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ON SOME PROBLEMS REGARDING SET VALUED (α, ψ)-F-CONTRACTIONS

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Abstract. In this paper, we introduce set valued (α, ψ) F-contraction mappings in the setting of a partial metric space. We obtain some common fixed point theorems for a pair of these mappings. These results generalize several recent results existing in the current literature.

1. Introduction

Let Ψ represent the class of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(1) ψ is nondecreasing;
(2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, ψ^n being n^{th} iterate of ψ.

These functions are known, in the literature, as c-comparison functions. Clearly, if ψ is a c-comparison function, then $\psi(t) < t$ for any $t > 0$.

Definition 1. [24] *Let $\alpha : X \times X \to [0, \infty)$ be a function. The mapping $T : X \to X$ is said to be α-admissible if it satisfies the condition:*

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(T(x), T(y)) \geq 1 \text{ for all } x, y \in X.$$

Very recently, Samet et al.[24] introduced a meaningful generalization of Banach Contraction Principle using the concept of α-admissible mappings.

Definition 2. [24] *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. The mapping $T : X \to X$ is said to be an (α, ψ)-contraction mapping if there exists $\psi \in \Psi$ such that*

$$\alpha(x, y)d(T(x), T(y)) \leq \psi(d(x, y)) \text{ for all } x, y \in X. \tag{1}$$

Samet et al. presented the following famous theorem.

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Theorem 3. [24] *Let (X,d) be a complete metric space and $T : X \to X$ be an $(\alpha-\psi)$-contractive mapping. If T is α-admissible and continuous, then T has a fixed point in X.*

Kumam et al.[20] considered Definition 2 in PMS.

Definition 4. [20] *Let (X,p) be a partial metric space and $\alpha : X \times X \to [0,\infty)$ be a function. The mapping $T : X \to X$ is said to be an (α,ψ)-contraction mapping in PMS if there exists $\psi \in \Psi$ such that*

$$\alpha(x,y)p(T(x),T(y)) \leq \psi(p(x,y)) \text{ for all } x,y \in X. \tag{2}$$

Clearly, Definition 2 is a particular case of Definition 4.

Theorem 5. [20] *Let (X,d) be a complete partial metric space and $T : X \to X$ be an $(\alpha-\psi)$-contractive mapping. If T is α-admissible and continuous, then T has a fixed point in X.*

Minak et al. [22] and Asl et al. [14] showed the existence of fixed points of $(\alpha-\psi)$ multi-valued-contractive mappings in metric spaces. In this article, motivated by Wardowski [25], Samet et al. [24] and Aydi et al. [7], we prove some common fixed point theorems for a pair of mappings satisfying set valued (α,ψ) F-contractions in a complete partial metric space. We also deduce several fixed point results under different contractive conditions. Some examples are presented to support the obtained results.

2. Preliminaries

Throughout this paper, we denote $(0,\infty)$ by R^+, $[0,\infty)$ by R_0^+, $(-\infty,+\infty)$ by R and set of natural numbers by N. The following concepts and results will be required in the sequel.

Definition 6. [25] *A mapping $T : M \to M$ is said an F-contraction if it satisfies the following condition*

$$(d(T(r_1),T(r_2)) > 0 \Rightarrow \tau + F(d(T(r_1),T(r_2)) \leq F(d(r_1,r_2))), \tag{3}$$

for all $r_1,r_2 \in M$ and some $\tau > 0$, where $F : R^+ \to R$ is a function satisfying the following properties:

(F_1) : *F is strictly increasing;*
(F_2) : *For each sequence $\{r_n\}$ of positive numbers,*

$$\lim_{n\to\infty} r_n = 0 \text{ if and only if } \lim_{n\to\infty} F(r_n) = -\infty;$$

(F_3) : *There exists $\theta \in (0,1)$ such that $\lim_{\alpha\to 0^+} (\alpha)^\theta F(\alpha) = 0$.*

Wardowski [25] established the following result using the concept of F-contractions.

Theorem 7. [25] *Let (M,d) be a complete metric space and $T: M \to M$ be an F-contraction. Then T has a unique fixed point $\upsilon \in M$ and for every $r_0 \in M$, the sequence $\{T^n(r_0)\}$ for all $n \in N$, converges to υ.*

For other results using F-contractions, see [3, 12, 13, 16, 18, 19]. Let (F_4): $F(\inf A) = \inf F(A)$ for all $A \subseteq (0,\infty)$ with $\inf A > 0$. We denote by F and F^*, the set of all functions satisfying the conditions $(F_1)-(F_3)$ and $(F_1)-(F_4)$, respectively. One can note that $F^* \subseteq F$ and

(a) $f(x) = \ln(x)$,
(b) $g(x) = x + \ln(x)$,
(c) $h(x) = \ln(x^2 + x)$,
(d) $k(x) = -\dfrac{1}{\sqrt{x}}$

are members of F^*. Also if we define

$$F(x) = \begin{cases} \ln(x) & \text{if } x \leq 1; \\ x & \text{if } x > 1, \end{cases}$$

then $F \in F - F^*$

Remark 8. *If F satisfies (F_1), then it satisfies (F_4) if and only if F is right-continuous.*

Let (X,d) be a metric space. Let $P(X)$, $CB(X)$, $K(X)$ denote the family of all non-empty subsets, bounded and closed subsets, compact subsets of X, respectively. It is obvious that $K(X) \subseteq CB(X) \subseteq P(X)$. For $x \in X$ and $A, B \in CB(X)$, we define

$$D(x,A) = \inf_{a \in A} d(x,a) \text{ and } D(A,B) = \sup_{a \in A} D(a,B).$$

Consider the mapping $H: CB(X) \times CB(X) \to [0,\infty)$ given by

$$H(A,B) = \max\left\{\sup_{x\in A} D(x,B), \sup_{b \in B} D(b,A)\right\},$$

for every $A, B \in CB(X)$. Then the mapping H is a metric and it is called a Hausdorff metric induced by the metric d.

