_{k_2} & 0 \end{array} \right] \tag{2}$$

where A, A_1, B_1, B_2, D, S and T are matrices over \mathbb{Z}_2 .

2. Standard form of the generator matrices of $\mathbb{Z}_p[u^r, u^s]$ -linear codes

In this section of the paper, we give standard form of the generator matrix of a $\mathbb{Z}_p[u^r, u^s]$ -linear code \mathcal{C} . A generator matrix G for a linear code \mathcal{C} is the matrix such that the rows are basis vectors of \mathcal{C} . We can put this generator matrix in a special form by elementary row operations, and we say this matrix is the standard form of the generator matrix. Generator matrices give us information about the structure of a linear code. If we know the standard form, we can easily read the type of a code and then find the number of elements. Further, we say any two codes are *permutation equivalent* or only *equivalent* if one can be obtained from the other by permutation of their coordinates or (if necessary) changing the coordinates by their unit multiples. The below theorem determines the generator matrix of a $\mathbb{Z}_p[u^r, u^s]$ -linear code \mathcal{C} .

Theorem 2.1. *Let \mathcal{C} be a $\mathbb{Z}_p[u^r, u^s]$ -linear code then \mathcal{C} is permutation equivalent to a $\mathbb{Z}_p[u^r, u^s]$ -linear code which has the following standard generator matrix of the form.*

$$G = \begin{bmatrix} R & A \\ B & S \end{bmatrix} \tag{3}$$

where

$$R = \begin{bmatrix} I_{k_0} & R_{01} & R_{02} & R_{03} & \cdots & R_{0,r-2} & R_{0,r-1} & R_{0,r} \\ 0 & uI_{k_1} & uR_{12} & uR_{13} & \cdots & uR_{1,r-2} & uR_{1,r-1} & uR_{1,r} \\ 0 & 0 & u^2I_{k_2} & u^2R_{23} & \cdots & u^2R_{2,r-2} & u^2R_{2,r-1} & u^2R_{2,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & u^{r-2}I_{k_{r-2}} & u^{r-2}R_{r-2,r-1} & u^{r-2}R_{r-2,r} \\ 0 & 0 & 0 & 0 & \cdots & 0 & u^{r-1}I_{k_{r-1}} & u^{r-1}R_{r-1,r} \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & \cdots & u^{s-r}A_{01} & u^{s-r}A_{02} & \cdots & u^{s-r}A_{0,r-2} & u^{s-r}A_{0,r-1} & u^{s-r}A_{0,r} \\ 0 & 0 & \cdots & 0 & u^{s-r+1}A_{12} & \cdots & u^{s-r+1}A_{1,r-2} & u^{s-r+1}A_{1,r-1} & u^{s-r+1}A_{1,r} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & u^{s-r+2}A_{2,r-2} & u^{s-r+2}A_{2,r-1} & u^{s-r+2}A_{2,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & u^{s-2}A_{r-2,r-1} & u^{s-2}A_{r-2,r} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & u^{s-1}A_{r-1,r} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & B_{01} & B_{02} & B_{03} & \cdots & B_{0,r-2} & B_{0,r-1} & B_{0,r} \\ 0 & B_{11} & B_{12} & B_{13} & \cdots & B_{1,r-2} & B_{1,r-1} & B_{1,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & B_{s-r-1,1} & B_{s-r-1,2} & B_{s-r-1,3} & \cdots & B_{s-r-1,r-2} & B_{s-r-1,r-1} & B_{s-r-1,r} \\ 0 & 0 & uB_{s-r,2} & uB_{s-r,3} & \cdots & uB_{s-r,r-2} & uB_{s-r,r-1} & uB_{s-r,r} \\ 0 & 0 & 0 & u^2B_{s-r+1,3} & \cdots & u^2B_{s-r+1,r-2} & u^2B_{s-r+1,r-1} & u^2B_{s-r+1,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & u^{r-1}B_{s-2,r} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} I_{l_0} & S_{01} & S_{02} & S_{03} & \cdots & S_{0,s-2} & S_{0,s-1} & S_{0,s} \\ 0 & uI_{l_1} & uS_{12} & uS_{13} & \cdots & uS_{1,s-2} & uS_{1,s-1} & uS_{1,s} \\ 0 & 0 & u^2I_{l_2} & u^2S_{23} & \cdots & u^2S_{2,s-2} & u^2S_{2,s-1} & u^2S_{2,s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & u^{s-3}S_{s-3,s-2} & u^{s-3}S_{s-3,s-1} & u^{s-3}S_{s-3,s} \\ 0 & 0 & 0 & 0 & \cdots & u^{s-2}I_{l_{s-2}} & u^{s-2}S_{s-2,s-1} & u^{s-2}S_{s-2,s} \\ 0 & 0 & 0 & 0 & \cdots & 0 & u^{s-1}I_{l_{s-1}} & u^{s-1}S_{s-1,s} \end{bmatrix}.$$

