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WRAPPED FLEXIBLE SKEW LAPLACE DISTRIBUTION

Abdullah YILMAZ *
Kirikkale University, Dept. of Actuarial Sciences, TURKEY

Abstract: We introduce a new circular distribution named as wrapped flexible skew Laplace distribution. This distribution is the generalization of wrapped Laplace which was introduced by Jammalamadaka and Kozubowski 2003 and has more flexibility properties in terms of skewness, kurtosis, unimodality or bimodality. We also derive expressions for characteristic function, trigonometric moments, coefficients of skewness and kurtosis. We analyzed two popular datasets from the literature to show the good modeling ability of the WFSL distribution.

Key words: Circular distribution; flexible skew laplace distribution; wrapped distribution; laplace distribution; skew-symmetric distribution
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1. Introduction

Circular or directional data is encountered in various fields of science such as meteorology, astronomy, medicine, biology, geology, physics and sociology. The first studies on the modeling of directional data are very old. The book "Statistics for Circular Data" written by Mardia 1972 can be regarded as the first work in this area. Other important works on this subject can be listed as "Statistical Analysis of Circular Data" [4], "Directional statistics" [10], "Topics in Circular Statistics" [7]. In the following years, many authors have proposed models and statistical methods for the analysis of circular data.

The von Mises distribution, also known as the circular normal or the Tikhonov distribution, is one of the principal symmetric distributions on the circle. However, most of the classical models such as Von-Mises, cardioid and wrapped Cauchy are symmetric-unimodal distributions and rarely applied in practice, since circular data is very often asymmetric and multimodal. Therefore, several new unimodal/multimodal circular distributions are capable modeling symmetry as well as asymmetry has been proposed, for example asymmetric Laplace distribution [5], nonnegative trigonometric sums distribution [3], asymmetric version of the von Mises distribution [13] and stereographic extreme–value distribution [12].

In recent years, studies on obtaining circular models have generally focused on wrapping linear probability models on a circle. In the literature, there are many wrapped models obtained by various well-known linear distributions. Pewsey 2000 obtained the wrapped skew normal distribution by using the Azzalini's skew normal distribution 1985. Jammalamadaka and Kozlowski 2004 studied the circular distributions obtained by exponential and Laplace distributions. Rao et al 2007 derived new circular models by wrapping the lognormal, logistic, Weibull, and extreme-value distributions.

In a previous paper [14] we introduced the flexible skew Laplace (FSL) distribution. This distribution is a member of skew-symmetric distribution family, and that means it has a pdf form that $h(x) = 2f(x)F(g(x))$ where f and F are the pdf and cdf of Laplace distribution and

$$g(x) = \left(\lambda_1 x + \lambda_3 x^3\right)\left(1 + \lambda_2 x^2\right)^{-\frac{1}{2}}, \quad \lambda_1, \lambda_3 \in \mathbb{R}, \lambda_2 \geq 0. \tag{1.1}$$

* Corresponding author. E-mail address: a.yilmaz@kku.edu.tr

We showed that, this distribution has remarkable flexibility properties in data modelling via contained parameters such as unimodality-bimodality, skewness or kurtosis. In this paper, the wrapped version of the flexible skew Laplace distribution will be presented.

2. Definition

A well-known approach to obtain circular distributions is wrapping method. In this approach, a known distribution is taken on the real line and wrapped around a unit circle. Namely, taking a real random variable (say Y) and wrapping it around the circle by transformation $Y(mod\ 2\pi)$. The new random variable $Y(mod\ 2\pi)$ can be named as the corresponding wrapped version of Y and has a probability density function (pdf) form that

$$f_{Y(mod\ 2\pi)}(\theta) = \sum_{r=-\infty}^{\infty} f_Y(\theta + 2\pi r),$$

where f_Y is the pdf of random variable Y.

Let Y be a $FSL(\mu, \sigma, \lambda_1, \lambda_2, \lambda_3)$ random variable, i.e. has a pdf

$$f_Y(y; \underline{\upsilon}) = \frac{1}{2\sigma} e^{-\frac{|y-\mu|}{\sigma}} \left[1 + \operatorname{sgn}\left(\frac{\lambda_1 (y-\mu) + \frac{\lambda_3}{\sigma^2}(y-\mu)^3}{\sqrt{\sigma^2 + \lambda_2 (y-\mu)^2}} \right) \left(1 - e^{-\left| \frac{\lambda_1 (y-\mu) + \frac{\lambda_3}{\sigma^2}(y-\mu)^3}{\left(\sigma^2 + \lambda_2 (y-\mu)^2\right)^{0.5}} \right|} \right) \right],$$

where $\underline{\upsilon} = (\mu, \sigma, \lambda_1, \lambda_2, \lambda_3)$. Then the corresponding circular random variable is defined as