Altun et al. [6] presented the multi-valued version of Theorem 7.

Theorem 9. [6] *Let (M,d) be a complete metric space and $T: M \to CB(M)$ be a mapping. If there exist $\tau > 0$ and $F \in F^*$ such that*

$$(H(T(r_1), T(r_2)) > 0 \Rightarrow \tau + F(H(T(r_1), T(r_2)) \leq F(d(r_1, r_2))).$$

Then T has a fixed point $\upsilon \in M$.

Altun et al. [6] also proved that the condition (F_4) can be removed if we replace $CB(M)$ by $K(M)$ in Theorem 9.

The notion of a partial metric space (PMS) was introduced by Matthews [21] to the model computation over a metric space. The PMS is a generalization of the usual metric space in which the self-distance is no longer necessarily zero. The notions such as convergence, completeness, Cauchy sequence in the setting of partial metric spaces can be found in [1, 2, 5, 9, 10, 11, 15, 17, 21, 23] and references therein.

Definition 10. [21] *Let M be a nonempty set. If the function $p : M \times M \to [0, \infty)$ satisfies the following properties:*

(p_1) $r_1 = r_2 \Leftrightarrow p(r_1, r_1) = p(r_1, r_2) = p(r_2, r_2)$;
(p_2) $p(r_1, r_1) \leq p(r_1, r_2)$;
(p_3) $p(r_1, r_2) = p(r_2, r_1)$;
(p_4) $p(r_1, r_3) \leq p(r_1, r_2) + p(r_2, r_3) - p(r_2, r_2)$;

for all $r_1, r_2, r_3 \in M$, then p is called a partial metric on M and the pair (M, p) is a partial metric space.

Example 11. *The classical partial metric space is known as $p(r_1, r_2) = \max\{r_1, r_2\}$ for all $r_1, r_2 \geq 0$.*

In [21], Matthews proved that every partial metric p on M induces a metric $p^s : M \times M \to R_0^+$ defined by

$$p^s(r_1, r_2) = 2p(r_1, r_2) - p(r_1, r_1) - p(r_2, r_2), \tag{4}$$

for all $r_1, r_2 \in M$.
Matthews [21] established that each partial metric p on M generates a T_0 topology $\tau(p)$ on M. The base of topology $\tau(p)$ is the family of open p-balls $\{B_p(r, \epsilon) : r \in M, \ \epsilon > 0\}$, where $B_p(r, \epsilon) = \{r_1 \in M : p(r, r_1) < p(r, r) + \epsilon\}$ for all $r \in M$ and $\epsilon > 0$. A sequence $\{r_n\}_{n \in \mathbb{N}}$ in (M, p) converges to a point $r \in M$ if and only if $p(r, r) = \lim\limits_{n \to \infty} p(r, r_n)$.

Definition 12. [21] *Let (M, p) be a partial metric space.*

(1) *A sequence $\{r_n\}_{n \in \mathbb{N}}$ in (M, p) is called Cauchy if $\lim\limits_{n,m \to \infty} p(r_n, r_m)$ exists and is finite.*
(2) *(M, p) is complete if every Cauchy sequence $\{r_n\}_{n \in \mathbb{N}}$ in M converges, with respect to $\tau(p)$, to a point $r \in X$ such that $p(r, r) = \lim\limits_{n,m \to \infty} p(r_n, r_m)$.*

The following lemma will be helpful in the sequel.

Lemma 13. [21] *Let (M, p) be a partial metric space.*

(1) *A sequence $\{r_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in (M, p) if and only if it is a Cauchy sequence in the metric space (M, p^s)*
(2) *The partial metric space (M, p) is complete if and only if the metric space (M, p^s) is complete.*
(3) *A sequence $\{r_n\}_{n \in \mathbb{N}}$ in M converges to a point $r \in M$, with respect to $\tau(p^s)$ if and only if $\lim_{n \to \infty} p(r, r_n) = p(r, r) = \lim_{n,m \to \infty} p(r_n, r_m)$.*

Definition 14. *Let (X, p) be a partial metric space and let $\alpha : X \times X \to R_0^+$ be a function. (X, p) is said to be α -regular if for any sequence $\{x_n\} \subset X$ such that*

$\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in N$ *and* $x_n \to x$ *as* $n \to \infty$, *we have* $\alpha(x_n, x) \geq 1$ *for all* $n \in N$.

Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of the PMS (X, p) induced by the partial metric p. Note that the closedness is taken from (X, τ_p) and the boundedness is given as follows: A is a bounded subset in (X, p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(x_0, x_0) + M$.

Definition 15. [7] *For* $A, B \in CB^p(X), x \in X, \delta_p : CB^p(X) \times CB^p(X) \to [0, \infty)$, *define*

(1) $D_p(x, A) = \inf\{p(x, a) : a \in A\}$;
(2) $\delta_p(A, B) = \sup\{p(a, B) : a \in A\}$;
(3) $\delta_p(B, A) = \sup\{p(b, A) : b \in B\}$;
(4) $H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}$.

