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ON $b-$COLORING OF CENTRAL GRAPH OF SOME GRAPHS

M.KALPANA AND D.VIJAYALAKSHMI

ABSTRACT. The $b-$chromatic number of G, denoted by $\varphi(G)$, is the maximum k for which G has a $b-$coloring by k colors. A $b-$coloring of G by k colors is a proper k-coloring of the vertices of G such that in each color class i there exists a vertex x_i having neighbors in all the other $k-1$ color classes. Such a vertex x_i is called a $b-$dominating vertex, and the set of vertices $\{x_1, x_2 \dots x_k\}$ is called a $b-$dominating system. In this paper, we are going to investigate on the $b-$chromatic number of Central graph of Triangular Snake graph, Sunlet graph, Helm Graph, Double Triangular Snake graph, Gear graph, and Closed Helm graph are denoted as $C(T_n)$, $C(S_n)$, $C(H_n)$, $C(DT_n)$, $C(G_n)$, $C(CH_n)$ respectively.

1. INTRODUCTION

Graph theory is the theory of graphs dealing with nodes and connections or vertices and edges. This subject has experienced explosive growth, due in large measure to its role as an essential structure underpinning modern applied mathematics. Configurations of nodes and connections has great diversity of applications. They may represented physical networks, such as electrical circuits, roadways, or organic molecules. They are also used in representing less tangible interactions as might occur in ecosystem, sociological relationships, database, or in the flow of control in a computer program. It is a fastest growing field in Mathematics mainly because of its applications in distinct areas. There are plenty of works have been done in different topics, like decomposition, domination, factoring, orienting, coloring etc., in the past decades. A graph $G = (V, E)$ is an ordered pair of two sets called V and E. Here elements of V are called vertices and elements of E are called edges. We consider the graphs here as undirected, finite and neither have multiple edges nor loops. The order of G is denoted by n and the size is denoted by m. All graph terminologies are referred from J.A. Bondy and U.S.R. Murty [1].

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In Graph theory, Graph coloring is a well known area which is widely studied by many researchers. Various types of graph coloring and many open problems were discussed in the wonderful books [2, 7]. Graph coloring deals with the general and widely applicable concept of partitioning the underlying set of a structure into parts, each of which satisfies a given requirement. Here coloring means a vertex coloring of a graph. A $k-$coloring of a graph $G = (V, E)$ is a mapping $C : V \to P$ where P is a set of k colors; thus $k-$coloring is an assignment of k colors to the vertices of G. Usually, the set P of colors is taken to be $\{1, 2, \dots k\}$. A coloring C is proper if no two adjacent vertices are assigned the same color. Only loop less graphs admits proper coloring. The minimum k for which a graph G is $k-$colorable is called its chromatic number, and denoted $\chi(G)$. If $\chi(G) = k$, the graph G is said to be $k-$chromatic.

The $b-$chromatic number of G, denoted by $\varphi(G)$, is the maximum k for which G has a $b-$coloring by k colors. A $b-$coloring of G by k colors is a proper k-coloring of the vertices of G such that in each color class i there exists a vertex x_i having neighbors in all the other $k-1$ color classes. Such a vertex x_i is called a $b-$dominating vertex, and the set of vertices $\{x_1, x_2 \dots x_k\}$ is called a $b-$dominating system. The $b-$coloring was introduced by R.W.Irving and D.F.Manlove in [6]. They proved that determining $\varphi(G)$ is NP-hard in general and polynomial for trees.

A Triangular Snake [8] is obtained from a path $x_1, x_2, \dots, x_n$ by joining x_i and x_{i+1} to a new vertex y_i for $1 \le i \le n$. That is, every edge of a path is replaced by a triangle C_3. The n-Sunlet graph S_n is a graph [11] with cycle C_n and each vertex of the cycle attached to one pendent vertex. Each n-sunlet graph consists $2n$ nodes and $2n$ edges. A Helm H_n, $n \ge 3$ is the graph [11] obtained from the Wheel W_n by adding a pendent edge at each vertex on the rim of the Wheel W_n. The Central graph [9] of G, denoted by $C(G)$ is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in $C(G)$.

