Smoothing Approximations to Non-smooth Functions

Ahmet SAHINER[1], Nurullah YILMAZ[1,*], and Shehab A. IBRAHEM[1,♯]

[1]Department of Mathematics,
Suleyman Demirel University, Isparta, TURKEY
ahmetsahiner@sdu.edu.tr, *nurullahyilmaz@sdu.edu.tr,
♯mullaiq@gmail.com

Received: 21.12.2018, Accepted: 10.01.2019, Published: 21.01.2019

Abstract — *In this study, a new smoothing method is proposed for non-smooth functions. The theoretical results and error estimates are presented about this new smoothing method. Finally, some numerical examples are given.*

Keywords: non-smooth functions, smoothing techniques.
Mathematics Subject Classification: 90C30, 49J52, 65D10.

1 Introduction

Non-smooth functions are studied for a long time [1, 2, 3]. Since they have been used in many areas such as optimization and data modeling [4, 5], they have attracted great attention in recent years. The basic lack of these functions is not completely smooth on the domain. Therefore, most of the efficient gradient-based approaches become unusable in their optimization process. To overcome these inadequacies, smoothing techniques have been proposed for some subclasses of non-smooth functions. Smoothing techniques are based on constructing the smooth approximations to the non-smooth function.

The smoothing approaches have been studied for a long time in optimization. The first study on smoothing approaches was proposed by Bertsekas to solve one of the major optimization problems called min-max [6]. For the same problem, another important smoothing function approach is proposed in [7]. In that study, the smoothing approach is constructed to smooth out the kink points of non-smooth function, locally.

Another idea is to create a set of smooth functions that converge to the non-smooth function, globally. One of the global smoothing approaches is the hyperbolic smoothing approach developed by Xavier [8]. The hyperbolic smoothing approach was used to solve min-max problems in [9]. In the penalty function method for constrained optimization, the smoothing functions are used [10, 11]. In [12], a class of smoothing functions considering the wavelet-based approach is presented. In recent years, the smoothing approach has also been used to solve regularization problems [13, 14, 15] and global optimization

Cite as: *A. Sahiner, N. Yilmaz, S. A. Ibrahem,Smoothing Approximations to Non-smooth Functions, Journal of Multidisciplinary Modeling and Optimization 1 (2) (2018), 69-74.*

problems [16, 17, 18]. For more information on the smoothing approach used in optimization problems, see [19, 20, 21, 22].

The next section is devoted to give some preliminaries. In Section 3, the smoothing technique is proposed and related error estimates among the optimal objective function values of the smoothed objective function is given. In Section 4, some numerical examples are given. In the last section, we present some concluding remarks.

2 Preliminaries

Throughout the paper, we use x_k^* to denote the $k-$th local minimizer of f whereas by x^* we mean the global minimizer. $\|x\| = \sqrt{\sum_{k=1}^{n} x_k^2}$ denotes the Euclidean norm in $\mathbb{R}^n$. For error estimation we use the following $L^1[a,b]-$norm defined as

$$\|E\|_{L^1[a,b]} = \int_a^b |E(t)|dt,$$

where E is a continuous function on the interval $[a,b]$.

Definition 1. *[19] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. The function $\tilde{f} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is called a smoothing function of f, if $\tilde{f}(\cdot, \beta)$ is continuously differentiable in $\mathbb{R}^n$ for any fixed $\beta > 0$, and for any $x \in \mathbb{R}^n$,*

$$\lim_{z \to x, \beta \to 0} \tilde{f}(z, \beta) = f(x).$$

3 Main Results

Let f and g be continuously differentiable functions on $\mathbb{R}^n$ and let us define the following function $F(x) = \max\{f(x), g(x)\}$. The function F is used in many optimization problems of min-max, penalty method, regularization and global optimization. We develop smooth relaxations of the above optimization problems involving F.