It is easy to show that $D_p(x, A) = 0$ implies $D_p^s(x, A) = 0$ where $D_p^s(x, A) = \inf\{p^s(x, a) : a \in A\}$.

Lemma 16. [4] *Let* (X, p) *be a* PMS *and* A *be any nonempty subset of* X, *then* $a \in \bar{A}$ *if and only if* $D_p(a, A) = p(a, a)$.

Proposition 17. [7, 8] *Let* (X, p) *be a* PMS. *For any* $A, B, C \in CB^p(X)$, *we have*

(1) $\delta_p(A, A) = \sup\{d(a, b) : a, b \in A\}$;
(2) $\delta_p(A, B) = \delta_p(B, A)$;
(3) $\delta_p(A, A) = 0 \Rightarrow A \subseteq B$;
(4) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$.

Proposition 18. [7] *Let* (X, p) *be a partial metric space.*

(1) $H_p(A, B) = 0$ *implies* $A = B$;
(2) $H_p(A, A) \leq H_p(A, B)$;
(3) $H_p(A, B) = H_p(B, A)$;
(4) $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$.

The function $H_p : CB^p(X) \times CB^p(X) \to [0, \infty)$ satisfying properties in Proposition 18 is called a partial Hausdorff metric. It is easy to show that any Hausdorff metric is a partial Hausdorff metric. The converse is not true (see [7, Example 2.6]).

3. Set valued (α, ψ) F-contractions

Let (M, p) be a partial metric space. Define

$$\mathcal{N}(r_1, r_2) = \max\left\{\begin{array}{l} p(r_1, r_2), \dfrac{D_p(r_1, T(r_1))D_p(r_2, T(r_2))}{1 + p(r_1, r_2)}, \\ \dfrac{D_p(r_1, T(r_1))D_p(r_2, T(r_2))}{1 + \delta_p(T(r_1), T(r_2))} \end{array}\right\};$$

$$\mathcal{M}(r_1,r_2)=\max\left\{\begin{array}{l} p(r_1,r_2),\dfrac{D_p(r_1,S(r_1))D_p(r_2,T(r_2))}{1+p(r_1,r_2)}, \\ \dfrac{D_p(r_1,S(r_1))D_p(r_2,T(r_2))}{1+\delta_p(S(r_1),T(r_2))} \end{array}\right\}.$$

Definition 19. *Let (M,p) be a partial metric space and $\alpha: M\times M\to[0,\infty)$ be a function. The mapping $T: M\to CB^p(M)$ is called a set valued (α,ψ) F-contraction, if there exist $F\in F^*$, $\tau>0$ and $\psi\in\Psi$ such that*

$$\tau+F(\alpha(r_1,r_2)H_p(T(r_1),T(r_2)))\le F\left(\psi(p(r_1,r_2))\right),$$

for all $r_1,r_2\in M$ such that $\alpha(r_1,r_2)\ge 1$, whenever $H_p(T(r_1),T(r_2))>0$.

Definition 20. *Let (M,p) be a partial metric space and $\alpha: M\times M\to[0,\infty)$ be a function. The mapping $T: M\to CB^p(M)$ is called a set valued (α,ψ) F-contraction of rational type, if there exist $F\in F^*$, $\tau>0$ and $\psi\in\Psi$ such that*

$$\tau+F(\alpha(r_1,r_2)H_p(T(r_1),T(r_2)))\le F\left(\psi(\mathcal{N}(r_1,r_2))\right),$$

for all $r_1,r_2\in M$ such that $\alpha(r_1,r_2)\ge 1$, whenever $H_p(T(r_1),T(r_2))>0$.

Obviously, Definition 19 is a particular case of Definition 20.

Definition 21. *Let (M,p) be a partial metric space and $\alpha: M\times M\to[0,\infty)$ be a function. The mappings $S,T: M\to CB^p(M)$ are called set valued (α,ψ) F-contractions, if there exist $F\in F^*$, $\tau>0$ and $\psi\in\Psi$ such that*

$$\tau+F(\alpha(r_1,r_2)H_p(T(r_1),S(r_2)))\le F\left(\psi(p(r_1,r_2))\right),$$

for all $r_1,r_2\in M$ with $\alpha(r_1,r_2)\ge 1$, whenever $H_p(S(r_1),T(r_2))>0$.

Definition 22. *Let (M,p) be a partial metric space and $\alpha: M\times M\to[0,\infty)$ be a function. The mappings $S,T: M\to CB^p(M)$ are called set valued (α,ψ) F-contractions of rational type, if there exist $F\in F^*$, $\tau>0$ and $\psi\in\Psi$ such that*

$$\tau+F(\alpha(r_1,r_2)H_p(T(r_1),S(r_2)))\le F\left(\psi(\mathcal{M}(r_1,r_2))\right), \tag{5}$$

for all $r_1,r_2\in M$ such that $\alpha(r_1,r_2)\ge 1$, whenever $H_p(T(r_1),S(r_2))>0$.

Definition 21 can be seen as a particular case of Definition 22. The following example shows the significance of (α,ψ) set valued F-contractions on partial metric spaces with respect to (α,ψ) set valued F-contractions on metric spaces.