$$\Theta = Y(mod\ 2\pi),$$

and has the density

$$f_\Theta(\theta; \underline{\upsilon}) = \frac{1}{2\sigma} \left[e^{-\frac{|\theta-\mu|}{\sigma}} + \frac{e^{\frac{\theta-\mu}{\sigma}} + e^{\frac{\mu-\theta}{\sigma}}}{e^{\frac{2\pi}{\sigma}} - 1} + A(\theta, \underline{\upsilon}) \right] \tag{2.1}$$

where

$$A(\theta, \underline{\upsilon}) = \sum_{r=-\infty}^{\infty} e^{-\frac{\left|\theta_\mu^r\right|}{\sigma}} \operatorname{sgn} g\left(\frac{\theta_\mu^r}{\sigma}\right) \left(1 - e^{-\left| g\left(\frac{\theta_\mu^r}{\sigma}\right) \right|} \right),$$

and $0 \le \theta < 2\pi$, $\theta_\mu^r = \theta + 2\pi r - \mu$. The parameters $\mu \in \mathbb{R}$ location, $\sigma > 0$ scale parameter and $\lambda_1, \lambda_3 \in \mathbb{R}$, $\lambda_2 \ge 0$ are shape parameters. The random variable Θ having wrapped flexible skew Laplace distribution is denoted by $\Theta \sim WFSL(\mu, \sigma, \lambda_1, \lambda_2, \lambda_3)$. Illustrations of the pdf of WFSL distribution for several values of parameters are shown in Figure 1.

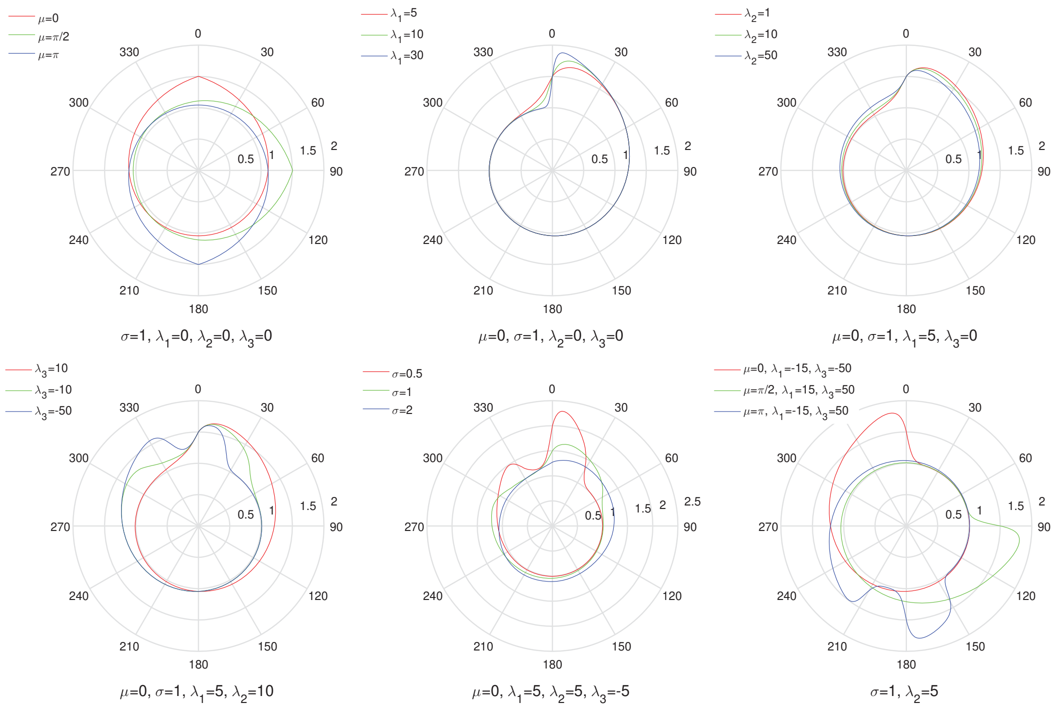

Figure 1. The pdf of WFSL distribution for several values of parameters.

The following sections of this article are organized as follows: In Section 3 we give the characteristic function of wrapped flexible skew Laplace distribution and some moments properties, i.e. location, dispersion, skewness and kurtosis. We also provide some results of limiting cases of parameters, and a simulation study in this section. In last section we will analyze two popular datasets from the literature.

3. Basic Properties

In this section, we obtain the equations for characteristic function, trigonometric moments, location, dispersion and coefficients of skewness and kurtosis. We also provide some properties and relations with other known distributions.