In this standard form of the generator matrix, R_{ij} 's are matrices over \mathcal{R}_r and A_{ij} 's are matrices over \mathcal{R}_s for $0 \leq i < j \leq r$. And for $0 \leq t < s - 1, 0 \leq p < q \leq s, B_{tj}$'s are matrices over \mathcal{R}_r and S_{pq} 's are matrices over \mathcal{R}_s . Further, I_k 's and I_l 's are identity matrices with given sizes.

Proof. Let $\mathcal{C} \subseteq \mathcal{R}_r^\alpha \times \mathcal{R}_s^\beta$ be a $\mathbb{Z}_p[u^r, u^s]$ -linear code of type $(\alpha, \beta; k_0, k_1, \dots, k_{r-1}; l_0, l_1, \dots, l_{s-1})$. Since the first α coordinates of \mathcal{C} is a submodule of \mathcal{R}_r and the last β coordinates of \mathcal{C} is a submodule of \mathcal{R}_s then from [9] we can write the generator matrix for \mathcal{C} as follows:

$$\left[\begin{array}{c|c} \bar{R} & \mathbf{1} \\ \hline \mathbf{2} & \bar{S} \end{array} \right]$$

where

$$\bar{R} = \begin{bmatrix} I_{k_0} & \bar{R}_{01} & \bar{R}_{02} & \bar{R}_{03} & \cdots & \bar{R}_{0,r-2} & \bar{R}_{0,r-1} & \bar{R}_{0,r} \\ 0 & uI_{k_1} & u\bar{R}_{12} & u\bar{R}_{13} & \cdots & u\bar{R}_{1,r-2} & u\bar{R}_{1,r-1} & u\bar{R}_{1,r} \\ 0 & 0 & u^2I_{k_2} & u^2\bar{R}_{23} & \cdots & u^2\bar{R}_{2,r-2} & u^2\bar{R}_{2,r-1} & u^2\bar{R}_{2,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & u^{r-2}I_{k_{r-2}} & u^{r-2}\bar{R}_{r-2,r-1} & u^{r-2}\bar{R}_{r-2,r} \\ 0 & 0 & 0 & 0 & \cdots & 0 & u^{r-1}I_{k_{r-1}} & u^{r-1}\bar{R}_{r-1,r} \end{bmatrix}$$

and

$$\bar{S} = \begin{bmatrix} I_{l_0} & \bar{S}_{01} & \bar{S}_{02} & \bar{S}_{03} & \cdots & \bar{S}_{0,s-2} & \bar{S}_{0,s-1} & \bar{S}_{0,s} \\ 0 & uI_{l_1} & u\bar{S}_{12} & u\bar{S}_{13} & \cdots & u\bar{S}_{1,s-2} & u\bar{S}_{1,s-1} & u\bar{S}_{1,s} \\ 0 & 0 & u^2I_{l_2} & u^2\bar{S}_{23} & \cdots & u^2\bar{S}_{2,s-2} & u^2\bar{S}_{2,s-1} & u^2\bar{S}_{2,s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & u^{s-2}I_{l_{s-2}} & u^{s-2}\bar{S}_{s-2,s-1} & u^{s-2}\bar{S}_{s-2,s} \\ 0 & 0 & 0 & 0 & \cdots & 0 & u^{s-1}I_{l_{s-1}} & u^{s-1}\bar{S}_{s-1,s} \end{bmatrix}.$$