Example 23. *Let $M=[0,1]$. Consider the partial metric $p(r_1,r_2)=\max\{r_1,r_2\}$ for all $r_1,r_2\in M$. The metric p^s induced by the partial metric p is given by $p^s(r_1,r_2)=|r_1-r_2|$ for all $r_1,r_2\in M$. Define the mappings $F: R^+\to R$ by $F(r)=\ln(r)$, $\psi(t)=\frac{t}{2}$, $\alpha: M\times M\to[0,\infty)$ and $T: M\to CB^p(M)$ by*

$$T(r)=\begin{cases}\left\{\dfrac{r}{5}\right\} & \text{if } r\in[0,1);\\[2ex] \left\{0,\dfrac{1}{7}\right\} & \text{if } r=1,\end{cases}$$

and

$$\alpha(r_1, r_2) = \begin{cases} 1 \ \textit{if}\ r_1, r_2 \in [\frac{5}{6}, 1]; \\ 0 \ \textit{otherwise.} \end{cases}$$

Then T is not a set valued (α, ψ) F- contraction on the metric space (M, p^s). Indeed, for $r_1 = 1$ and $r_2 = \frac{5}{6}$, we have $\alpha(r_1, r_2) = 1$ and $H_{p^s}(T(r_1), T(r_2)) > 0$. Also, for all $\tau > 0$,

$$\begin{aligned} &\tau + F\left(H_{p^s}(T(r_1), T(r_2))\right) \\ &= \tau + F\left(H_{p^s}(T(1), T(\frac{5}{6}))\right) \\ &= \tau + F\left(\frac{1}{6}\right) \\ &> F\left(\frac{1}{6}\right) \\ &= F\left(p^s(1, \frac{5}{6})\right) \\ &= F\left(p^s(r_1, r_2)\right), \end{aligned}$$

On the other hand, let $r_1, r_2 \in M$ be such that $\alpha(r_1, r_2) \geq 1$ and $H_p(T(r_1), T(r_2)) > 0$. Then $r_1, r_2 \in [\frac{5}{6}, 1]$. We have the following:
If $r_1, r_2 \in [\frac{5}{6}, 1)$, then for $\tau \leq \ln\left(\frac{5}{2}\right)$, we have

$$\tau + \ln\left(H_p(T(r_1), T(r_2))\right) \leq \ln\left(\frac{5}{2}\right) + \ln\left(\frac{p(r_1, r_2)}{5}\right) = \ln\left(\frac{1}{2}p(r_1, r_2)\right).$$

If $r_1 \in [\frac{5}{6}, 1)$ and $r_2 = 1$, for $\tau \leq \ln\left(\frac{5}{2}\right)$, we have

$$\begin{aligned} \tau + \ln\left(H_p(T(r_1), T(r_2))\right) &\leq \ln\left(\frac{5}{2}\right) + \ln\left(\frac{r_1}{5}\right) \\ &= \ln\left(\frac{r_1}{2}\right) \\ &\leq \ln(\frac{1}{2}) \\ &= \ln\left(\frac{1}{2}p(r_1, r_2)\right). \end{aligned}$$

The case $r_2 \in [\frac{5}{6}, 1)$ and $r_1 = 1$ is similarly. Consequently, we obtained that

$$\tau + F\left(\alpha(r_1, r_2) H_p(T(r_1), T(r_2))\right) \leq F\left(\psi(p(r_1, r_2))\right).$$

Thus, T is an (α, ψ) set valued F-contraction on partial metric spaces.

4. Fixed point of set valued (α,ψ) F-Contractions

We begin with the following definition.

Definition 24. *Let (M,p) be a partial metric space. Let $S,T: M\to CB(M)$ be two set valued mappings and $\alpha: M\times M\to[0,+\infty)$ be a function. The pair (S,T) is said to be triangular α_*-admissible if the following conditions hold:*

(1) *(S,T) is α_*-admissible; that is,*

$$\alpha(r_1,r_2)\geq 1 \text{ implies } \alpha_*(Sr_1,Tr_2)\geq 1 \text{ and } \alpha_*(Tr_1,Sr_2)\geq 1, \text{ where}$$

$$\alpha_*(A,B)=\inf\{\alpha(a,b): a\in A,\ b\in B\},$$

(2) *$\alpha(r_1,u)\geq 1$ and $\alpha(u,r_2)\geq 1$ imply $\alpha(r_1,r_2)\geq 1$.*

It is easy to show that every α_*-admissible mapping is also α-admissible, but the converse is not true (see [22, Example 15]).
Our first main result is

Theorem 25. *Let (M,p) be a complete partial metric space and $S,T: M\to CB^p(M)$ be a pair of mappings. Assume that*

(1) *(S,T) is a pair of set valued (α,ψ) F-contractions of rational type;*
(2) *(S,T) is a triangular α_*-admissible pair of mappings;*
(3) *there exists $r_0\in M$ such that $\alpha_*(r_0,S(r_0))\geq 1$;*
(4) *M is α-regular.*

Then there exists a common fixed point of the pair (S,T) in M.

Proof. By assumption (3), there exits $r_0\in M$ such that $\alpha_*(r_0,S(r_0))\geq 1$. There exists $r_1\in S(r_0)$ such that $\alpha(r_0,r_1)\geq\alpha_*(r_0,S(r_0))\geq 1$. By assumption (2), $\alpha_*(S(r_0),T(r_1))\geq 1$. So there exists $r_2\in T(r_1)$ such that $\alpha(r_1,r_2)\geq\alpha_*(S(r_0), T(r_1))\geq 1$. Also, $\alpha_*(T(r_1),S(r_2))\geq 1$.