First of all, we can reformulate F by

$$F(x) = \frac{1}{2}\left(\left(f(x) - g(x)\right)\phi\left(t\right) + f(x) + g(x)\right), \tag{1}$$

where

$$\phi\left(t\right) = \begin{cases} 1, & t \geq 0, \\ -1, & t < 0, \end{cases} \tag{2}$$

and $t = f(x) - g(x)$. It can be observed that the function F may be non-smooth. If we smooth out the non-smooth function $\phi\left(t\right)$ as

$$\tilde{\phi}\left(t, \varepsilon\right) = \begin{cases} 1, & t > \varepsilon, \\ S_1(t, \varepsilon), & -\varepsilon \leq t \leq \varepsilon, \\ -1, & t < -\varepsilon, \end{cases} \tag{3}$$

where $S_1(t, \varepsilon) = \frac{-1}{2\varepsilon^3} t^3 + \frac{3}{2\varepsilon} t$ and $\varepsilon > 0$ then, we obtain smoothing function of $F\left(x\right)$ as

$$\tilde{F}(x, \varepsilon) = \frac{1}{2}\Big(\left(f(x) - g(x)\right)\tilde{\phi}\left(t, \varepsilon\right) + f(x) + g(x)\Big). \tag{4}$$

If f and g second-order continuously differentiable and it is needed to second-order continuously differentiability of $\tilde{F}$, the function $S_2(t,\varepsilon) = \frac{3}{8\varepsilon^5}t^5 - \frac{5}{4\varepsilon^3}t^3 + \frac{15}{8\varepsilon}t$ can be use instead of $S_1(t,\varepsilon)$.

Remark 1. *Since* $\min\{f(x), g(x)\} = -\max\{-f(x), -g(x)\}$ *and* $|f(x)| = \max\{f(x), -f(x)\}$, *the smoothing approach is also valid for the operators* min *and* $|\cdot|$.

Lemma 3.1. *Let the function* $\phi(x)$ *is defined as in* (2) *and* $\tilde{\phi}(x,\varepsilon)$ *be the smoothing function of it. Then, we have*

$$\|\tilde{\phi}(t,\varepsilon) - \phi(t)\|_{L^1} = \frac{3}{4}\varepsilon,$$

for any $\varepsilon > 0$.

Proof. Since the functions $\tilde{\phi}(t,\varepsilon)$ and $\phi(t)$ are equal for $t \le -\varepsilon$ and $t \ge \varepsilon$, the difference between $\tilde{\phi}(t,\varepsilon)$ and $\phi(t)$ equal to 0. So, we handle only the case $-\varepsilon \le t \le \varepsilon$ and

$$\begin{aligned} \left\|\tilde{\phi}(t,\varepsilon) - \phi(t)\right\|_{L^1_{[-\varepsilon,\varepsilon]}} &= \int_{-\varepsilon}^{\varepsilon} \left|\tilde{\phi}(t,\varepsilon) - \phi(t)\right| dt \\ &= \int_{-\varepsilon}^{0} |S_1(t,\varepsilon) - (-1)|\, dt + \int_{0}^{\varepsilon} |S_1(t,\varepsilon) - 1|\, dt \\ &= \frac{3}{4}\varepsilon. \end{aligned}$$

Therefore, the proof is completed. □

Theorem 3.1. *Let* f *and* g *be continuously differentiable functions on* $\mathbb{R}^n$ *and,* $F(x)$ *and the function* $\tilde{F}(x,\varepsilon)$ *is defined as* (1) *and* (4), *respectively. Then, we have*

$$\|\tilde{F}(x,\varepsilon) - F(x)\|_{L^1} \le \frac{3}{8}\varepsilon^2, \tag{5}$$

for any $\varepsilon > 0$ *and* $x \in \mathbb{R}^n$.