3.1. Trigonometric Moments

The characteristic function defines the entire probability distribution in the circular models as well as in the models defined on the real line. Note that, since the random variables with such distributions are periodic, have the same distribution when shifted by 2π. So if we consider $\Theta \overset{dist}{=} \Theta + 2\pi$, it must be

$$\varphi_\Theta(p) = E(e^{ip\Theta}) = E(e^{ip(\Theta+2\pi)}) = e^{ip2\pi}\varphi_\Theta(p).$$

Hence p must be an integer. The value of the characteristic function at an integer p is called the pth trigonometric moment of Θ. One can also write pth trigonometric moments in terms of α_p and β_p

$$\varphi_p = \varphi_\Theta(p) = \alpha_p + i\beta_p, \quad p = 0, \pm 1, \pm 2, \ldots.$$

where α_p is pth cosine moment and defined as $\alpha_p = E(\cos p\Theta)$, β_p is pth sine moment and defined as $\beta_p = E(\sin p\Theta)$. In order to obtain pth cosine and sine moments of $WFSL(\mu,\sigma,\lambda_1,\lambda_2,\lambda_3)$ distribution, we define two signum functions

$$\nabla = \begin{cases} \operatorname{sgn}(\lambda_1) & ,\text{if } \lambda_1\lambda_3 < 0 \\ 0 & ,\text{if } \lambda_1\lambda_3 \geq 0 \end{cases} \text{ and } \Delta = \begin{cases} 0 & ,\text{if } \lambda_1 = 0 \text{ and } \lambda_3 = 0 \\ \operatorname{sgn}(\lambda_3) & ,\text{if } \lambda_1 = 0 \text{ and } \lambda_3 \neq 0 \\ \operatorname{sgn}(\lambda_1) & ,\text{if } \lambda_1 \neq 0 \end{cases},$$

and quantities

$$C_p = \frac{1}{2\sigma}\int_0^{2\pi} \cos p\theta \sum_{r=-\infty}^{\infty} e^{-\left|\theta_\mu^r\sigma^{-1}\right|} \operatorname{sgn} g\left(\theta_\mu^r\sigma^{-1}\right) e^{-\left|g\left(\theta_\mu^r\sigma^{-1}\right)\right|} d\theta$$

and

$$S_p = \frac{1}{2\sigma}\int_0^{2\pi} \sin p\theta \sum_{r=-\infty}^{\infty} e^{-\left|\theta_\mu^r\sigma^{-1}\right|} \operatorname{sgn} g\left(\theta_\mu^r\sigma^{-1}\right) e^{-\left|g\left(\theta_\mu^r\sigma^{-1}\right)\right|} d\theta.$$

It follows that the pth cosine and sine moments are

$$\alpha_p = \frac{\cos p\mu + 2\nabla e^{-k}\sin(p\mu)\,\xi_p}{p^2\sigma^2+1} - \frac{\Delta p\sigma\sin p\mu}{p^2\sigma^2+1} - \Delta^2 C_p, \tag{3.1}$$

$$\beta_p = \frac{\sin p\mu - 2\nabla e^{-k}\cos(p\mu)\,\xi_p}{p^2\sigma^2+1} + \frac{\Delta p\sigma\cos p\mu}{p^2\sigma^2+1} - \Delta^2 S_p, \tag{3.2}$$

where

$$\xi_p = \sin kp\sigma + p\sigma\cos kp\sigma,$$

and

$$k = \begin{cases} \left(-\lambda_1\lambda_3^{-1}\right)^{0.5} & ,\text{if } \lambda_1\lambda_3 < 0 \\ 0 & ,\text{if } \lambda_1\lambda_3 \geq 0 \end{cases}.$$

Using these trigonometric values, an alternative representation for the density of Θ can be written as

$$f_\Theta(\theta;\underline{\upsilon}) = \frac{1}{2\pi} - \frac{1}{\pi}\sum_{p=1}^{\infty}\left\{ \begin{array}{c} \left(\frac{2\nabla e^{-k}\sin p(\theta-\mu)\xi_p}{p^2\sigma^2+1}\right) \\ -\frac{\cos p(\theta-\mu)+\Delta p\sigma\sin p(\theta-\mu)}{p^2\sigma^2+1} \\ +\Delta^2\left(C_p\cos p\theta + S_p\sin p\theta\right) \end{array}\right\}.$$

Thus, the first two trigonometric moments of $WFSL(0,1,\lambda_1,\lambda_2,\lambda_3)$ are

$$\varphi_1 = \frac{1+i\Delta}{2} - \nabla i e^{-k}\left[\cos k + \sin k\right] - i\Delta^2 S_1,$$

$$\varphi_2 = \frac{1+2i\Delta}{5} - \frac{2}{5}\nabla i e^{-k}\left[2\cos 2k + \sin 2k\right] - i\Delta^2 S_2.$$

Since the clear analytical form of both C_p and S_p cannot be found, they need to be evaluated numerically. However, the following two lemmas provide the values of C_p and S_p in some special cases of parameters.