A Double Triangular Snake [12] is a graph formed by two Triangular Snakes having a common path. i.e., a Double Triangular Snake with k blocks is obtained from a path x_1,x_2,$\dots x_n$ by joining x_i and x_{i+1} to two new vertices y_i and z_i for $i = 1, 2, \dots n$. A Gear graph[11] is obtained from the Wheel W_n by adding a vertex between every pair of adjacent vertices of rim of the Wheel W_n. A Closed Helm CH_n is the graph obtained by taking a Helm H_n and adding edges between the pendent vertices. In this paper we have used algorithmic approach to prove the results of $C(T_n)$, $C(H_n)$ and $C(DT_n)$.

2. b-Coloring of $C(T_n)$

2.1. b-Coloring Algorithm of $C(T_n)$.

Input: $C(T_n)$, $n \ge 2$.

V $\leftarrow \{x_1, x_2, \dots x_n, y_1, y_2, \dots y_n, a_1, a_2, \dots a_n, b_1, b_2, \dots b_n, c_1, c_2, \dots c_n\}$.
for $i = 1$ to n
$x_i \leftarrow i$;
end for
for $i = 1$ to n
$y_i \leftarrow i + n$;
end for
for $i = 1$ to n
$a_i \leftarrow i + n$;
end for
for $i = 1$ to n
$b_i \leftarrow i$;
end for
for $i = 1$ to n
$c_i \leftarrow i + n + 1$;
end for
end procedure
Output: vertex colored $C(T_n)$.

Theorem 1. *For a Triangular Snake graph T_n, $n \geq 2$, the b - chromatic number of Central Graph of Triangular Snake graph is 2n+1.*

$$i.e., \varphi\left[C\left(T_n\right)\right] = 2n + 1.$$

Proof. Let the vertex set of Triangular Snake graph as

$$V\left(T_n\right) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n + 1\}$$

By the definition of Central graph, the edge joining x_i and y_{i+1} has been subdivided by the newly introduced vertex $a_i (1 \leq i \leq n)$ and the edge joining y_i and y_{i+1} has been subdivided by the newly introduced vertex $b_i (1 \leq i \leq n)$.Denote the newly added vertex on the edge joining y_i and x_i as $c_i (1 \leq i \leq n)$. Then the vertex set of central graph of Triangular Snake graph is

$$\begin{aligned} V\left[C\left(T_n\right)\right] &= \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n + 1\} \cup \{a_i : 1 \leq i \leq n\} \\ &\quad \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq n\} \end{aligned}$$

The vertices of $C\left(T_n\right)$ are colored as given in the algorithm 2.1.
By algorithm $\ni$ $(2n + 1)$ vertices $\{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n + 1\}$ with color class

$$C = \{i : 1 \leq i \leq 2n + 1\}.$$

By algorithm the set of vertices $\{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n + 1\}$ with cardinality 2n+1 has the color classes

$$C\ \left[\{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n + 1\}\right] = \{i : 1 \leq i \leq 2n + 1\}$$

Since

$$|\{x_i : 1 \le i \le n\} \cup \{y_i : 1 \le i \le n+1\}| = 2n+1 = |\{i : 1 \le i \le 2n+1\}|,$$

the vertices of $\{x_i : 1 \le i \le n\} \cup \{y_i : 1 \le i \le n+1\}$ receive distinct colors. $C\,[N(x_i)]$ and $C\,[N(y_i)]$ have $2n$ distinct colors for each i. It implies that the coloring is $b-$coloring. To prove it is maximum, let us suppose that $\varphi[C(T_n)] > 2n+1$. Then there exists at least $2n+2$ vertices of degree $2n+1$. The $C[T_n]$ has $2n+1$ vertices of degree $2n$, and other vertices are of degree 2. i.e., $\ni$ no vertex of degree $2n+2$, which is a contradiction to the fact that $\varphi[C(T_n)] > 2n+1$. Therefore $\varphi\,[C\,(T_n)] \le 2n+1$. Hence $\varphi\,[C\,(T_n)] = 2n+1$.

