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ON THE EVOLUTE OFFSETS OF RULED SURFACES USING THE DARBOUX FRAME

GÜLSÜM YELIZ ŞENTÜRK AND SALIM YÜCE

Abstract. In this study, using Darboux frame $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$ of ruled surface $\varphi(s,v)$, the evolute offsets $\varphi^*(s,v)$ with Darboux frame $\{\mathbf{T}^*,\mathbf{g}^*,\mathbf{n}^*\}$ of $\varphi(s,v)$ are defined. Characteristic properties of $\varphi^*(s,v)$ as a striction curve, distribution parameter and orthogonal trajectory are investigated using the Darboux frame. The distribution parameters of ruled surfaces $\varphi^*_{\mathbf{T}^*}, \varphi^*_{\mathbf{g}^*}$ and $\varphi^*_{\mathbf{n}^*}$ are given. By using Darboux frame of the surfaces we have given the relations between the instantaneous Pfaffian vectors of motions $H/H^{'}$ and $H^*/H^{*'}$, where $H=\{\mathbf{T},\mathbf{g},\mathbf{n}\}$ be the moving space along the base curve of $\varphi(s,v)$, $H^*=\{\mathbf{T}^*,\mathbf{g}^*,\mathbf{n}^*\}$ be the moving space along the base curve of $\varphi^*(s,v)$, H' and $H^{*'}$ be fixed Euclidean spaces.

1. Introduction

Differential geometry of the ruled surface is a important subject of the geometry. A ruled surface can always be easily parameterized. These surfaces can be described by moving a straight line along a chosen curve. Therefore, the equation of the ruled surface can be written as

$$\varphi(s,v)=\alpha(s)+v\mathbf{e}(s), \|\mathbf{e}(s)\|=1$$

where (α) is curve which is called the *base curve* of the ruled surface and the curve $\mathbf{e}$ is also called the *spherical indicatrix vector* of the ruled surface. The ruled surfaces are very useful in many areas of sciences for instance Computer-Aided Manufacturing (CAM), Computer-Aided Geometric Design (CAGD), geometric modeling and kinematics.

Another special subject of geometry is differential geometry of the curves. The geometers are defined some different offsets of curves for example the involute-evolute, Bertrand, Mannheim and Smarandache. Offsets of curves generally more complicated than their progenitor curve. The involute and evolute of a curve were discovered by Christian Huygens in 1673, [1]. An evolute offsets of a given curve is

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some curves that always remains perpendicular to the tangent line to the progenitor curve. In this case, the progenitor curve is called an involute. These curves also have many applications in gear industry and business.

Properties of ruled surfaces and their offset surfaces have been examined in Euclidean and non-Euclidean spaces. Ravani and Ku studied Bertrand offsets of the ruled surfaces, [2]. Kasap and Kuruoğlu (2006) investigated Bertrand offsets of the ruled surfaces in Minkowski 3-space, [3]. Moreover, the involute-evolute offsets of the ruled surface have been introduced by Kasap et al, [4]. Mannheim offsets of the ruled surface is defined by Orbay et al., [5]. Yoon classified evolute offsets of the ruled surface with constant Gaussian curvature and mean curvature and investigated linear Weingarten evolute offsets in Minkowski 3-space, [6].

The ruled surfaces with Darboux frame (RSDF) are defined by Şentürk and Yüce, [7]. Bertrand offsets of the ruled surfaces with Darboux frame (RSDF) and their properties are studied by Şentürk and Yüce, [8].
The involute-evolute curves which lying on the surfaces have studied by Bektaş and Yüce by using the Darboux frame of curves (D-curves, [9]. They obtained the relations between $\kappa_g, \tau_g, \kappa_n$ and κ_n^* for a curve to be the special involute partner D-curves. κ_g^* and τ_g^* of this special involute partner D-curve are found.