Now, we have to determine the forms of the matrices in $\mathbf{1}$ and $\mathbf{2}$. We will put codewords to $\mathbf{1}$ such that they do not change the form of the matrix \bar{R} . So we have

$$\left[\begin{array}{c|cccc} & u^{s-r}\bar{A}_{01} & u^{s-r}\bar{A}_{02} & \cdots & u^{s-r}\bar{A}_{0,s+1} \\ & u^{s-r+1}\bar{A}_{11} & u^{s-r+1}\bar{A}_{12} & \cdots & u^{s-r+1}\bar{A}_{1,s+1} \\ \bar{R}_1 & \vdots & \vdots & \vdots & \vdots \\ & u^{s-2}\bar{A}_{r-2,1} & u^{s-2}\bar{A}_{r-2,2} & \cdots & u^{s-2}\bar{A}_{r-2,s+1} \\ & u^{s-1}\bar{A}_{r-1,1} & u^{s-1}\bar{A}_{r-1,2} & \cdots & u^{s-1}\bar{A}_{r-1,s+1} \\ \hline \mathbf{2} & & & & \bar{S}_1 \end{array} \right].$$

Similarly, putting codewords to $\mathbf{2}$ by protecting the form of the matrix \bar{S} we get

$$\left[\begin{array}{c|cccc} & u^{s-r}\bar{A}_{01} & u^{s-r}\bar{A}_{02} & \cdots & u^{s-r}\bar{A}_{0,s+1} \\ & u^{s-r+1}\bar{A}_{11} & u^{s-r+1}\bar{A}_{12} & \cdots & u^{s-r+1}\bar{A}_{1,s+1} \\ & \vdots & \vdots & \vdots & \vdots \\ & u^{s-2}\bar{A}_{r-2,1} & u^{s-2}\bar{A}_{r-2,2} & \cdots & u^{s-2}\bar{A}_{r-2,s+1} \\ & u^{s-1}\bar{A}_{r-1,1} & u^{s-1}\bar{A}_{r-1,2} & \cdots & u^{s-1}\bar{A}_{r-1,s+1} \\ \hline & \bar{B}_{01} & \bar{B}_{02} & \cdots & \bar{B}_{0,r+1} \\ & \bar{B}_{11} & \bar{B}_{12} & \cdots & \bar{B}_{1,r+1} \\ & \vdots & \vdots & \vdots & \vdots \\ & \bar{B}_{s-r-1,1} & \bar{B}_{s-r-1,2} & \cdots & \bar{B}_{s-r-1,r+1} \\ & u\bar{B}_{s-r,1} & u\bar{B}_{s-r,2} & \cdots & u\bar{B}_{s-r,r+1} \\ & u^2\bar{B}_{s-r+1,1} & u^2\bar{B}_{s-r+1,2} & \cdots & u^2\bar{B}_{s-r+1,r+1} \\ & \vdots & \vdots & \vdots & \vdots \\ & u^{r-1}\bar{B}_{s-2,1} & u^{r-1}\bar{B}_{s-2,2} & \cdots & u^{r-1}\bar{B}_{s-2,r+1} \\ & 0 & 0 & \cdots & 0 \end{array} \right].$$

Finally, by applying necessary elementary row operations to above matrix, we have the standard form in (3). □

Example 2.2. Let \mathcal{C} be a $\mathbb{Z}_2[u^2, u^3]$ -linear code with the following generator matrix

$$\left[\begin{array}{ccc|ccc} 1 & u & 1+u & 1+u+u^2 & 0 & u \\ u & 0 & u & u & u+u^2 & u \\ 1+u & u & 1 & 1+u^2 & 1 & u+u^2 \\ 1 & 0 & u & u & u+u^2 & 0 \\ 1+u & u & u & u^2 & u^2 & u^2 \end{array} \right]. \tag{4}$$

Then, by applying elementary row operations we have the standard form of this generator matrix as,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & u & 0 & 0 & 0 \\ \hline 0 & 0 & 1+u & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & u \end{array} \right].$$

Therefore,

- \mathcal{C} is of type $(3, 3; 1, 1; 2, 1, 0)$.

- \mathcal{C} has $2^{2 \cdot 1} \cdot 2^1 \cdot 2^{3 \cdot 2} \cdot 2^{2 \cdot 1} \cdot 2^0 = 2^{11} = 2048$ codewords.