Again, there exists $r_3\in S(r_2)$ such that $\alpha(r_2,r_3)\geq\alpha_*(T(r_1),S(r_2))\geq 1$ which implies that $\alpha_*(S(r_2),T(r_3))\geq 1$. Continuing in this way, we construct an iterative sequence $\{r_n\}$ of points in M such that $r_{2i+1}\in S(r_{2i})$, $r_{2i+2}\in T(r_{2i+1})$ such that $\alpha(r_{2i+1},r_{2i+2})\geq 1$ and $\alpha(r_{2i},r_{2i+1})\geq 1$ for all $i\geq 0$. Hence $\alpha(r_n,r_{n+1})\geq 1$ for all $n\geq 0$. Let $r_{2i}\notin S(r_{2i})$ and $r_{2i+1}\notin T(r_{2i+1})$ such that $\alpha(r_{2i},r_{2i+1})\geq 1$. Since $T(r_{2i+1})$ is a closed set, by Lemma 16, we get $D_p(r_{2i+1},T(r_{2i+1}))>0$. On the other hand, as $0<D_p(r_{2i+1},T(r_{2i+1}))\leq H_p(S(r_{2i}),T(r_{2i+1}))$, by the contractive condition (5) and (F_1), we get

$$F(D_p(r_{2i+1},T(r_{2i+1})))\leq F(\alpha(r_{2i},r_{2i+1})H_p(S(r_{2i}),T(r_{2i+1})))\leq F(\psi(\mathcal{M}(r_{2i},r_{2i+1})))-\tau,$$

for all $i \geq 0$, where

$$\begin{aligned}
\mathcal{M}(r_{2i}, r_{2i+1}) &= \max\left\{\begin{array}{l} p(r_{2i}, r_{2i+1}), \dfrac{D_p(r_{2i}, S(r_{2i}))D_p(r_{2i+1}, T(r_{2i+1}))}{1+p(r_{2i}, r_{2i+1})}, \\ \dfrac{D_p(r_{2i}, S(r_{2i}))D_p(r_{2i+1}, T(r_{2i+1}))}{1+\delta_p(S(r_{2i}), T(r_{2i+1}))} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} p(r_{2i}, r_{2i+1}), \dfrac{p(r_{2i}, r_{2i+1})p(r_{2i+1}, r_{2i+2})}{1+p(r_{2i}, r_{2i+1})}, \\ \dfrac{p(r_{2i}, r_{2i+1})p(r_{2i+1}, r_{2i+2})}{1+p(r_{2i+1}, r_{2i+2})} \end{array}\right\} \\
&\leq \max\left\{p(r_{2i}, r_{2i+1}), p(r_{2i+1}, r_{2i+2})\right\}.
\end{aligned}$$

If for some i, $M(r_{2i}, r_{2i+1}) \leq p(r_{2i+1}, r_{2i+2})$, then

$$F\left(D_p(r_{2i+1}, T(r_{2i+1}))\right) \leq F\left(\psi(p(r_{2i+1}, r_{2i+2}))\right) - \tau. \tag{6}$$

The axiom (F_4) implies that $F(D_p(r_{2i+1}, T(r_{2i+1}))) = \inf_{r\in T(r_{2i+1})} F(p(r_{2i+1}, r))$. Thus, there exists $r = r_{2i+2} \in T(r_{2i+1})$ such that $F(D_p(r_{2i+1}, T(r_{2i+1}))) = F(p(r_{2i+1}, r_{2i+2}))$ and the inequality (6) implies that

$$F\left(p(r_{2i+1}, r_{2i+2})\right) \leq F\left(\psi(p(r_{2i+1}, r_{2i+2}))\right) - \tau,$$

which is a contradiction with respect to (F_1) and definition of ψ. Therefore,

$$F\left(p(r_{2i+1}, r_{2i+2})\right) \leq F\left(p(r_{2i}, r_{2i+1})\right) - \tau,$$

for all $i \geq 0$. Similarly, we have

$$F\left(p(r_{2i+2}, r_{2i+3})\right) \leq F\left(p(r_{2i+1}, r_{2i+2})\right) - \tau,$$

for all $i \geq 0$. Hence,

$$F\left(p(r_n, r_{n+1})\right) \leq F\left(p(r_{n-1}, r_n)\right) - \tau, \tag{7}$$

for all $n \in N$. By (7), we obtain

$$F\left(p(r_{n-1}, r_n)\right) \leq F\left(p(r_{n-2}, r_{n-1})\right) - 2\tau.$$

Repeating these steps, we get

$$F\left(p(r_n, r_{n+1})\right) \leq F\left(p(r_0, r_1)\right) - n\tau.$$

This yields that $\lim_{n\to\infty} F\left(p(r_n, r_{n+1})\right) = -\infty$. By (F_2), we have

$$\lim_{n\to\infty} p(r_n, r_{n+1}) = 0. \tag{8}$$

By (F_3), there exists $\kappa \in (0,1)$ such that

$$\lim_{n\to\infty} \left((p(r_n, r_{n+1}))^{\kappa} F\left(p(r_n, r_{n+1})\right)\right) = 0. \tag{9}$$

We have

$$\left(p(r_n, r_{n+1})\right)^{\kappa}\left(F\left(p(r_n, r_{n+1})\right) - F\left(p(r_0, x_1)\right)\right) \leq -\left(p(r_n, r_{n+1})\right)^{\kappa} n\tau \leq 0. \tag{10}$$

Using (8), (9) and letting $n\to\infty$ in (10), we have

$$\lim_{n\to\infty}\left(n\left(p(r_n,r_{n+1})\right)^{\kappa}\right)=0. \tag{11}$$