In this paper, the evolute offsets of ruled surfaces with Darboux frame are defined. The distribution parameters of ruled surfaces $\varphi^*_{\mathbf{T}^*}, \varphi^*_{\mathbf{g}^*}$ and $\varphi^*_{\mathbf{n}^*}$ are given. By using Darboux frame of the surfaces $\varphi(s,v)$ and $\varphi^*(s,v)$, we give the relations between the instantaneous Pfaffian vectors of motions $H/H^{'}$ and $H^*/H^{*'}$, where $H = \{\mathbf{T},\mathbf{g},\mathbf{n}\}$ be the moving space along the base curve of $\varphi(s,v)$, $H^* = \{\mathbf{T}^*,\mathbf{g}^*,\mathbf{n}^*\}$ be the moving space along the base curve of $\varphi^*(s,v)$, H' and $H^{*'}$ be fixed Euclidean spaces.

2. Preliminaries

Differential geometry of ruled surface and Darboux frame. A ruled surface M in $\mathbb{E}^3$ is generated by a one-parameter family of straight lines. The straight lines are called the *rulings*. The equation of the ruled surface can be written as,

$$\varphi(s,v) = \alpha(s) + v\mathbf{e}(s), \|\mathbf{e}(s)\| = 1$$

where (α) is curve which is called the *base curve* of the ruled surface and the curve, which is drawn by $\mathbf{e}(s)$ on the unit sphere S^2 is called the *spherical indicatrix curve* and $\mathbf{e}$ is also called the *spherical indicatrix vector* of the ruled surface, [2]. The ruled surface is said to be a *noncylindrical ruled surface* provided that $< \mathbf{e}_s, \mathbf{e}_s > \neq 0$.

The *striction point* on ruled surface is the foot of the common perpendicular line of the successive rulings on the main ruling. The set of striction points of the noncylindrical ruled surface generates its striction curve, [2]. It is given by

$$c(s) = \alpha(s) - \frac{< \alpha_s, \mathbf{e}_s >}{< \mathbf{e}_s, \mathbf{e}_s >}\mathbf{e}(s). \tag{2.1}$$

Theorem 1. *If successive rulings intersect, the ruled surface is called developable,* [2].

The *distribution parameter* of the noncylindrical ruled surface is identified by, [2]

$$P_e = \frac{det(\alpha_s, \mathbf{e}, \mathbf{e}_s)}{< \mathbf{e}_s, \mathbf{e}_s >}. \tag{2.2}$$

Theorem 2. *The ruled surface is developable if and only if $P_e = 0$,* [2].

A curve which intersects perpendicularly each one of rulings is called *an orthogonal trajectory* of the ruled surface. It is calculated by

$$< \mathbf{e}, d\varphi >= 0. \tag{2.3}$$

Since $\alpha(s)$ is a space curve, there exists the moving Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ along the curve. The unit normal vector field of the ruled surface is $\mathbf{n}$. We can define $\mathbf{g} = \mathbf{n} \times \mathbf{T}$ unit vector, which satisfies $< \mathbf{T}, \mathbf{g} >=< \mathbf{n}, \mathbf{g} >= 0$. (Darboux, 1896) Therefore $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ is a Darboux frame the ruled surface. The derivative formulae of the ruled surface with Darboux frame can be defined by

$$\begin{bmatrix} \dot{\mathbf{T}} \\ \dot{\mathbf{g}} \\ \dot{\mathbf{n}} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} \tag{2.4}$$

where κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the relative (also called geodesic) torsion of $\alpha(s)$. In this paper, we prefer using "dot" to denote the derivative with respect to the arc length parameter of a curve and "prime" to denote the derivative with respect to the arbitrary parameter of a curve.

3. On The Evolute Offsets of The ruled Surfaces with the Darboux Frame

Let $\varphi(s, v)$ with Darboux frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ and $\varphi^*(s, v)$ with Darboux frame $\{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\}$ be two ruled surfaces in $\mathbb{E}^3$. $\varphi(s, v)$ is said to be an involute offset of $\varphi^*(s, v)$ or $\varphi^*(s, v)$ is said to be an evolute offset of $\varphi(s, v)$, if there exists a one-to-one correspondence between their points such that $\mathbf{T}$ of $\varphi(s, v)$ and $\mathbf{g}^*$ of $\varphi^*(s, v)$ are linearly dependent.