Example 2.3. Now, we consider any $\mathbb{Z}_p[u^2, u^3]$ -linear code of type $(\alpha, \beta; k_0, k_1; l_0, l_1, l_2)$. So, $r = 2$ and $s = 3$. Then the generator matrix of \mathcal{C} is permutation equivalent to a matrix of the form

$$G = \begin{bmatrix} I_{k_0} & R_{01} & R_{02} & 0 & 0 & uA_{01} & uA_{02} \\ 0 & uI_{k_1} & uR_{12} & 0 & 0 & 0 & u^2A_{12} \\ 0 & B_{01} & B_{02} & I_{l_0} & S_{01} & S_{02} & S_{03} \\ 0 & 0 & uB_{12} & 0 & uI_{l_1} & uS_{12} & uS_{13} \\ 0 & 0 & 0 & 0 & 0 & u^2I_{l_2} & u^2S_{23} \end{bmatrix}. \tag{5}$$

Corollary 2.4. Taking $p = 2$, $r = 1$ and $s = 2$ we have $\mathbb{Z}_2[u, u^2]$ -linear code which is equal to a $\mathbb{Z}_2\mathbb{Z}_2[u]$ -linear code introduced in [3]. Then if \mathcal{C} is a $\mathbb{Z}_2[u, u^2]$ -linear code of type $(\alpha, \beta; k_0; l_0, l_1)$, its generator matrix is

$$G = \begin{bmatrix} I_{k_0} & R_{01} & 0 & 0 & uA_{01} \\ 0 & B_{01} & I_{l_0} & S_{01} & S_{02} \\ 0 & 0 & 0 & uI_{l_1} & uS_{12} \end{bmatrix} \tag{6}$$

which is permutation equivalent to a matrix of the form (1).

3. Duality on $\mathbb{Z}_p[u^r, u^s]$ -linear codes and parity-check matrices

We have well known concept of duality over finite fields and rings. In this part, we define an inner product for codes over $\mathcal{R}_r^\alpha \times \mathcal{R}_s^\beta$ and we determine the structure of the dual space of a $\mathbb{Z}_p[u^r, u^s]$ -linear code \mathcal{C} using this inner product.

The inner product for any two vectors $v, w \in \mathcal{R}_r^\alpha \times \mathcal{R}_s^\beta$ is defined by

$$v \cdot w = u^{s-r} \left(\sum_{i=1}^{\alpha} v_i w_i \right) + \sum_{j=\alpha+1}^{\alpha+\beta} v_j w_j.$$

Moreover, we can easily define the dual code \mathcal{C}^\perp of $\mathbb{Z}_p[u^r, u^s]$ -linear code \mathcal{C} in the following standard way:

$$\mathcal{C}^\perp = \{w \in \mathcal{R}_r^\alpha \times \mathcal{R}_s^\beta \mid v \cdot w = 0 \text{ for all } v \in \mathcal{C}\}.$$

It is also clear that \mathcal{C}^\perp is an \mathcal{R}_r -submodule of $\mathcal{R}_r^\alpha \times \mathcal{R}_s^\beta$. So, \mathcal{C}^\perp is also a $\mathbb{Z}_p[u^r, u^s]$ -linear code.

We know that the dual space of any linear code is generated by a parity-check matrix of \mathcal{C} or equivalently generator matrix of \mathcal{C}^\perp . To establish the parity-check matrix of a $\mathbb{Z}_p[u^r, u^s]$ -linear code \mathcal{C} we first give the following definitions.

Let $k(R)$ and $l(S)$ be the number of rows of the matrices R and S , respectively. For $i = 0, 1, \dots, r - 1(s - 1)$, let $k_i(R)$ ($l_i(S)$) denote the number of rows of R (S) that are divisible by u^i but not by u^{i+1} . Then, $k(R) = \sum_{i=0}^{r-1} k_i(R)$ and $l(S) = \sum_{i=0}^{s-1} l_i(S)$. Therefore, we give the following theorem that determines the standard form of the parity-check matrix of a $\mathbb{Z}_p[u^r, u^s]$ -linear code \mathcal{C} .

Theorem 3.1. The parity-check matrix for a $\mathbb{Z}_p[u^r, u^s]$ -linear code \mathcal{C} of type $(\alpha, \beta; k_0, k_1, \dots, k_{r-1}; l_0, l_1, \dots, l_{s-1})$ with generator matrix in (3) is given by the following standard form matrix.

$$H = \begin{bmatrix} \bar{R} + F & M \\ N & \bar{S} + E \end{bmatrix}$$

where \bar{R} , F , M , N , \bar{S} and E are matrices of the following forms.