By (11), there exists $n_1\in N$ such that for all $n\left(p(r_n,r_{n+1})\right)^{\kappa}\le 1$, $n\ge n_1$ or

$$p(r_n,r_{n+1})\le\frac{1}{n^{\frac{1}{\kappa}}}\text{ for all }n\ge n_1. \tag{12}$$

The inequality (12) implies that, for $m>n\ge n_1$,

$$\begin{aligned}p(r_n,r_m) &\le p(r_n,r_{n+1})+p(r_{n+1},r_{n+2})+p(r_{n+2},r_{n+3})+\ldots+p(r_{m-1},r_m)\\ &- \sum_{j=n+1}^{m-1}p(r_j,r_j)\\ &\le p(r_n,r_{n+1})+p(r_{n+1},r_{n+2})+p(r_{n+2},r_{n+3})+\ldots+p(r_{m-1},r_m)\\ &= \sum_{i=n}^{m-1}p(r_i,r_{i+1})\le\sum_{i=n}^{\infty}p(r_i,r_{i+1})\le\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}.\end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{\kappa}}}$ entails $\lim_{n,m\to\infty}p(r_n,r_m)=0$. Hence $\{r_n\}$ is a Cauchy sequence in (M,p). By Lemma 13(1), $\{r_n\}$ is a Cauchy sequence in (M,p^s). Since (M,p) is a complete partial metric space, so (M,p^s) is a complete metric space and as a result, there exists $\upsilon\in M$ such that $\lim_{n\to\infty}p^s(r_n,\upsilon)=0$. Moreover, Lemma 13(3) implies

$$\lim_{n\to\infty}p(\upsilon,r_n)=p(\upsilon,\upsilon)=\lim_{n,m\to\infty}p(r_n,r_m). \tag{13}$$

Since $\lim_{n,m\to\infty}p(r_n,r_m)=0$, therefore, we deduce from (13) that

$$p(\upsilon,\upsilon)=0=\lim_{n\to\infty}p(\upsilon,r_n). \tag{14}$$

Now by (14), it follows that $r_{2n+1}\to\upsilon$ and $r_{2n+2}\to\upsilon$ as $n\to\infty$ with respect to $\tau(p)$. We show that υ is a common fixed point of pair (S,T). By hypothesis (4), there exists a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ such that $\alpha(r_{2n_k},\upsilon)\ge 1$ for all k. Now, by using (5) for all k, we have

$$\begin{aligned}F\left(D_p(r_{2n_k+1},T(\upsilon))\right) &\le F(\alpha(r_{2n_k},\upsilon)H_p(S(x_{2n_k}),T(\upsilon))\\ &\le F\left(\psi\left(\mathcal{M}(r_{2n_k},\upsilon)\right)\right)-\tau.\end{aligned}$$

This implies that

$$D_p(r_{2n_k+1},T(\upsilon))<\mathcal{M}(r_{2n_k},\upsilon). \tag{15}$$

Note that

$$\begin{aligned}\mathcal{M}(r_{2n_k},\upsilon) &= \max\left\{\begin{array}{l} p(r_{2n_k},\upsilon), \dfrac{D_p(r_{2n_k},S(r_{2n_k}))D_p(\upsilon,T(\upsilon))}{1+p(r_{2n_k},\upsilon)}, \\ \dfrac{D_p(r_{2n_k},S(r_{2n_k}))D_p(\upsilon,T(\upsilon))}{1+\delta_p(S(r_{2n_k}),T(\upsilon))} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{l} p(r_{2n_k},\upsilon), \dfrac{p(r_{2n_k},r_{2n_k+1})D_p(\upsilon,T(\upsilon))}{1+p(r_{2n_k},\upsilon)}, \\ \dfrac{p(r_{2n_k},r_{2n_k+1})D_p(\upsilon,T(\upsilon))}{1+\delta_p(S(r_{2n_k}),T(\upsilon))} \end{array}\right\}.\end{aligned}$$

Thus,

$$\lim_{k\to\infty}\mathcal{M}(r_{2n_k},\upsilon)=0.$$

Letting $k \to \infty$ in (15), we have $D_p(\upsilon,T(\upsilon)) = 0$. Thus, $\upsilon \in \overline{T(\upsilon)} = T(\upsilon)$. Similarly, $\upsilon \in S(\upsilon)$. Hence, υ is a common fixed point of the mappings S and T. □

Proposition 26. *In addition to assumptions (1)-(4) in Theorem 25, if $\alpha(\cdot,\cdot) \geq 1$ for every common fixed point of S and T, then υ is the unique common fixed point of S and T.*

Proof. Assume the contrary, that is, there exists $\omega \in M$ such that $\upsilon \neq \omega$ such that $\omega \in T(\omega)$ and $\omega \in S(\omega)$. From the contractive condition (5) and (F_1), we have

$$F(p(\upsilon,\omega)) \leq F(\alpha(\upsilon,\omega)H_p(S(\upsilon),T(\omega))) \leq F\left(\psi(\mathcal{M}(\upsilon,\omega))\right)-\tau, \tag{16}$$

where

$$\mathcal{M}(\upsilon,\omega)=\max\left\{\begin{array}{l} p(\upsilon,\omega), \dfrac{D_p(\upsilon,S(\upsilon))D_p(\omega,T(\omega))}{1+p(\upsilon,y)}, \\ \dfrac{D_p(\upsilon,S(\upsilon))D_p(\omega,T(\omega))}{1+\delta_p(S(\upsilon),T(\omega))} \end{array}\right\}.$$

From (16), we have

$$\tau+F(p(\upsilon,\omega)) \leq F\left(p(\upsilon,\omega)\right), \tag{17}$$

The inequality (17) leads to

$$p(\upsilon,\omega)<p(\upsilon,\omega),$$

which is a contradiction. Hence, $\upsilon=\omega$ and υ is the unique common fixed point of the pair (S,T). □

Following Theorem 25, we have

Corollary 27. *Let (M,p) be a complete partial metric space and $T: M \to CB^p(M)$ be a set valued mapping such that*

(1) *T is a set valued (α,ψ) F-contraction;*
(2) *T is a triangular α_*-admissible mapping;*
(3) *there exists $r_0 \in M$ such that $\alpha_*(r_0,T(r_0)) \geq 1$;*
(4) *M is α-regular.*

Then there exists a fixed point of T in M.