Lemma 1. *When $\mu = 0$ for each integer p,*
(a) $C_p = 0$.
(b) $S_p = 0$ when $\Delta = 0$.

PROOF. (a) When $\Delta = 0$, it is immediate $g\left(\theta_0^r \sigma^{-1}\right) = 0$ and thus $C_p = 0$. When $\Delta \neq 0$, denote

$$\Lambda(\theta) = \sum_{r=-\infty}^{\infty} e^{-\left|\frac{\theta-\pi+2\pi r}{\sigma}\right|} \operatorname{sgn} g\left(\theta_0^r \sigma^{-1}\right) e^{-\left|g\left(\theta_0^r \sigma^{-1}\right)\right|}.$$

It's easy to see $\Lambda(\theta)$ is an odd function, i.e. $\Lambda(\theta) + \Lambda(-\theta) = 0$, $\theta \in (-\pi, \pi)$. One can rewrite

$$\begin{aligned} C_p &= \frac{1}{2\sigma} \int_0^{2\pi} \cos p\theta \sum_{r=-\infty}^{\infty} e^{-\left|\frac{\theta+2\pi r}{\sigma}\right|} \operatorname{sgn} g\left(\theta_0^r \sigma^{-1}\right) e^{-\left|g\left(\theta_0^r \sigma^{-1}\right)\right|} d\theta \\ &= \frac{(-1)^p}{2\sigma} \int_{-\pi}^{\pi} \Lambda(\theta) \cos p\theta d\theta = 0. \end{aligned}$$

(b) Proof is clear since $g\left(\theta_0^r \sigma^{-1}\right) = 0$ for $\Delta = 0$.

LEMMA 2. *For each integer p,*
(a) When $\lambda_1 \to \infty$ or $\lambda_3 \to \infty$, $C_p = 0$ and $S_p = 0$.
(b) When $\nabla = 0$ and $\lambda_2 \to \infty$,

$$C_p = -\frac{\Delta p \sigma \sin p\mu}{p^2 \sigma^2 + 1} \text{ and } S_p = \frac{\Delta p \sigma \cos p\mu}{p^2 \sigma^2 + 1}.$$

PROOF. (a) Let's just consider $\lambda_3 \to \infty$

$$\begin{aligned} \lim_{\lambda_3 \to \infty} C_p &= \lim_{\lambda_3 \to \infty} \frac{1}{2\sigma} \int_0^{2\pi} \cos p\theta \sum_{r=-\infty}^{\infty} e^{-\left|\theta_\mu^r \sigma^{-1}\right|} \operatorname{sgn} g\left(\theta_\mu^r \sigma^{-1}\right) e^{-\left|g\left(\theta_\mu^r \sigma^{-1}\right)\right|} d\theta \\ &= \frac{1}{2\sigma} \int_0^{2\pi} \cos p\theta \sum_{r=-\infty}^{\infty} \left[e^{-\left|\theta_\mu^r \sigma^{-1}\right|} \left\{ \lim_{\lambda_3 \to \infty} \operatorname{sgn} g\left(\theta_\mu^r \sigma^{-1}\right) e^{-\left|g\left(\theta_\mu^r \sigma^{-1}\right)\right|} \right\} \right] d\theta \\ &= 0. \end{aligned}$$

The situation is the same for $\lambda_1 \to \infty$ or $\lambda_1 \to \infty$, $\lambda_3 \to \infty$ and the proof is similar for S_p.

(b) Just consider C_p since the proof is similar for S_p. While $\lambda_2 \to \infty$, $e^{-\left|g\left(\theta_\mu^r \sigma^{-1}\right)\right|}$ tends to 1. Thus,

$$\begin{aligned} \lim_{\lambda_2 \to \infty} C_p &= \frac{1}{2\sigma} \int_0^{2\pi} \cos p\theta \lim_{\lambda_2 \to \infty} \sum_{r=-\infty}^{\infty} e^{-\left|\theta_\mu^r \sigma^{-1}\right|} \operatorname{sgn} g\left(\theta_\mu^r \sigma^{-1}\right) e^{-\left|g\left(\theta_\mu^r \sigma^{-1}\right)\right|} d\theta \\ &= \frac{1}{2\sigma} \left[\int_0^{2\pi} \cos p\theta \lim_{\lambda_2 \to \infty} \sum_{r=-\infty}^{\infty} e^{-\left|\theta_\mu^r \sigma^{-1}\right|} \operatorname{sgn} g\left(\theta_\mu^r \sigma^{-1}\right) \right] \\ &= -\left(\Delta p \sigma \sin p\mu\right) \left(p^2 \sigma^2 + 1\right)^{-1}. \end{aligned}$$