$$\bar{R} = \begin{bmatrix} \bar{R}_{0,r} & \bar{R}_{0,r-1} & \bar{R}_{0,r-2} & \cdots & \bar{R}_{0,3} & \bar{R}_{0,2} & \bar{R}_{0,1} & I_{\alpha-k(R)} \\ u\bar{R}_{1,r} & u\bar{R}_{1,r-1} & u\bar{R}_{1,r-2} & \cdots & u\bar{R}_{1,3} & u\bar{R}_{1,2} & uI_{k_{r-1}(R)} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u^{r-2}\bar{R}_{r-2,r} & u^{r-2}\bar{R}_{r-2,r-1} & u^{r-2}I_{k_2(R)} & \cdots & 0 & 0 & 0 & 0 \\ u^{r-1}\bar{R}_{r-1,r} & u^{r-1}I_{k_1(R)} & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} F_{0,r-2} & F_{0,r-3} & \cdots & F_{0,2} & F_{0,1} & 0 & 0 & 0 \\ uF_{1,r-2} & uF_{1,r-3} & \cdots & uF_{1,2} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u^{r-4}F_{r-4,r-2} & u^{r-4}F_{r-4,r-3} & \cdots & 0 & 0 & 0 & 0 & 0 \\ u^{r-3}F_{r-3,r-2} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} M_{0,s-1} & M_{0,s-2} & \cdots & M_{0,3} & M_{0,2} & M_{0,1} & 0 & 0 \\ uM_{1,s-1} & uM_{1,s-2} & \cdots & uM_{1,3} & uM_{1,2} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u^{r-2}M_{r-2,s-1} & u^{r-2}M_{r-2,s-2} & \cdots & 0 & 0 & 0 & 0 & 0 \\ u^{r-1}M_{r-1,s-1} & u^{r-1}M_{r-1,s-2} & \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N = \begin{bmatrix} N_{0,r} & N_{0,r-1} & N_{0,r-2} & \cdots & N_{0,3} & N_{0,2} & N_{0,1} & 0 \\ uN_{1,r} & uN_{1,r-1} & uN_{1,r-2} & \cdots & uN_{1,3} & uN_{1,2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u^{r-1}N_{r-1,r} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{S} = \begin{bmatrix} \bar{S}_{0,s} & \bar{S}_{0,s-1} & \bar{S}_{0,s-2} & \cdots & \bar{S}_{0,3} & \bar{S}_{0,2} & \bar{S}_{0,1} & I_{\beta-l(S)} \\ u\bar{S}_{1,s} & u\bar{S}_{1,s-1} & u\bar{S}_{1,s-2} & \cdots & u\bar{S}_{1,3} & u\bar{S}_{1,2} & uI_{l_{s-1}(S)} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u^{s-2}\bar{S}_{s-2,s} & u^{s-2}\bar{S}_{s-2,s-1} & u^{s-2}I_{l_2(S)} & \cdots & 0 & 0 & 0 & 0 \\ u^{s-1}\bar{S}_{s-1,s} & u^{s-1}I_{l_1(S)} & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} E_{0,s-2} & E_{0,s-3} & E_{0,s-4} & \cdots & E_{0,2} & E_{0,1} & 0 & 0 & 0 \\ uE_{1,s-2} & uE_{1,s-3} & uE_{1,s-4} & \cdots & uE_{1,2} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u^{r-2}E_{r-2,s-2} & u^{r-2}E_{r-2,s-3} & u^{r-2}E_{r-2,s-4} & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 3.2. Let C be a $\mathbb{Z}_p[u^2, u^3]$ -linear code of type $(\alpha, \beta; k_0, k_1; l_0, l_1, l_2)$ with the standard form of the generator matrix in (5). We calculate the parity-check matrix of this code step by step as follows.

$$H = \begin{bmatrix} \bar{R} + F & M \\ N & \bar{S} + E \end{bmatrix}.$$

Since $r = 2$, then $F = 0$. Now,

$$\bar{R} = \begin{bmatrix} \bar{R}_{02} & \bar{R}_{01} & I_{\alpha-k_0-k_1} \\ u\bar{R}_{12} & uI_{k_1} & 0 \end{bmatrix} \text{ where } \bar{R}_{01} = -R_{12}^t, \bar{R}_{02} = -\bar{R}_{01}R_{01}^t - R_{02}^t, \bar{R}_{12} = -R_{01}^t.$$

$$N = \begin{bmatrix} N_{02} & N_{01} & 0 \\ uN_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ where } N_{01} = -A_{12}^t, N_{02} = -N_{01}R_{01}^t - \bar{S}_{01}A_{01}^t - A_{02}^t, N_{12} = -A_{01}^t.$$