Corollary 28. *Let (M,p) be a complete partial metric space and $T : M \to CB^p(M)$ be a set valued mapping such that*

(1) *T is a set valued (α,ψ) F-contraction of rational type;*
(2) *T is a triangular α_*-admissible mapping;*
(3) *there exists $r_0 \in M$ such that $\alpha_*(r_0, T(r_0)) \geq 1$;*
(4) *M is α-regular.*

Then there exists a fixed point of T in M.

Corollary 29. *Let (M,p) be a complete partial metric space and $S,T : M \to CB^p(M)$ be a pair of mappings. Assume that*

(1) *(S,T) is a pair of set valued (α,ψ) F-contractions;*
(2) *(S,T) is a triangular α_*-admissible pair of mappings;*
(3) *there exists $r_0 \in M$ such that $\alpha_*(r_0, S(r_0)) \geq 1$;*
(4) *M is α-regular.*

Then there exists a common fixed point of the pair (S,T) in M.

Remark 30. *The arguments for the proof of Corollary 27, Corollary 28 and Corollary 29 follow as the same lines in proof of Theorem 25. In addition we have to consider $S = T$ in the proof of Theorem 25 for the proofs of Corollary 27 and Corollary 28.*

The following example illustrates Theorem 25 and shows that the condition (5) is more general than the corresponding condition in metric spaces.

Example 31. *Let $M = \{0,1,4\}$ be endowed with the partial metric $p : X \times X \to [0,\infty)$ defined by*

$$p(r_1,r_2) = \frac{1}{4}|r_1 - r_2| + \frac{1}{2}\max\{r_1,r_2\}, \text{ for all } r_1,r_2 \in M.$$

Note that $p(0,0) = 0$, $p(1,1) = \frac{1}{2}$ and $p(4,4) = 2$, so p is not a metric on M. As $p^s(r_1,r_2) = |r_1 - r_2|$, thus (M,p) is complete partial metric space. We observe that $\{0\}$, $\{1\}$ and $\{0,1\}$ are closed sets in the partial metric space (M,p). Indeed, if $r \in M$, then

$$\begin{aligned} r \in \overline{\{1\}} &\Leftrightarrow p(r,\{1\}) = p(r,r) \\ &\Leftrightarrow \frac{1}{4}|r-1| + \frac{1}{2}\max\{r,1\} = \frac{r}{2} \\ &\Leftrightarrow r \in \{1\}. \end{aligned}$$

Hence, $\{1\}$ is closed in (M,p). Similarly, $\{0\}$ is closed in (M,p). Also,

$$\begin{aligned} r \in \overline{\{0,1\}} &\Leftrightarrow p(r,\{0,1\}) = p(r,r) \\ &\Leftrightarrow \min\{p(r,0),p(r,1)\} = p(r,r) \\ &\Leftrightarrow r \in \{0,1\}. \end{aligned}$$

Hence, $\{0,1\}$ is closed in (M,p). Now, define the mappings $S,T: M \to CB^p(M)$ by

$$T(0)=T(1)=\{0\}, T(4)=\{0,1\} \text{ and } S(r)=\begin{cases}\{0\} & \text{if } r\in\{0,1\};\\ \{1\} & \text{if } r=4.\end{cases}$$

Define the function $\alpha: M\times M \to [0,\infty)$ by

$$\alpha(r_1,r_2)=\begin{cases}0 & \text{if } r_2\in\{0,1\} \text{ and } r_1=4;\\ 1 & \text{otherwise.}\end{cases}$$

Clearly, the pair (S,T) is a triangular α_*-admissible pair of mappings and $\alpha_*(1,S(1)) \geq 1$. Let $r_1,r_2\in M$ be such that $\alpha(r_1,r_2)\geq 1$ and $H_p(T(r_1),S(r_2))>0$, then we have the three cases: [$r_1,r_2\in\{0,1\}$ with $(r_1,r_2)\neq(0,0)$], [$r_1\in\{0,1\}$ with $r_2=4$] and [$r_1=r_2=4$]. We shall show that the contractive condition (5) is satisfied for all possible cases.