□

3. $b-$Coloring of $C\,(S_n)$

Theorem 2. *For a Sunlet graph S_n, $n \ge 3$, the b - chromatic number of Central Graph of Sunlet graph is $(3n-1)/2$ for odd n and $n + (n/2)$ for even n.*

$$i.e., \varphi[C(S_n)] = \begin{cases} (3n-1)/2, & \text{for } n \text{ is odd} \\ n+(n/2), & \text{for } n \text{ is even.} \end{cases}$$

Proof. Let

$$V\,(S_n) = \{x_i : 1 \le i \le n\} \cup \{y_i : 1 \le i \le n\}\,.$$

By the definition of Central graph, the edge joining x_i and $x_{i+1}(1 \le i \le n)$ has been subdivided by the newly introduced vertex $a_i(1 \le i \le n-1)$, the remaining edge $x_n x_1$ has been subdivided by the vertex a_n and the edge joining x_i and y_i has been subdivided by the newly introduced vertex $b_i(1 \le i \le n)$. Then the vertex set of central graph of Sunlet graph is

$$\begin{aligned} V\,[C\,(S_n)] &= \{x_i : 1 \le i \le n\} \cup \{y_i : 1 \le i \le n\} \cup \{a_i : 1 \le i \le n\} \\ &\quad \cup \{b_i : 1 \le i \le n\} \end{aligned}$$

Now let us assign the b-coloring to the vertex set of $[C\,(S_n)]$ by using the following function,

$$f : V\,[C\,(S_n)] \to C_i : 1 \le i \le n.$$

Case (i): When n is odd

$$\begin{aligned} f\,(x_i) &= f\,(x_{i+1}) = C_{n+i},\ 1 \le i \le n-3, \\ f\,(x_{n-2}) &= f\,(x_{n-1}) = f\,(x_n) = C_{(3n-1)/2},\ n-2 \le i \le n. \end{aligned}$$

Next consider the vertex set of $y_i\,(1 \le i \le n)$. The coloring functions of y_i are,

$$f\,(y_i) = C_i,\ 1 \le i \le n.$$

The coloring functions of newly introduced vertex set $a_i\,(1 \le i \le n)$ are,

$$f\,(a_i) = C_i,\ 1 \le i \le n.$$

and $b_i\,(1 \le i \le n)$ are,

$$f\,(b_i) = C_{1+i},\ 1 \le i \le n.$$

Then the vertices of $C(S_n)$ are colored by the above coloring process. By the above coloring $\ni$ $(3n-1)/2$ vertices of $\{x_i : 1 \le i \le n\} \cup \{y_i : 1 \le i \le n\}$ with cardinality $2n$ has the color class

$$\begin{aligned} C &= \{i : 1 \le i \le (3n-1)/2\}\,. \\ N[x_i] &= \{x_1, x_2, \ldots, x_{n-2}, x_{n-1}, x_n\} \\ C\ [N[x_i]] &= \{n+i, \ldots, (3n-1)/2\} \\ N[y_i] &= \{y_i : 1 \le i \le n\}\,. \\ C\ [N[y_i]] &= \{i : 1 \le i \le n\} \end{aligned}$$

Then

$$C\ \ [N\,[x_i]]\ \cup\ C\ [N[y_i]] = (3n-1)/2.$$

It implies the $b-$coloring. To prove it is maximum. Let us assume that, $\varphi[C(S_n)] > (3n-1)/2$. The $(3n)/2$ color does not have neighbors of other colors. Then it does not satisfying $b-$coloring, which is a contradiction to the fact that $\varphi[C(S_n)] > (3n)/2$. Therefore, $\varphi[C(S_n)] \le (3n+1)/2$. Hence $\varphi[C(S_n)] = (3n+1)/2$.

Case (ii): When n is even

$$\begin{aligned} f\,(x_i) &= f\,(x_{i+1}) = C_{n+i},\ 1 \le i \le n-2, \\ f\,(x_{n-1}) &= f\,(x_n) = C_{n+(n/2)},\ n-1 \le i \le n. \end{aligned}$$