$$M = \begin{bmatrix} M_{02} & M_{01} & 0 & 0 \\ uM_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ where } M_{01} = -uB_{12}^t, M_{02} = -M_{01}S_{01}^t - u(\bar{R}_{01}B_{01}^t + B_{02}^t), M_{12} = -uB_{01}^t.$$

$$E = \begin{bmatrix} E_{01} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ where } E_{01} = -uN_{01}^tB_{01}^t, \text{ and}$$

$$\bar{S} = \begin{bmatrix} \bar{S}_{03} & \bar{S}_{02} & \bar{S}_{01} & I_{\beta-l_0-l_1-l_2} \\ u\bar{S}_{13} & u\bar{S}_{12} & uI_{l_2} & 0 \\ u^2\bar{S}_{23} & u^2I_{l_1} & 0 & 0 \end{bmatrix}$$

where

$$\begin{aligned} \bar{S}_{01} &= -S_{23}^t, \bar{S}_{02} = -\bar{S}_{01}S_{12}^t - S_{13}^t, \\ \bar{S}_{03} &= -(\bar{S}_{01}S_{02}^t + \bar{S}_{02}S_{01}^t) - S_{03}^t, \\ \bar{S}_{12} &= -S_{12}^t, \bar{S}_{13} = -\bar{S}_{12}S_{01}^t - S_{02}^t, \\ \bar{S}_{23} &= -S_{01}^t. \end{aligned}$$

So,

$$H = \left[\begin{array}{ccc|cccc} \bar{R}_{02} & \bar{R}_{01} & I_{\alpha-k_0-k_1} & M_{02} & M_{01} & 0 & 0 \\ u\bar{R}_{12} & uI_{k_1} & 0 & uM_{12} & 0 & 0 & 0 \\ \hline N_{02} & N_{01} & 0 & \bar{S}_{03} + E_{01} & \bar{S}_{02} & \bar{S}_{01} & I_{\beta-l_0-l_1-l_2} \\ uN_{12} & 0 & 0 & u\bar{S}_{13} & u\bar{S}_{12} & uI_{l_2} & 0 \\ 0 & 0 & 0 & u^2\bar{S}_{23} & u^2I_{l_1} & 0 & 0 \end{array} \right].$$

We can easily determine the type of C^\perp from the above matrix as $(\alpha, \beta; \alpha - k_0 - k_1, k_1; \beta - l_0 - l_1 - l_2, l_2, l_1)$.

Example 3.3. Now, let C be a $\mathbb{Z}_2[u^2, u^3]$ -linear code with the generator matrix in (4). We have found the standard form of this generator matrix before in Example 2.2. Therefore,

$$G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & u & 0 & 0 & 0 \\ \hline 0 & 0 & 1+u & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & u \end{array} \right] = \left[\begin{array}{ccc|cc} I_{k_0} & R_{01} & R_{02} & 0 & 0 \\ 0 & uI_{k_1} & uR_{12} & 0 & 0 \\ \hline 0 & B_{01} & B_{02} & I_{l_0} & S_{01} \\ 0 & 0 & uB_{12} & 0 & uI_{l_1} \end{array} \right].$$

Hence, with the help of previous example we have the parity-check matrix of \mathcal{C} as

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & u+u^2 & 0 & 0 \\ 0 & u & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u^2 \end{array} \right].$$

So, \mathcal{C}^\perp is of type $(3, 3; 1, 1; 0, 0, 1)$ and has $|\mathcal{C}^\perp| = 2^{2 \cdot 1} 2^1 2^1 = 16$ codewords.

Corollary 3.4. Let \mathcal{C} be a $\mathbb{Z}_2[u, u^2]$ -linear code of type $(\alpha, \beta; k_0; l_0, l_1)$ with generator matrix in (6) then the parity-check matrix of \mathcal{C} is

$$\left[\begin{array}{ccc|cc} -R_{01}^t & I_{\alpha-k_0} & -uB_{01}^t & 0 & 0 \\ -A_{01}^t & 0 & -S_{02}^t + S_{12}^t S_{01}^t & -S_{12}^t & I_{\beta-l_0-l_1} \\ 0 & 0 & -uS_{01}^t & uI_{l_1} & 0 \end{array} \right].$$

Note that this matrix is permutation equivalent to a matrix in (2).