A unit direction vector of a straight line $\mathbf{e}^*$ of φ^* is spanned by the orthonormal system $\{\mathbf{T}^*, \mathbf{g}^*\}$. So $\mathbf{e}^*$ can be written as:

$$\mathbf{e}^* = \mathbf{T}^* \cos\phi^* + \mathbf{g}^* \sin\phi^* \tag{3.1}$$

where ϕ^* is the angle between the vectors $\mathbf{T}^*$ and $\mathbf{e}^*$.

The Darboux vectors of involute offset φ^* of φ are given by

$$\begin{bmatrix} \mathbf{T}^* \\ \mathbf{g}^* \\ \mathbf{n}^* \end{bmatrix} = \begin{bmatrix} 0 & \cos\psi & -\sin\psi \\ 1 & 0 & 0 \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix}, \tag{3.2}$$

where ψ is the angle between $\mathbf{n}$ and $\mathbf{n}^*$. The equation of the offset surface φ^*, in the terms of the base curve of φ and $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$ Darboux frame, can be written as

$$\begin{aligned} \varphi^*(s,v) &= \alpha^*(s)+ve^*(s) \\ &= \underbrace{\alpha(s)+R(s)\,\mathbf{T}(s)}_{\alpha^*(s)}+v\underbrace{\left[\mathbf{T}\sin\phi^*+\mathbf{g}\cos\phi^*\cos\psi-\mathbf{n}\cos\phi^*\sin\psi\right]}_{e^*(s)}\ , \end{aligned} \tag{3.3}$$

where $R(s)$ is the distance function between the corresponding points. The distance function is calculated as $R(s)=(c-s)$, [9].

Theorem 3. *Let (φ,φ^*) be a pair of the involute-evolute RSDF. Even if φ is a closed ruled surface, φ^* is never a closed surface.*

Proof. Let φ be a closed RSDF. Then, $\alpha(s)$ and $\mathbf{e}(s)$ are closed curves. From the equality $\alpha^*(s)=\alpha(s)+(c-s)\,\mathbf{T}(s)$, $R(s)=c-s$ distance function is never closed. □

We can find the vectors $\mathbf{e}^*$ and ${\mathbf{e}_s}^*$ in terms of Darboux frame $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$. $\mathbf{e}^*$ can be written as

$$\mathbf{e}^*=\mathbf{T}\sin\phi^*+\mathbf{g}\cos\phi^*\cos\psi-\mathbf{n}\cos\phi^*\sin\psi, \tag{3.4}$$

${\mathbf{e}_s}^*$ provides the following equation

$$\begin{aligned} {\mathbf{e}_s}^* = \mathbf{T}\cos\phi^*&\left({\phi^*}'-\kappa_g\cos\psi+\kappa_n\sin\psi\right) \\ +\mathbf{g}&\left[\sin\phi^*\left(\kappa_g-{\phi^*}'\cos\psi\right)+\cos\phi^*\sin\psi\left(\tau_g-\psi'\right)\right] \\ +\mathbf{n}&\left[\sin\phi^*\left(\kappa_n+{\phi^*}'\sin\psi\right)+\cos\phi^*\cos\psi\left(\tau_g-\psi'\right)\right]. \end{aligned} \tag{3.5}$$