Next consider the vertex set of $y_i\,(1 \le i \le n)$. The coloring functions of y_i are,

$$f\,(y_i) = C_i,\ 1 \le i \le n.$$

The coloring functions of newly introduced vertex set $a_i\,(1 \le i \le n)$ are,

$$f\,(a_i) = C_i,\ 1 \le i \le n.$$

and $b_i\,(1 \le i \le n)$ are,

$$f\,(b_i) = C_{1+i},\ 1 \le i \le n.$$

Then the vertices of $C(S_n)$ are colored by the above coloring process. By the above coloring $\ni$ $n+(n/2)$ vertices of $\{x_i : 1 \le i \le n\} \cup \{y_i : 1 \le i \le n\}$ with cardinality $2n$ has the color class

$$\begin{aligned} C &= \{i : 1 \le i \le n+(n/2)\} \\ N[x_i] &= \{x_1, x_2, \ldots, x_{n-1}, x_n\}\,. \\ C\ (N[x_i]) &= \{n+i, \ldots, n+(n/2)\} \\ N[y_i] &= \{y_i : 1 \le i \le n\}\,. \\ C\ (N[y_i]) &= \{i : 1 \le i \le n\} \end{aligned}$$

Then $C\ (N[x_i]) \cup C\ (N[y_i]) = n+(n/2)$. It implies the b-coloring. To prove it is maximum. Let us assume that, $\varphi[C(S_n)] > n+(n/2)$. The $n+(n/2)+1$ color is not adjacent to all other colors. Then it does not satisfy $b-$coloring, which is a contradiction to the fact that $\varphi[C(S_n)] > n + (n/2)$. Therefore, $\varphi[C(S_n)] \leq n + (n/2)$. Hence $\varphi[C(S_n)] = n + (n/2)$. □

4. $b-$Coloring of $C(H_n)$

4.1. b-Coloring Algorithm of $C(H_n)$.

Input: $C(H_n), n \geq 3$.
$V \leftarrow \{x, x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_n, a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n, c_1, c_2, \ldots c_n.\}$
$x \leftarrow n+1$;
for $i = 1$ to n
if $i = 1, 2, n$
$x_i \leftarrow n+2$;
else
$x_i \leftarrow i$;
end for
for $i = 1$ to n
$y_i \leftarrow i$;
end for
for $i = 1$ to n
if $i = 1$,
$a_i \leftarrow i$;
else
$a_i \leftarrow n+1$;
end for
for $i = 1$ to n
$b_i \leftarrow n+1$;
end for
for $i = 1$ to n
if $i = 1, 2, n$
$c_i \leftarrow n$;
else
$c_i \leftarrow n+2$;
end for
end procedure
Output: vertex colored $C(H_n)$.

Theorem 3. *For a Helm graph H_n, $n \geq 3$, the b-chromatic number of Central Graph of Helm graph is n+2.*

$$i.e., \varphi[C(H_n)] = n+2.$$

Proof. Let us consider the vertex set of helm graph

$$V\left(H_n\right) = x \cup \left\{x_i : 1 \leq i \leq n\right\} \cup \left\{y_i : 1 \leq i \leq n\right\}.$$

By the definition of Central graph, the edge joining x_i and x_{i+1} has been subdivided by the newly introduced vertex $a_i\,(1 \leq m \leq n-1)$, the remaining edge $x_n x_1$ has been subdivided by the vertex a_n and the edge joining x_i and y_i has been subdivided by the new vertex $b_i\,(1 \leq i \leq n)$. Denote the newly added vertex on the edge joining x_i and x as $c_i\,(1 \leq i \leq n)$. Then the vertex set of central graph of Helm graph is

$$\begin{aligned} V\left[C\left(H_n\right)\right] &= x \cup \left\{x_i : 1 \leq i \leq n\right\} \cup \left\{y_i : 1 \leq i \leq n\right\} \cup \left\{a_i : 1 \leq i \leq n\right\} \\ &\cup \left\{b_i : 1 \leq i \leq n\right\} \cup \left\{c_i : 1 \leq i \leq n\right\}. \end{aligned}$$