4. Binary images of $\mathbb{Z}_2[u^r, u^s]$ -linear codes

Binary linear codes are the most important member of the family of linear codes. So, taking $p = 2$ we have $\mathbb{Z}_2[u^r, u^s]$ -linear codes. Then we can look at $\mathbb{Z}_2[u^r, u^s]$ -linear codes as a binary codes under the special Gray map, defined as follows.

Definition 4.1. Let $\mathcal{R}_s = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2 + \dots + u^{s-1}\mathbb{Z}_2$ with $u^s = 0$. Define the mapping

$$\begin{aligned} \Phi_s : \mathcal{R}_s &\longrightarrow \mathbb{Z}_2^{2^{s-1}} \\ (a_0 + ua_1 + \dots + u^{s-1}a_{s-1}) &\longrightarrow (a_{s-1}, a_0 \oplus a_{s-1}, a_1 \oplus a_{s-1}, \dots, a_{s-2} \oplus a_{s-1}, \\ &\quad a_0 \oplus a_1 \oplus a_{s-1}, \dots, a_0 \oplus a_{s-2} \oplus a_{s-1}, \dots, \\ &\quad a_0 \oplus a_1 \oplus \dots \oplus a_{s-1}) \end{aligned}$$

where $a_i \oplus a_j = a_i + a_j \pmod 2$. This is a linear Gray map from \mathcal{R}_s to $\mathbb{Z}_2^{2^{s-1}}$.

As an example, let us consider

$$\begin{aligned} \Phi_3 : \mathcal{R}_3 &\longrightarrow \mathbb{Z}_2^4 \\ 0 &\longrightarrow (0000) \\ 1 &\longrightarrow (0101) \\ u &\longrightarrow (0011) \\ 1 + u &\longrightarrow (0110) \\ u^2 &\longrightarrow (1111) \\ 1 + u^2 &\longrightarrow (1010) \\ u + u^2 &\longrightarrow (1100) \\ 1 + u + u^2 &\longrightarrow (1001). \end{aligned}$$

This map can be extended to $\mathcal{R}_r^\alpha \times \mathcal{R}_s^\beta$ as follows.

Definition 4.2. Let $x = (x_0, x_1, \dots, x_{\alpha-1}) \in \mathcal{R}_r^\alpha$ and $y = (y_0, y_1, \dots, y_{\beta-1}) \in \mathcal{R}_s^\beta$ and let $\Phi : \mathcal{R}_r^\alpha \times \mathcal{R}_s^\beta \longrightarrow \mathbb{Z}_2^n$ be the map defined by

$$\Phi(x, y) = (\Phi_r(x_0), \dots, \Phi_r(x_{\alpha-1}), \Phi_s(y_0), \dots, \Phi_s(y_{\beta-1})).$$

We called the binary image $\Phi(\mathcal{C}) = C$ as a $\mathbb{Z}_2[u^r, u^s]$ -linear code of length $n = 2^{r-1}\alpha + 2^{s-1}\beta$.

Example 4.3. Let C be a $\mathbb{Z}_2[u, u^3]$ -linear code of type $(7, 7; 0; 0, 1, 3)$ with the following generator matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & u + u^2 & u & u + u^2 & u + u^2 & u & u & u \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & u^2 & u^2 & u^2 & 0 & u^2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & u^2 & u^2 & u^2 & 0 & u^2 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & u^2 & u^2 & u^2 & 0 & u^2 \end{bmatrix}.$$

Using the Gray map that we defined above, we have the Gray image $\Phi(C) = C$ is a binary linear code with the parameters $[35, 5, 16]$. It is worth mentioning that C with this parameters is an optimal code, that is, C has the best minimum distance $d = 16$ compared to the existing and known bounds for $n = 35$ and $k = 5$.

5. Self-dual $\mathbb{Z}_2[u^2, u^3]$ -linear codes

We know that if $C = C^\perp$ then C is called a self-dual code. In this section, we investigate the structure of self-dual codes over $\mathbb{Z}_2[u]/\langle u^2 \rangle \times \mathbb{Z}_2[u]/\langle u^3 \rangle$. We give some conditions for a $\mathbb{Z}_2[u^2, u^3]$ -linear code C to be a self-dual. Further, we present some examples of self-dual codes.