Proof. For any $\varepsilon > 0$ and $t \notin I_\varepsilon = [-\varepsilon, \varepsilon]$, we have $\tilde{F}(x,\varepsilon) = F(x)$. Therefore, it is sufficient to prove that the inequality (5) holds on I_ε. For any $t \in I_\varepsilon$, we have $t = f(x) - g(x) \le \varepsilon$ and

$$\begin{aligned} \|\tilde{F}(x,\varepsilon) - F(x)\|_{L^1} &= \int_{-\varepsilon}^{\varepsilon} \left|\Big((f(x) - g(x))\,\tilde{\phi}\,(t,\varepsilon) - (f(x) - g(x))\,\phi\,(t)\Big)\right| dt \\ &= \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} \left|(f(x) - g(x))\left(\tilde{\phi}\,(t,\varepsilon) - \phi\,(t)\right)\right| dt \\ &\le \frac{\varepsilon}{2}\int_{-\varepsilon}^{\varepsilon} \left|\tilde{\phi}\,(t,\varepsilon) - \phi\,(t)\right| dt. \end{aligned}$$

From Lemma 3.1 we have

$$\|\tilde{F}(x,\varepsilon) - F(x)\|_{L^1} \le \frac{3}{8}\varepsilon^2.$$

The proof is completed. □

Theorem 3.2. *Let f and g be continuously differentiable functions on $\mathbb{R}^n$ and, $F(x)$ and the function $\tilde{F}(x,\varepsilon)$ is defined as* (1) *and* (4)*, respectively. Then, the function $\tilde{F}(x,\varepsilon)$ is a smooth function and for any fixed x, we have*

$$\lim_{\varepsilon \to 0} \tilde{F}(x,\varepsilon) = F(x)$$

for $\varepsilon > 0$.

Proof. The smoothing function $\tilde{\phi}(x,\varepsilon)$ is smooth for $\varepsilon > 0$. Since f and g are smooth, according to equation (4), $\tilde{F}(x,\varepsilon)$ is smooth. From the Theorem 3.1, it can be seen that $\tilde{F}(x,\varepsilon)$ approaches to $F(x)$ as $\varepsilon \to 0$. □

4 Numerical Examples

Example 1. *Let us define the following function*

$$F(x) = \max\{f(x), g(x)\},$$

where $f(x) = \exp(-\frac{x}{5}) + 1$ and $g(x) = 3x + 5$. For any $\varepsilon > 0$, the smoothing function of $F(x)$ is defined as

$$\tilde{F}(x,\varepsilon) = \frac{1}{2}\Big((f(x) - g(x))\tilde{\phi}(t,\varepsilon) + f(x) + g(x)\Big),$$

where $t = f(x) - g(x)$. The graphs of the function $F(x)$ and smoothing function $\tilde{F}(x,\varepsilon)$ in Fig. 1*.*

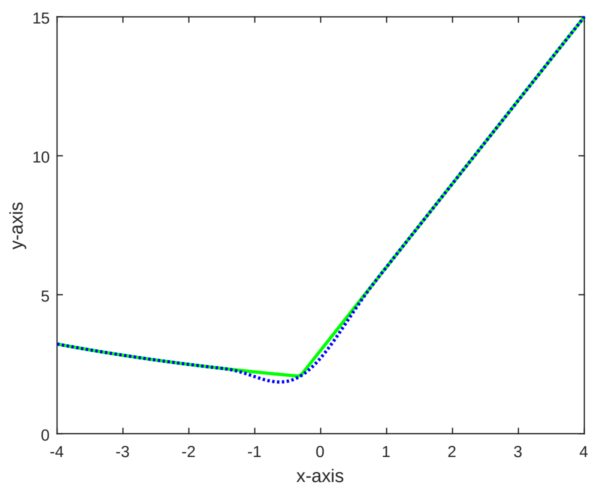

Figure 1: The graphs of $F(x)$ (green and solid) and $\tilde{F}(x,\varepsilon)$ (blue and dotted).