3.2. Location and Dispersion

Resultant vector length and direction for pth trigonometric moment of a circular distribution are

$$\rho_p = \sqrt{\alpha_p^2 + \beta_p^2} \text{ and } \mu_p = \operatorname{atan}\left(\alpha_p \beta_p^{-1}\right) \tag{3.3}$$

respectively, where atan(.) is quadrant inverse tangent function and defined as

$$\operatorname{atan}(y/x) = \begin{cases} \tan^{-1}(x/y) & , y>0, x\geq 0 \\ \pi/2 & , y=0, x>0 \\ \tan^{-1}(x/y)+\pi & , y<0 \\ \tan^{-1}(x/y)+2\pi & , y\geq 0, x<0 \\ \text{undefined} & , y=0, x=0 \end{cases}.$$

The pth trigonometric moment can be expressed in $\varphi_p = \rho_p e^{i\mu_p}$ and has a special meaning for $p=1$. The values of ρ_1 and μ_1 obtained from (3.3) are called the angular concentration and the mean direction, respectively. Mean direction of $WFSL(0,\sigma,\lambda_1,\lambda_2,\lambda_3)$ distribution is

$$\begin{aligned} \mu_1 &= \operatorname{atan}\left[\left(\sigma\Delta - \Delta^2 S_1\left(1+\sigma^2\right) - 2\nabla e^{-k}\left[\sin k\sigma + \sigma\cos k\sigma\right]\right)^{-1}\right] \\ &= \operatorname{atan}\left[\left(\sigma\Delta - \Delta^2 S_1\left(1+\sigma^2\right) - 2\nabla e^{-k}\xi_1\right)^{-1}\right]. \end{aligned} \tag{3.4}$$

The mean direction vector gives information about the mean of the distribution as an analogy of the mean in the linear models. The length of this vector is a measure of its dispersion around the mean and corresponds to the usual standard deviation or variance in linear models. Square of angular concentration for $WFSL$ distribution is

$$\begin{aligned} \rho_1^2 = & \frac{\nabla^2}{\varsigma}\xi_1^2\left(4e^{-2k}\right) + \frac{\nabla}{\varsigma}\xi_1\left(4\Delta e^{-k}\right)\left(\Delta S_1 - \sigma + \sigma^2\Delta S_1\right) \\ & + \frac{1}{\varsigma}\left(\Delta S_1\left(\sigma^2+1\right)\left(\Delta S_1 - 2\sigma + \sigma^2\Delta S_1\right) + \sigma^2\Delta^2 + 1\right), \end{aligned}$$

or with value of μ_1

$$\begin{aligned} \rho_1 &= -\Delta^2 S_1 \sin\mu_1 + \frac{\Delta\sigma\sin\mu_1}{\sigma^2+1} - \frac{2\nabla e^{-k}\xi_1\sin\mu_1}{\sigma^2+1} + \frac{\cos\mu_1}{\sigma^2+1} \\ &= \left[\Delta\frac{\sigma}{\sigma^2+1} - \Delta^2 S_1 - \nabla\frac{2e^{-k}\xi_1}{\sigma^2+1}\right]\sin\mu_1 + \frac{\cos\mu_1}{\sigma^2+1}, \end{aligned} \tag{3.5}$$

where $\varsigma = \sigma^4 + 2\sigma^2 + 1$.

COROLLARY 1. *When $\mu = 0$ and $\Delta = 0$, $\varphi_p = \left(p^2\sigma^2+1\right)^{-1} e^{ip\mu}$, for each integer p. Hence, $\mu_1 = \mu$ and $\rho_1 = \left(\sigma^2+1\right)^{-1}$ for $WFSL(\mu,\sigma,0,\lambda_2,0)$ distribution.*

COROLLARY 2. *When $\lambda_1 \to \infty$ or $\lambda_3 \to \infty$, μ_1 tends to $[\mu + \operatorname{atan}(\sigma)] \bmod (2\pi)$ and ρ_1 tends to $\left(\sigma^2+1\right)^{-0.5}$ for $WFSL(\mu,\sigma,\lambda_1,\lambda_2,\lambda_3)$ distribution.*

COROLLARY 3. *When $\lambda_2 \to \infty$, μ_1 tends to μ and ρ_1 tends to $\left(\sigma^2+1\right)^{-1}$ for $WFSL(\mu,\sigma,\lambda_1,\lambda_2,\lambda_3)$ distribution.*

It is clear that $0 \leq \rho_1 \leq 1$ and tends to maximum value when the concentration increases around the mean. The effect of μ and σ parameters on the angular concentration and the mean direction of the Θ random variable are shown in Figure 2. If Θ is rotated by θ_0 degrees, the value of angular concentration does not change but mean direction is shifted by θ_0.