If $r_1,r_2\in\{0,1\}$, we have $\alpha(r_1,r_2)=1$ and

$$\begin{aligned}H_p(S(r_1),T(r_2)) &= H_p(\{0\},\{0\})\\ &= p(0,0)\leq \frac{3}{4}\mathcal{M}(r_1,r_2).\end{aligned}$$

If $r_1\in\{0,1\}$ and $r_2=4$, we have $\alpha(r_1,r_2)=1$ and

$$\begin{aligned}H_p(S(r_1),T(r_2)) &= H_p(\{0\},\{0,1\})=\max\{\delta_p(\{0\},\{0,1\}),\delta_p(\{0,1\},\{0\})\}\\ &= \max\left\{0,\frac{3}{4}\right\}=\frac{3}{4}<\frac{3}{4}p(r_1,r_2)\leq\frac{3}{4}\mathcal{M}(r_1,r_2).\end{aligned}$$

If $r_1=4$ and $r_2=4$, we have $\alpha(4,4)=1$ and

$$\begin{aligned}H_p(S(4),T(4)) &= H_p(\{1\},\{0,1\})=\max\{\delta_p(\{1\},\{0,1\}),\delta_p(\{0,1\},\{1\})\}\\ &= \max\{p(1,1),p(0,1)\}=\frac{3}{4}<\frac{3}{4}p(4,4)\leq\frac{3}{4}\mathcal{M}(4,4).\end{aligned}$$

We conclude that there exist $F\in F$ defined by $F(r)=\ln(r)$, $\tau=\ln(\frac{4}{3})>0$ and $\psi\in\Psi$ defined by $\psi(t)=t$ such that

$$\tau+F(\alpha(r_1,r_2)H_p(S(r_1),T(r_2)))\leq F(\psi(\mathcal{M}(r_1,r_2))).$$

We observe that $r=0$ is a common fixed point of mappings S and T.

On the other hand, Theorem 25 is not applicable for the Hausdorff metric H_{p^s}. Indeed, note that $p^s(0,0)=0$, $p^s(1,1)=0$, $p^s(4,4)=0$, $p^s(0,1)=1=p^s(1,0)$, $p^s(4,0)=4=p^s(0,4)$ and $p^s(1,4)=3=p^s(4,1)$. For $r_1=r_2=4$, we have

$$\begin{aligned}H_{p^s}(S(4),T(4)) &= H_{p^s}(\{0\},\{0,1\})=\max\{\delta_{p^s}(\{0\},\{0,1\}),\delta_{p^s}(\{0,1\},\{0\})\}\\ &= \max\{0,1\}=1>\frac{3}{4}p^s(4,4)=\frac{3}{4}\mathcal{M}^s(4,4).\end{aligned}$$

That is,

$$\tau+F(\alpha(r_1,r_2)H_{p^s}(T(r_1),S(r_2)))\nleq F(\psi(\mathcal{M}^s(r_1,r_2))) \text{ for } r_1=r_2=4.$$

Example 32. *Let $M=[0,1]$ and define $p(r_1,r_2)=\max\{r_1,r_2\}$. Then (M,p) is a complete partial metric space. Moreover, the metric induced by p is given by $p^s(r_1,r_2)=|r_1-r_2|$, so (M,p^s) is a complete metric space. Define the mappings $S,T:M\to CB^p(M)$ as follows:*

$$T(r)=\begin{cases}\left\{\dfrac{r}{5}\right\} & \text{if } r\in[0,1);\\[2ex] \left\{0,\dfrac{1}{7}\right\} & \text{if } r=1\end{cases}\quad \text{and } S(r)=\left\{\frac{3r}{7}\right\} \text{ for all } r\in M$$

Define the function $F:R^+\to R$ by $F(r)=\ln(r)$ for all $r\in R^+$, $\psi(t)=t$ for all $t>0$ and $\alpha:M\times M\to[0,\infty)$ by

$$\alpha(r_1,r_2)=\begin{cases}1 & \text{if } r_1,r_2\in[0,\frac{1}{2}];\\ 0 & \text{otherwise.}\end{cases}$$

Let $r_1,r_2\in M$ be such that $\alpha(r_1,r_2)\geq 1$ and $H_p(S(r_1),T(r_2))>0$. Then the case $r_1=r_2=0$ is excluded and $r_1,r_2\in[0,\frac{1}{2}]$. In this case, we have the following:
If $r_1\leq r_2$, we have

$$\mathcal{M}(r_1,r_2)=\max\left\{r_2,\frac{r_1r_2}{1+r_2},\frac{r_1r_2}{1+\max\left\{\frac{3r_2}{7},\frac{r_1}{5}\right\}}\right\}.$$

Since $\frac{r_1}{1+r_2}<1$ and $\frac{r_1}{1+\max\left\{\frac{3r_2}{7},\frac{r_1}{5}\right\}}<1$, we have that $M(r_1,r_2)=r_2>0$.
In a similar way, if $r_1\geq r_2$, we obtain that $M(r_1,r_2)=r_1>0$.
Consequently, $M(r_1,r_2)=p(r_1,r_2)>0$. Let $\tau\leq\ln(\frac{7}{3})$. Then

$$\begin{aligned}\tau+F(\alpha(r_1,r_2)\,(H_p(S(r_1),T(r_2)))) &= \tau+\ln\left(\max\left\{\frac{3r_1}{7},\frac{r_2}{5}\right\}\right)\\ &\leq \ln(\frac{7}{3})+\ln\left(\max\left\{\frac{3p(r_1,r_2)}{7},\frac{p(r_1,r_2)}{5}\right\}\right)\\ &= \ln(\frac{7}{3})+\ln\left(\frac{3p(r_1,r_2)}{7}\right)=\ln\left(p(r_1,r_2)\right)\\ &= F\left(\psi(\mathcal{M}(r_1,r_2))\right).\end{aligned}$$

Thus, the contractive condition (5) is satisfied for all $r_1,r_2\in M$ such that $\alpha(r_1,r_2)\geq 1$ and $H_p(S(r_1),T(r_2))>0$. Hence, all the hypotheses of Theorem 25 are satisfied. Note that S and T have a common fixed point, which is $r=0$.

Conflict of Interests:

The authors declare that they have no competing interests regarding the publication of this paper.

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