Example 2. *Let us define the function $F : \mathbb{R}^2 \to \mathbb{R}$ as*

$$F(x,y) = \max\{f(x,y), g(x,y)\},$$

where $f(x,y) = \exp(-\frac{x}{4}) + 2y - 1$ *and* $g(x,y) = 3x - \frac{1}{2}y^2$*. For any* $\varepsilon > 0$*, the smoothing function of* $F(x,y)$ *is defined as*

$$\tilde{F}(x,y,\varepsilon) = \frac{1}{2}\Big((f(x,y) - g(x,y))\tilde{\phi}(t,\varepsilon) + f(x,y) + g(x,y)\Big),$$

where $t = f(x,y) - g(x,y)$*. The graphs of the function* $F(x)$ *and smoothing function* $\tilde{F}(x,y,\varepsilon)$ *is given in Figs.* 2 *(a) and (b), respectively. In Figs.* 2 *(c) and (d), the graph of contours of the functions* $F(x,y)$ *and* $F(x,y,\varepsilon)$ *are presented, respectively.*

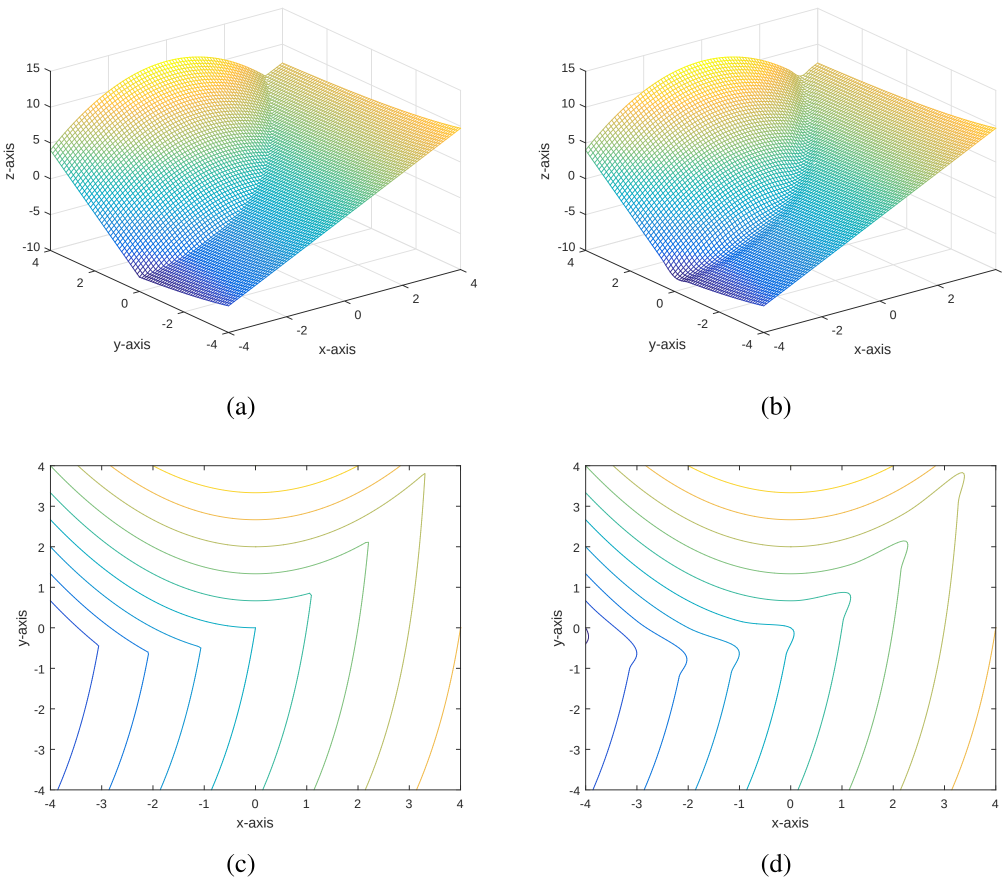

Figure 2: (a) The graph of $F(x,y)$, (b) The graph of $F(x,y,\varepsilon)$, (c) The contour graph of $F(x,y)$, (d) The contour graph of $\tilde{F}(x,y,\varepsilon)$.

5 Conclusion

In this study, we have introduced a new smoothing technique for non-smooth functions. The presented smoothing process is useful for problems which contain any of “max, min and $|\cdot|$ and operators. The smoothing technique is controlled by a parameter. This parameter gives an opportunity to get a sensitive approximation to the original non-smooth function. By this study, well-known gradient based optimization techniques are available to solve non-smooth optimization problems such as min-max, penalty methods and regularization.