The vertices of $C(H_n)$ are colored as given in the algorithm 4.1. By algorithm $\ni n{+}2$ vertices $\{x\} \cup \{x_1\} \cup \{y_i : 1 \leq i \leq n\}$ with color class $C = (i : 1 \leq i \leq n+2)$. By algorithm the set of vertices $\{x\} \cup \{x_1\} \cup \{y_i : 1 \leq i \leq n\}$ with cardinality n+2 has the color class

$$C[\ x \cup x_1 \cup y_i : 1 \leq i \leq n] = (i : 1 \leq i \leq n+2)$$

and the vertices of $x \cup x_1 \cup y_i : 1 \leq i \leq n$ receive distinct colors. This implies the fact that, the coloring is b-coloring. To prove it is maximum. Let us suppose that $\varphi[C(H_n)] > n+2$. Then there must be at least $n+3$ vertices having degree $n+2$ in $C\left(H_n\right)$, with distinct colors and also adjacent to all other colors. Then only we can assign $n+3$ colors to the vertex set of $\varphi\left[C\left(H_n\right)\right]$. Here the vertex set $\{y_i : 1 \leq i \leq n\}$ and x in $C\left(H_n\right)$ forms a clique of order n. If we assign the $n+3$ color, then it does not satisfy the $b-$coloring condition. Therefore we can assign n+1 colors to the clique and $n+2$ color to the vertex x_1. Which produce $b-$coloring. Therefore $\varphi\left[C\left(H_n\right)\right] \leq n+2$. Hence $\varphi\left[C\left(H_n\right)\right] = n+2$. □

5. $b-$Coloring of $C\left(DT_n\right)$

5.1. $b-$Coloring Algorithm of $C\left(DT_n\right)$.

Input: $C\left(DT_n\right), n \geq 1$.
$V \leftarrow \{x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_n, z_1, z_2, \ldots z_n, a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n\}$.
$V \leftarrow \{c_1, c_2, \ldots c_n, d_1, d_2, \ldots d_n, e_1, e_2, \ldots e_n.\}$
for $i = 1$
$x_1 \leftarrow 2(n+1)$;
end for
for $i = 2$
$x_2 \leftarrow 2n+1$;
end for
for i = 3 to n
$x_i \leftarrow 2k, k = 1, 2, \ldots, n$;
end for

for $i = 1$ to n
$y_i \leftarrow i$;
end for
for $i = 1$ to n
$z_i \leftarrow i + n$;
end for
for $i = 1$ to n
$a_i \leftarrow k,\ k = 1, 3, \dots n$;
end for
for $i = 1$ to 2
$b_i \leftarrow i + 2n$;
end for
for $i = 3$ to n
$b_i \leftarrow 2n + 1$;
end for
for $i = 1$ to 2
if $i = 1$
$c_i \leftarrow 2(n + 1)$;
else
$c_i \leftarrow 2n + 1$;
end for
for $i = 3$ to n
$c_i \leftarrow 2(n + 1)$;
end for
for $i = 1$ to 2
if $i = 1$
$d_i \leftarrow 2n + 1$;
else
$d_i \leftarrow 2(n + 1)$;
end for
for $i = 3$ to n
$d_i \leftarrow 2n + 1$;
end for
for $i = 1$ to 2
if i = 1
$e_i \leftarrow 2(n + 1)$;
else
$e_i \leftarrow 2n + 1$;
end for
for $i = 3$ to n
$e_i \leftarrow 2(n + 1)$;
end for

end procedure
Output: vertex colored $C\left(DT_n\right)$.

Theorem 4. *For a Double Triangular Snake graph DT_n, $n \geq 2$, the b - chromatic number of Central Graph of Double Triangular Snake graph is 2(n+1).*

$$i.e., \varphi\left[C\left(DT_n\right)\right] = 2(n+1).$$

Proof. Let DT_n be a Double Triangular Snake with i blocks on n vertices. Let

$$\begin{aligned} V_1 &= \{x_i : 1 \leq i \leq n+1\} \\ V_2 &= \{y_i : 1 \leq i \leq n\} \\ V_3 &= \{z_i : 1 \leq i \leq n\} \end{aligned}$$

The vertex set of Double Triangular Snake is

$$\begin{aligned} V(DT_n) &= V_1 \cup V_2 \cup V_3. \\ E_1 &= \{x_i x_{i+1} : 1 \leq i \leq n\} \\ E_2 &= \{y_i x_i : 1 \leq i \leq n\} \\ E_3 &= \{y_i x_{i+1} : 1 \leq i \leq n\} \\ E_4 &= \{z_i x_i : 1 \leq i \leq n\} \\ E_5 &= \{z_i x_{i+1} : 1 \leq i \leq n\} \end{aligned}$$