We can write the striction curve of the noncylindrical RSDF φ^* as follows:

$$\begin{aligned} c^*(s) &= \alpha(s)+(c-s)\,\mathbf{T}(s)-(c-s) \\ &\times\frac{\begin{pmatrix} \cos\phi^*\left(\tau_g-\psi'\right)\left(\kappa_g\sin\psi+\kappa_n\cos\psi\right) \\ +\sin\phi^*\left[{\kappa_g}^2+{\kappa_n}^2+{\phi^*}'\left(\kappa_n\sin\psi-\kappa_g\cos\psi\right)\right] \end{pmatrix}}{\begin{pmatrix} \sin 2\phi^*\left(\tau_g-\psi'\right)\left(\kappa_g\sin\psi+\kappa_n\cos\psi\right) \\ +{\phi^*}'\left[{\phi^*}'+2\left(\kappa_n\sin\psi-\kappa_g\cos\psi\right)\right]+\sin^2\phi^*\left({\kappa_g}^2+{\kappa_n}^2\right) \\ +\cos^2\phi^*\left[\left(\tau_g-\psi'\right)^2+\left(\kappa_g\cos\psi-\kappa_n\sin\psi\right)^2\right] \end{pmatrix}}e^*(s) \end{aligned} \tag{3.6}$$

where ${\phi^*}'=\frac{d\phi}{ds}$, $\psi'=\frac{d\psi}{ds}.(s)$

We can write the distribution parameter of the noncylindrical RSDF φ^* as follows:

$$P_{e^*}=\frac{\det\left({\alpha_s}^*,\mathbf{e}^*,{\mathbf{e}_s}^*\right)}{\|{\mathbf{e}_s}^*\|^2}=\frac{\left\langle (c-s)\,\kappa_g\mathbf{g}+(c-s)\,\kappa_n\mathbf{n}\times\mathbf{e},{\mathbf{e}_\mathbf{s}}^*\right\rangle}{\|{\mathbf{e}_\mathbf{s}}^*\|^2} \tag{3.7}$$

or

$$P_{e^*} = (c-s)\frac{\begin{pmatrix} \cos\phi^*\sin\phi^*\left(\tau_g-\psi'\right)\left(\kappa_n\sin\psi-\kappa_g\cos\psi\right) \\ -\left(\phi^*\right)'\left(\kappa_n\cos\psi+\kappa_g\sin\psi\right) \\ +\cos^2\phi^*\left(\kappa_g\sin\psi+\kappa_n\cos\psi\right)\left(\kappa_g\cos\psi-\kappa_n\sin\psi\right) \end{pmatrix}}{\begin{pmatrix} \sin 2\phi^*\left(\tau_g-\psi'\right)\left(\kappa_g\sin\psi+\kappa_n\cos\psi\right) \\ +\phi^{*\prime}\left[\phi^{*\prime}+2\left(\kappa_n\sin\psi-\kappa_g\cos\psi\right)\right]+\sin^2\phi^*\left({\kappa_g}^2+{\kappa_n}^2\right) \\ +\cos^2\phi^*\left[\left(\tau_g-\psi'\right)^2+\left(\kappa_g\cos\psi-\kappa_n\sin\psi\right)^2\right] \end{pmatrix}}. \tag{3.8}$$

where $\phi^{*\prime} = \frac{d\phi}{ds}$, $\psi' = \frac{d\psi}{ds}$.

Corollary 1. *Let (φ, φ^*) be a pair of the involute-evolute RSDF. In the case of $\phi^* = 90°$, φ^* ruled surface is developable.*

We can find the orthogonal trajectories of RSDF φ^* and the distribution parameters of the ruled surfaces $\varphi^*_{\mathbf{T}^*}, \varphi^*_{\mathbf{g}^*}$ and $\varphi^*_{\mathbf{n}^*}$. From the equation (2.3), we can write the orthogonal trajectories of φ^* as follows:

$$\begin{aligned} &\left\langle \mathbf{e}^*, {\alpha_s}^* \right\rangle ds = -dv, \\ &(c-s)\cos\phi^*\left(\kappa_g\cos\psi-\kappa_n\sin\psi\right) ds = -dv. \end{aligned} \tag{3.9}$$

and from the equation (2.2), we can write the distribution parameters of the ruled surfaces $\varphi^*_{\mathbf{T}^*}, \varphi^*_{\mathbf{g}^*}$ and $\varphi^*_{\mathbf{n}^*}$ as follows:

$$\begin{aligned} P_{T^*} &= \tfrac{\det\left({\alpha_s}^*, \mathbf{T}^*, {\mathbf{T}_s}^*\right)}{\left\|{\mathbf{T}_s}^*\right\|^2} = 0, \\ P_{g^*} &= \tfrac{\det\left({\alpha_s}^*, \mathbf{g}^*, {\mathbf{g}_s}^*\right)}{\left\|{\mathbf{g}_s}^*\right\|^2} = \tfrac{\det\left((c-s)\kappa_g\mathbf{g}+(c-s)\kappa_n\mathbf{n}, \mathbf{T}, \kappa_g\mathbf{g}+\kappa_n\mathbf{n}\right)}{\left({\kappa_g}^2+{\kappa_n}^2\right)} = 0, \\ P_{n^*} &= \tfrac{\det\left({\alpha_s}^*, \mathbf{n}^*, {\mathbf{n}_s}^*\right)}{\left\|{\mathbf{n}_s}^*\right\|^2} = -\tfrac{(c-s)\left(\kappa_g\sin\psi+\kappa_n\cos\psi\right)\left(\kappa_g\cos\psi-\kappa_n\sin\psi\right)}{\left(\dot{\psi}-\tau_g\right)^2+\left(\kappa_g\sin\psi+\kappa_n\cos\psi\right)^2}. \end{aligned} \tag{3.10}$$

Corollary 2. *$\varphi^*_{\mathbf{g}^*}$ ruled surface is developable.*

Corollary 3. *Let (φ, φ^*) be a pair of the oriented $(\psi = 0°)$ involute-evolute ruled surfaces. Then,*

$$P_{n^*} = \frac{\det\left({\alpha_s}^*, \mathbf{n}^*, {\mathbf{n}_s}^*\right)}{\left\|{\mathbf{n}_s}^*\right\|^2} = -\frac{(c-s)\,\kappa_g\kappa_n}{{\tau_g}^2+{\kappa_n}^2}. \tag{3.11}$$

Corollary 4. *Let (φ, φ^*) be a pair of the oriented $(\psi = 90°)$ involute-evolute ruled surfaces. Then,*

$$P_{n^*} = \frac{\det\left({\alpha_s}^*, \mathbf{n}^*, {\mathbf{n}_s}^*\right)}{\left\|{\mathbf{n}_s}^*\right\|^2} = \frac{(c-s)\,\kappa_g\kappa_n}{{\tau_g}^2+{\kappa_g}^2}. \tag{3.12}$$

We can find the relations between the instantaneous Pfaffian vectors of motions $H/H^{'}$ and $H^*/H^{*'}$, where $H = sp\left\{\mathbf{T}, \mathbf{g}, \mathbf{n}\right\}$ be the moving space along the base curve of $\varphi(s, v)$, $H^* = sp\left\{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\right\}$ be the moving space along the base curve of $\varphi^*(s,v)$, H' and $H^{*'}$ be fixed Euclidean spaces. Let $\varphi(s,v)$ with Darboux frame $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$ and $\varphi^*(s,v)$ with Darboux frame $\{\mathbf{T}^*,\mathbf{g}^*,\mathbf{n}^*\}$ be two ruled surfaces, which are the involute-evolute offsets.

Because of the equation (2.4), we get the Pfaffian forms (connection forms) of the system $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$

$$\left.\begin{array}{l}\omega_1=\tau_g\\ \omega_2=-\kappa_n\\ \omega_3=\kappa_g.\end{array}\right\}\tag{3.13}$$

For the instantaneous Pfaffian vector of motion H/H', we have

$$\boldsymbol{\omega}=\omega_1\mathbf{T}+\omega_2\mathbf{g}+\omega_3\mathbf{n}.\tag{3.14}$$