References

[1] R. T. Rockafellar, R. J-B. Wets R. J-B., Variational Analysis, Springer, Berlin, 1998.

[2] F. H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.

[3] V. F. Demyanov, A. M. Rubinov, Constructive Nonsmooth Analysis. Verlag Peter Lang, Frankfurt, 1995.

[4] B. Akteke-Ozkurt, G. W. Weber, G. Koksal, Optimization of generalized desirability functions under model uncertainty, Optimization, 66 (12) 2017, 2157–2169.

[5] A. Sahiner, N. Yilmaz and G. Kapusuz, A novel modeling and smoothing technique in global optimization, J. Ind. Manage. Optim., 15 (1) 2019, 113-130.

[6] D. Bertsekas, Nondifferentiable optimization via approximation, Math. Program. Stud., 3 1975, 1–25.

[7] I. Zang, A smoothing out technique for min-max optimization, Math. Programm., 19 1980, 61–77.

[8] A. E. Xavier, The hyperbolic smoothing clustering method, Patt. Recog., 43 2010, 731–737.

[9] A. M. Bagirov, A. Al Nuamiat and N. Sultanova, Hyperbolic smoothing functions for nonsmooth minimization, Optimization, 62 (6) 2013, 759–782.

[10] M. Jiang, R. Shen, X. Xu, Z. Meng, Second-order smoothing objective penalty function for constrained optimization problems, Numer. Func. Anal. Opt., 35 2014, 294–309.

[11] S. J. Lian Smoothing approximation to l_1 exact penalty for inequality constrained optimization, Appl. Math. Comput., 219 2012, 3113–3121.

[12] C. Chen and O.L. Mangasarian, A class of smoothing functions for nonlinear and mixed complementarity problem, Comput. Optim. Appl., 5 1996, 97–138.

[13] X. Chen and W. Zhou, Smoothing nonlinear conjugate gradient method for image restoration using nonsmooth nonconvex minimization, SIAM J. Imaging Sciences, 3(4) 2010, 765–790.

[14] G.W. Weber, I. Batmaz and G. Koksal, CMARS: a new contribution to nonparametric regression with multivariate adaptive regression splines supported by continuous optimization, Inverse. Probl. Eng., 20(3) 2012, 371–400.

[15] P. Taylan, G.-W. Weber, F. Yerlikaya Ozkurt, A new approach to multivariate adaptive regression splines by using Tikhonov regularization and continuous optimization. TOP 18(2) 2010, 377–395.

[16] A. Sahiner, N. Yilmaz and G. Kapusuz, A new global optimization method and applications, Carpathian Math. J., 33(3) 2017, 373-380.

[17] A. Sahiner, S. A. Ibrahem, A new global optimization technique by auxiliary function method in a directional search, Optim. Lett. 2018, doi: 10.1007/s11590-018-1315-1.

[18] J. Lee, D. Skipper, Virtuous smoothing for global optimization, J. Glob. Optim., 69(3) 2017, 677-697.

[19] X. Chen, Smoothing methods for nonsmooth, nonconvex minimzation, Math. Program. Ser. B, 134 2012, 71–99.

[20] C. Grossmann, Smoothing techniques for exact penalty function methods, Contemporary Mathematics, 658 2016, 249–265.

[21] F. G. Vazquez, H. Gunzel and J. J. Ruckmann, On logarithmic smoothing of the maximum function, Ann. Oper. Res., 101 2001, 209–220.

[22] H. Th. Jongen and D. Pallaschke, On linearization and continuous selection of functions, Optimization, 19(3) 1988, 343–353.

Ahmet Sahiner, ORCID: https://orcid.org/0000-0002-4945-2476
Nurullah Yilmaz, ORCID: https://orcid.org/0000-0001-6429-7518
Shehab Ahmed Ibrahem, ORCID: https://orcid.org/0000-0003-2428-2703