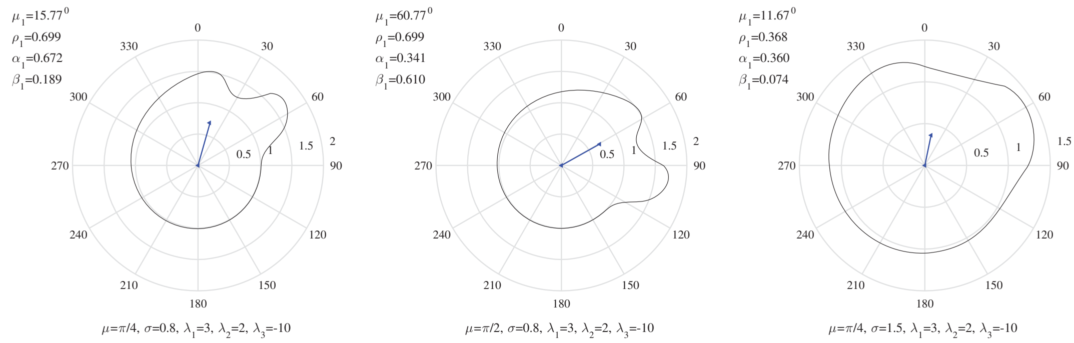

Figure 2. Amount of concentration and average direction change.

Another circular dispersion measure is the circular variance and defined as $V = 1 - \rho_1$. Using (3.4) and (3.5), the circular variance of $WFSL\left(0, \sigma, \lambda_1, \lambda_2, \lambda_3\right)$ is

$$\begin{aligned} V &= 1 - \beta_1 \sin \mu_1 - \alpha_1 \cos \mu_1 \\ &= 1 + S_1 \sin \mu_1 \Delta^2 - \frac{\Delta \sigma \sin \mu_1}{\sigma^2 + 1} + \frac{2 \nabla e^{-k} \xi_1 \sin \mu_1}{\sigma^2 + 1} - \frac{\cos \mu_1}{\sigma^2 + 1}. \end{aligned} \tag{3.6}$$

Circular variance is interpreted as the opposite of angular concentration. That is, the circular variance decreases while the concentration around the mean direction increases and vice versa. In Figure 3, it can be seen (σ, V) plots for different $\lambda_1, \lambda_2, \lambda_3$ values $(\mu = 0)$. In generally, according to Figure 3 it can be said that the circular variance increases with the increase of σ. When $\lambda_1 \lambda_3 \geq 0$, the circular variance decreases with increasing λ_1 or λ_3 and increases with λ_2. But this is only valid for some values of σ when $\lambda_1 \lambda_3 < 0$.

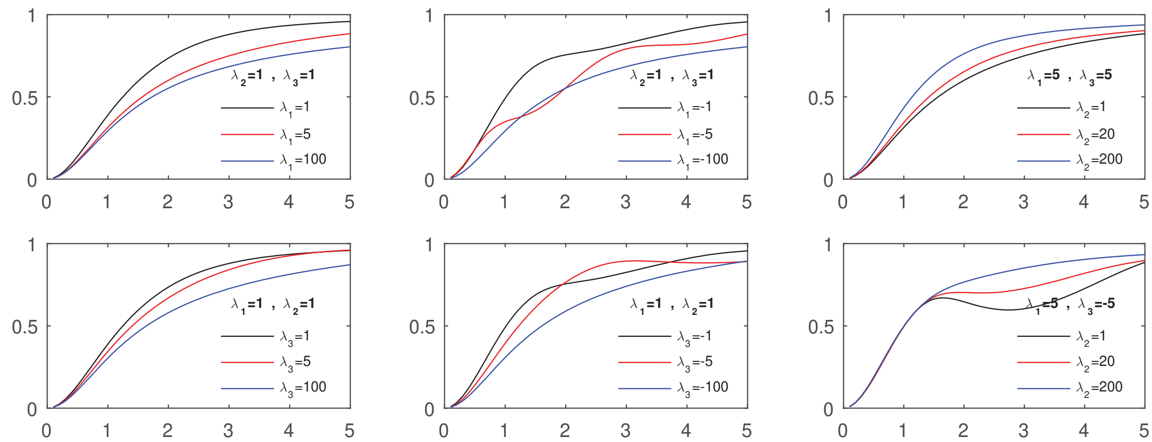

Figure 3. (σ, V) plots for different $\lambda_1, \lambda_2, \lambda_3$ values $(\mu = 0)$.