Similarly, we can give the following equations:

$$\begin{bmatrix}\mathbf{T}^{*\prime}\\ \mathbf{g}^{*\prime}\\ \mathbf{n}^{*\prime}\end{bmatrix}=\begin{bmatrix}0 & (\kappa_n\sin\psi-\kappa_g\cos\psi)\,s' & (\tau_g-\psi')\,s'\\ (\kappa_g\cos\psi-\kappa_n\sin\psi)\,s' & 0 & (\kappa_g\sin\psi+\kappa_n\cos\psi)\,s'\\ (\psi'-\tau_g)\,s' & -(\kappa_g\sin\psi+\kappa_n\cos\psi)\,s' & 0\end{bmatrix}\times\begin{bmatrix}\mathbf{T}^*\\ \mathbf{g}^*\\ \mathbf{n}^*\end{bmatrix}\tag{3.15}$$

where $\frac{ds}{ds^*}=s'$. Thus for the Pfaffian forms of the system $\{\mathbf{T}^*,\mathbf{g}^*,\mathbf{n}^*\}$, we get

$$\begin{array}{l}{\omega_1}^*=(\kappa_g\sin\psi+\kappa_n\cos\psi)\,s'=(\omega_3\sin\psi-\omega_2\cos\psi)\,s'\\ {\omega_2}^*=(\psi'-\tau_g)\,s'=(\psi'-\omega_1)\,s'\\ {\omega_3}^*=(\kappa_n\sin\psi-\kappa_g\cos\psi)\,s'=-(\omega_2\sin\psi+\omega_3\cos\psi)\,s'.\end{array}\tag{3.16}$$

For the instantaneous Pfaffian vector of motion $H^*/H^{*'}$, we have

$$\boldsymbol{\omega}^*=\left[(\omega_3\sin\psi-\omega_2\cos\psi)\,\mathbf{T}^*+(\psi'-\omega_1)\,\mathbf{g}^*-(\omega_2\sin\psi+\omega_3\cos\psi)\,\mathbf{n}^*\right]s'.\tag{3.17}$$

From the equation (3.2) we get

$$\boldsymbol{\omega}^*=\left(-\boldsymbol{\omega}+\psi'\mathbf{g}^*\right)s'.\tag{3.18}$$

Example 1. *We can write the hyperboloid of one sheet as*

$$\varphi(s,v)=\left(3\cos\frac{s}{3},3\sin\frac{s}{3},0\right)+v\frac{1}{5}\left(-3\sin\frac{s}{3},3\cos\frac{s}{3},4\right).\tag{3.19}$$

If we take $\phi^*=45°$ and $\psi=45°$, the involute offsets of this ruled surface are

$$\varphi^*(s,v)=\left(3\cos\frac{s}{3}-(c-s)\sin\frac{s}{3},3\sin\frac{s}{3}+(c-s)\cos\frac{s}{3},0\right)$$

$$+v\frac{1}{2}\left(-\sqrt{2}\sin\frac{s}{3}-\cos\frac{s}{3},\sqrt{2}\cos\frac{s}{3}-\sin\frac{s}{3},1\right). \tag{3.20}$$

We can see these surfaces in fig. 1, here we take $c = 3$ in the equation(3.20).

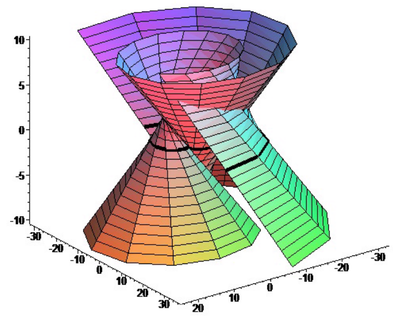

FIGURE 1. The involute offset of hyperboloid of one sheet and its base curve with $\phi^* = 45°$ and $\psi = 45°$

Example 2. *We can write the cylinder as*

$$\varphi(s,v) = (\cos s, \sin s, 0) + v(0,0,1). \tag{3.21}$$

If we take $\phi^* = 45°$ and $\psi = 45°$, the involute offsets of the cylinder are

$$\begin{aligned}\varphi^*(s,v) &= (\cos s-(c-s)\sin s, \sin s+(c-s)\cos s, 0)\\ &-\tfrac{v}{2}\left(\sqrt{2}\sin s+\cos s, -\sqrt{2}\cos s+\sin s, -1\right).\end{aligned} \tag{3.22}$$

We can see these surfaces in fig. 2, here we take $c = 3$ in the equation (3.22).