Lemma 5.1. Let C be a self-dual $\mathbb{Z}_2[u^2, u^3]$ -linear code. Then C is of type

$$(2k_0 + k_1, 2(l_0 + l_1); k_0, k_1; l_0, l_1, l_2).$$

Proof. Since C is self-dual then the dual code C^\perp has the same type with C . Therefore, we have

$$\begin{aligned} (\alpha, \beta; k_0, k_1; l_0, l_1, l_2) &= (\alpha, \beta; \alpha - k_0 - k_1, k_1; \beta - l_0 - l_1 - l_2, l_2, l_1) \\ \alpha - k_0 - k_1 &= k_0, \quad \beta - l_0 - l_1 - l_2 = l_0, \quad l_1 = l_2 \\ \alpha &= 2k_0 + k_1, \quad \beta = 2l_0 + 2l_1. \end{aligned}$$

□

Corollary 5.2. If C is a self-dual code of type $(\alpha, \beta; k_0, k_1; l_0, l_1, l_2)$, then β is even.

Corollary 5.3. If C is a separable $\mathbb{Z}_2[u^2, u^3]$ -linear code of type $(\alpha, \beta; k_0, k_1; l_0, l_1, l_2)$, then it has the standard form of the following generator matrix

$$G = \left[\begin{array}{ccc|cccc} I_{k_0} & R_{01} & R_{02} & 0 & 0 & 0 & 0 \\ 0 & uI_{k_1} & uR_{12} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{l_0} & S_{01} & S_{02} & S_{03} \\ 0 & 0 & 0 & 0 & uI_1 & uS_{12} & uS_{13} \\ 0 & 0 & 0 & 0 & 0 & u^2I_2 & u^2S_{23} \end{array} \right].$$

Theorem 5.4. Let C be a self-dual $\mathbb{Z}_2[u^2, u^3]$ -linear code of type $(2k_0 + k_1, 2(l_0 + l_1); k_0, k_1; l_0, l_1, l_2)$. Then the following statements are equivalent.

- i) C_α is a self-dual code over $\mathbb{Z}_2[u]/\langle u^2 \rangle$.
- ii) C_β is a self-dual code over $\mathbb{Z}_2[u]/\langle u^3 \rangle$.
- iii) C is separable and $|C_\alpha| = 2^{2k_0+k_1}$, $|C_\beta| = 2^{3(l_0+l_1)}$.

Corollary 5.5. Let C_1 be a self-dual code of length α over $\mathbb{Z}_2[u]/\langle u^2 \rangle$ and C_2 be a self-dual code of length β over $\mathbb{Z}_2[u]/\langle u^3 \rangle$. Then, $C_1 \times C_2$ is a self-dual $\mathbb{Z}_2[u^2, u^3]$ -linear code of length $\alpha + \beta$.

Example 5.6 (Separable). Let C be a $\mathbb{Z}_2[u^2, u^3]$ -linear code of type $(3, 4; 1, 1; 1, 1, 1)$ with the following generator matrix:

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & u & 0 & u \\ 0 & 0 & 0 & 0 & 0 & u^2 & u^2 \end{array} \right].$$

Hence, C is a separable self-dual code.

Example 5.7 (Non-separable). A $\mathbb{Z}_2[u^2, u^3]$ -linear code C with the below generator matrix is a non-separable self-dual code of type $(4, 4; 1, 2; 1, 1, 1)$.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1+u & 0 & 0 & u+u^2 & u \\ 0 & u & 0 & u & 0 & 0 & 0 & u^2 \\ 0 & 0 & u & u & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1+u^2 \\ 0 & 0 & u & u & 0 & u & 0 & u \\ 0 & 0 & 0 & 0 & 0 & 0 & u^2 & u^2 \end{array} \right].$$

6. Conclusion

This paper generalizes $\mathbb{Z}_2\mathbb{Z}_2[u]$ -linear codes to $\mathbb{Z}_p[u^r, u^s]$ -linear codes. The original study was introduced for the special case $p = 2$, $r = 1$ and $s = 2$. We give the standard forms of the generator and parity-check matrices of these codes. Further, we study $\mathbb{Z}_2[u^r, u^s]$ -linear codes, for the special case $p = 2$ and relate these codes to binary codes by using a special Gray map. Also, we investigate self-dual codes over $\mathbb{Z}_2[u]/\langle u^2 \rangle \times \mathbb{Z}_2[u]/\langle u^3 \rangle$. Since this family of linear codes has been introduced recently many more properties of these families of codes await explorations, such as cyclicity.

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