3.3. Skewness and Kurtosis

In a circular model, the pth central cosine moment and sine moments are $\bar{\alpha}_p = E\left[\cos p\left(\theta - \mu_1\right)\right]$ and $\bar{\beta}_p = E\left[\sin p\left(\theta - \mu_1\right)\right]$ respectively. Both of coefficients kurtosis and skewness are obtained using the second central moment and aren't affected by parameter μ. So, it is enough to calculate both coefficients according to $\mu = 0$. Thus,

$$\bar{\alpha}_2 = E\left[\cos 2\left(\theta - \mu_1\right)\right] \tag{3.7}$$

$$= \left[\Delta \frac{2\sigma}{4\sigma^2+1} - \Delta^2 S_2 - \nabla \frac{2e^{-k}\xi_2}{4\sigma^2+1} \right] \sin 2\mu_1 + \frac{\cos 2\mu_1}{4\sigma^2+1}$$

and

$$\begin{aligned} \bar{\beta}_2 &= E\left[\sin 2\left(\theta - \mu_1\right)\right] \\ &= \left[\Delta \frac{2\sigma}{4\sigma^2+1} - \Delta^2 S_2 - \nabla \frac{2e^{-k}\xi_2}{4\sigma^2+1} \right] \cos 2\mu_1 - \frac{\sin 2\mu_1}{4\sigma^2+1}. \end{aligned} \tag{3.8}$$

As a measure of asymmetry, skewness coefficient of a circular distribution is calculated by $\gamma_1 = \bar{\beta}_2 V^{-3/2}$ [9]. Using the values of (3.6) and (3.8) the skewness of $WFSL$ distribution is

$$\gamma_1 = \frac{\left(\Delta 2\sigma - \Delta^2 S_2 \left(4\sigma^2+1\right) - 2\nabla e^{-k}\xi_2\right)\cos 2\mu_1 - \sin 2\mu_1}{\left(4\sigma^2+1\right) V^{3/2}}.$$

If the distribution is symmetric and unimodal, the skewness coefficient will be zero.

Corollary 4. *$WFSL\left(\mu,\sigma,0,\lambda_2,0\right)$ is unimodal and symmetric about μ. From Corollary 2 mean direction is $\mu_1 = 0$, when $\mu = \lambda_1 = \lambda_3 = 0$, since $\Delta = \nabla = 0$ and $\mu_1 = 0$, $\gamma_1 = 0$.*

Kurtosis of a circular distribution is $\gamma_2 = \left(\bar{\alpha}_2 - \rho_1^4\right)\left(1-\rho_1\right)^{-2}$ [9]. Using the given values (3.5), (3.6) and (3.7) kurtosis of $WFSL\left(0,\sigma,\lambda_1,\lambda_2,\lambda_3\right)$ is

$$\gamma_2 = \frac{\left(\Delta 2\sigma - \Delta^2 S_2 \left(4\sigma^2+1\right) - 2\nabla e^{-k}\xi_2\right)\sin 2\mu_1 + \cos 2\mu_1 - \left(4\sigma^2+1\right)\rho_1^4}{\left(4\sigma^2+1\right) V^2}.$$

Corollary 5. *Kurtosis of $WFSL\left(\mu,\sigma,0,\lambda_2,0\right)$ is $\gamma_2 = \left(\sigma^4+4\sigma^2+6\right)\left(4\sigma^2+1\right)^{-1}\left(\sigma^2+1\right)^{-2}$.*

3.4. Some Propositions

In this section, we provide some properties related to the introduced distribution and the relationships with other distributions.

Proposition 1. *$WFSL\left(0,\sigma,0,\lambda_2,0\right) \overset{dist}{=} WL\left(\sigma^{-1},1\right)$ where $WL\left(\lambda,\kappa\right)$ denotes the wrapped Laplace distribution [5].*

Proof. Let denote φ_p^{WFSL} and φ_p^{WL} are characteristic functions of $WFSL\left(0,\sigma,0,\lambda_2,0\right)$ and $WL\left(\sigma^{-1},1\right)$ respectively. When $\lambda_1 = 0$, $\lambda_3 = 0$ and $\mu = 0$, it's easy to see for each integer p, $\varphi_p^{WFSL} = \varphi_p^{WL} = \left(p^2\sigma^2+1\right)^{-1}$.

Proposition 2. *Let $WE\left(\lambda\right)$ denotes the wrapped exponential distribution and $WL\left(\lambda\right)$ denotes the wrapped Laplace distribution [5].*

(a) $\lim_{\lambda_1\to\infty} WFSL\left(0,\sigma,\lambda_1,\lambda_2,\lambda_3\right) \overset{dist}{=} WE\left(\sigma^{-1}\right)$.
(b) $\lim_{\lambda_3\to\infty} WFSL\left(0,\sigma,\lambda_1,\lambda_2,\lambda_3\right) \overset{dist}{=} WE\left(\sigma^{-1}\right)$.
(c) $\lim_{\lambda^*\to\infty} WFSL\left(0,\sigma,\lambda^*,\lambda_2,\lambda^*\right) \overset{dist}{=} WE\left(\sigma^{-1}\right)$.
(d) $\lim_{\lambda_2\to\infty} WFSL\left(0,\sigma,\lambda_1,\lambda_2,\lambda_3\right) \overset{dist}{=} WL\left(\sigma^{-1},1\right)$.