If we take $\phi^* = 30°$ and $\psi = 60°$, the involute offsets of the cylinder are

$$\begin{aligned}\varphi^*(s,v) &= (\cos s-(c-s)\sin s, \sin s+(c-s)\cos s, 0)\\ &-\tfrac{v}{2}\left(\sin s+\tfrac{3}{2}\cos s, -\cos s+\tfrac{3}{2}\sin s, -\tfrac{\sqrt{3}}{2}\right)\end{aligned} \tag{3.23}$$

We can see these surfaces in fig. 3, here we take $c = 3$ in the equation (3.23).

Theorem 4. *Let α and α^* be two curves in RSDF φ. α and α^* are the involute-evolute offsets if and only if α is a asymptotic curve.*

Proof. α and α^* are the involute-evolute offsets in φ, hence we can write $\alpha^*(s) = \alpha(s) + (c-s)\mathbf{T}(s)$. Since α^* is a curve in φ, then the normal of the ruled surface $\mathbf{n}(s)$ must be orthogonal to $\alpha^{*\prime}(s)$,

$$\left\langle \alpha^{*\prime}(s), \mathbf{n}(s)\right\rangle = \left\langle (c-s)\left[\kappa_g\mathbf{g}(s)+\kappa_n\mathbf{n}(s)\right], \mathbf{n}(s)\right\rangle = (c-s)\kappa_n = 0.$$

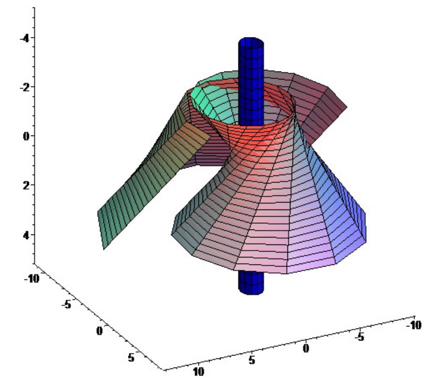

FIGURE 2. Cylinder (colored with blue) and the involute offset of the cylinder with $\phi^* = 45°$ and $\psi = 45°$

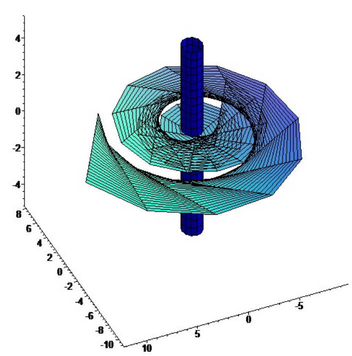

FIGURE 3. Cylinder (colored with blue) and the involute offset of the cylinder with $\phi^* = 30°$ and $\psi = 60°$.

If the distance function $R(s) = c - s$ is equal to zero, then α and α^* are the same curves. For this reason, κ_n is equal to zero and so α is a asymptotic curve.
Conversely, α is a asymptotic curve. In this case, we can write $\mathbf{T}(s)' = \kappa_g \mathbf{g}\,(s)$. The following equation must be available to α and α^* be the involute-evolute offsets

$$\alpha^*\,(s) = \alpha\,(s) + (c - s)\,\mathbf{T}(s).$$

We can find the vector $\alpha^{*\prime}\,(s)$ from this equation as $\alpha^{*\prime}\,(s) = (c - s)\,\kappa_g \mathbf{g}\,(s)$. The normal of the ruled surface $\mathbf{n}\,(s)$ is orthogonal to $\alpha^{*\prime}\,(s)$. So α and α^* are the involute-evolute offsets in RSDF φ. $\square$

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