The edge set of Double Triangular Snake is

$$E(DT_n) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5.$$

By using central graph definition, let us subdivide each edge of DT_n by introducing a new vertex between each edge. The edge $x_i x_{i+1} (1 \leq i \leq n)$, has been subdivided by the vertex $a_i (1 \leq i \leq n)$. The edges $y_i x_i$ and $y_i x_{i+1}$ are subdivided by the vertex $b_i (1 \leq i \leq n)$ and $c_i (1 \leq i \leq n)$ respectively. Denote the newly introduced vertex of $z_i x_i$ and $z_i x_{i+1}$ as $d_i (1 \leq i \leq n)$ and $e_i (1 \leq i \leq n)$. Then the vertex set of $C(DT_n)$ is

$$\begin{aligned} V[C(DT_n)] = & \{x_i : 1 \leq i \leq n+1\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\} \\ & \cup \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq n\} \\ & \cup \{d_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \end{aligned}$$

The vertices of $C(DT_n)$ are colored as given in the algorithm 5.1.
By algorithm $\ni 2(n+1)$ vertices

$$\begin{aligned} & \{x_i : 1 \leq i \leq n+1\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\} \cup \{a_i : 1 \leq i \leq n\} \\ & \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq n\} \cup \{d_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \end{aligned}$$

with color class $C = \{1 \leq i \leq 2(n+1)\}$. The set of vertices

$$\{x_i : 1 \leq i \leq 2\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\}$$

have cardinality $2(n+1)$ with color class

$$C[\{x_i : 1 \leq i \leq 2\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\}] = \{i : 1 \leq i \leq 2(n+1)\}$$

Since

$$|\{x_i : 1 \leq i \leq 2\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\}| \\ = 2(n+1) = |\{i : 1 \leq i \leq 2(n+1)\}|$$

the vertices of $\{x_i : 1 \leq i \leq 2\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\}$ receive distinct colors. $C[N(x_i)]$, $C[N(y_i)]$ and $C[N(z_i)]$ have $2(n+1)$ distinct colors for each i. It implies that, the coloring is $b-$coloring. To prove it is maximum. Let us suppose that $\varphi[C(DT_n)] > 2(n+1)$.

For assigning $2(n+2)$ colors to $C(DT_n)$ we need $2(n+3)$ vertices of degree $2(n+2)$, all are having distinct colors and adjacent to all other colors. Here in $C(DT_n)$ we have a clique formed by the vertex sets $\{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$. so that we can assign $2n$ colors to the vertex sets of y_i and z_i. In x_i the vertex set $\{x_i : 1 \leq i \leq n+1\}$ having degree $3n$, so we can assign two more colors. Therefore $\varphi[C(DT_n)] \leq 2(n+1)$. Hence $\varphi[C(DT_n)] = 2(n+1)$. □

6. $b-$Coloring of $C(G_n)$

Theorem 5. *For a Gear graph G_n, $n \geq 3$, the b - chromatic number of Central Graph of Gear graph is $n + \lfloor (2n-1)/2 \rfloor$ for n is 3 and $2(n-1)$ for $n \geq 4$.*

$$i.e., \varphi[C(G_n)] = \begin{cases} n + \lfloor (2n-1)/2 \rfloor, & \text{for } n = 3 \\ 2(n-1), & \text{for } n \geq 4. \end{cases}$$

7. $b-$Coloring of $C(CH_n)$

Theorem 6. *For a Closed Helm graph CH_n, $n \geq 3$, the b - chromatic number of Central Graph of Closed Helm graph is $n + \lceil n/2 \rceil$.*

$$i.e., \varphi[C(CH_n)] = \begin{cases} n + \lceil n/2 \rceil, & \text{for } n \text{ is odd.} \end{cases}$$

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Current address: M.Kalpana : Department of Mathematics, Kongunadu Arts and Science College Coimbatore - 641 029, Tamilnadu India
E-mail address: kalpulaxmi@gmail.com
ORCID Address: http://orcid.org/0000-0002-3303-6590
Current address: D.Vijayalakshmi : Department of Mathematics, Kongunadu Arts and Science College Coimbatore - 641 029, Tamilnadu India
E-mail address: vijikasc@gmail.com
ORCID Address: http://orcid.org/0000-0002-8925-1134