Proof. (a) The characteristic function of $WE\left(\sigma^{-1}\right)$ distribution is

$$\varphi_p^{WE} = \left(1+ip\sigma\right)\left(p^2\sigma^2+1\right)^{-1}.$$

Using Lemma 3a pth cosine moment of $WFSL\left(0,\sigma,\lambda_1,\lambda_2,\lambda_3\right)$ is

$$\alpha_p = \left(p^2\sigma^2+1\right)^{-1},$$

when $\lambda_1 \to \infty$. If $\lambda_3 < 0$, e^{-k} tends to 0 otherwise $\nabla = 0$. So in both cases pth sine moment will be equal to

$$\beta_p = p\sigma \left(p^2\sigma^2 + 1\right)^{-1}.$$

Thus

$$\varphi_p^{WFSL} = \varphi_p^{WE} = (1 + ip\sigma)\left(p^2\sigma^2 + 1\right)^{-1}.$$

(b) In a similar way, when λ_3 tends to ∞, α_p tends to

$$\alpha_p = \left(p^2\sigma^2 + 1\right)^{-1}.$$

If $\lambda_1 < 0$, e^{-k} tends to 1, $\nabla = \Delta = -1$ and $\xi_p \to p\sigma$. Otherwise if $\lambda_1 > 0$, ∇ equals to 0. So in both cases

$$\beta_p = p\sigma \left(p^2\sigma^2 + 1\right)^{-1}.$$

(c) Proof is clear, since $\nabla = 0$ when λ_1 and λ_3 tends to ∞.

(d) When $\lambda_2 \to \infty$, from Lemma 3b and (3.1) it can be seen immediately that $\alpha_p = \left(p^2\sigma^2 + 1\right)^{-1}$ and

$$\begin{aligned}\beta_p &= -\frac{p\sigma\Delta(1-\Delta)(1+\Delta)}{p^2\sigma^2 + 1} \\ &= 0,\end{aligned}$$

for all cases of Δ. Thus

$$\varphi_p^{WFSL} = \alpha_p = \left(p^2\sigma^2 + 1\right)^{-1} = \varphi_p^{WL}.$$

Proposition 3. $\Theta \sim WFSL(0, \sigma, \lambda_1, \lambda_2, \lambda_3) \Leftrightarrow -\Theta \sim WFSL(0, \sigma, -\lambda_1, \lambda_2, -\lambda_3)$.

Proof. Let's use $g_{\lambda_1,\lambda_2,\lambda_3}(x)$ notation instead of $g(x)$ notation in equation (1.1). It's easy to see

$$g_{-\lambda_1,\lambda_2,-\lambda_3}(x) = g_{\lambda_1,\lambda_2,\lambda_3}(-x) = -g_{\lambda_1,\lambda_2,\lambda_3}(x).$$

Thus,

$$A(\theta, \mu, \sigma, -\lambda_1, \lambda_2, -\lambda_3) = A(-\theta, \mu, \sigma, \lambda_1, \lambda_2, \lambda_3)$$

and

$$f_\Theta(\theta; 0, \sigma, -\lambda_1, \lambda_2, -\lambda_3) = f_\Theta(-\theta; 0, \sigma, \lambda_1, \lambda_2, \lambda_3).$$

Corollary 6. *Mean direction of* $-\Theta \sim WFSL(0, \sigma, -\lambda_1, \lambda_2, -\lambda_3)$ *is* $2\pi - \mu_1$, *where* μ_1 *is the mean direction of* $\Theta \sim WFSL(0, \sigma, \lambda_1, \lambda_2, \lambda_3)$.

Proposition 4. *Let* X *and* Y *be independent Laplace* (η) *random variables. Define the random variable* Θ *as*

$$\Theta = \{X | Y < g(X)\} \, (\mathrm{mod}\, 2\pi)$$

where $g(.)$ *defined as* (1.1). *Then* $\Theta \sim WFSL(0, \eta, \lambda_1, \lambda_2, \lambda_3)$.

Proof. Proof is clear since $X | Y < g(X) \sim FSL(\eta, \lambda_1, \lambda_2, \lambda_3)$ [14].

Corollary 7. $\{-X | Y > g(X)\} \, (\mathrm{mod}\, 2\pi) \sim WFSL(0, \eta, \lambda_1, \lambda_2, \lambda_3)$.

Corollary 8. $\Theta = [IX - (1 - I)X] \, (\mathrm{mod}\, 2\pi) \sim WFSL(0, \eta, \lambda_1, \lambda_2, \lambda_3)$ *where*

$$I = \begin{cases} 1, & Y < g(X) \\ 0, & Y \geq g(X) \end{cases}.$$

3.5. Simulation

The result of last proposition can be used to generate random numbers from $WFSL$ distribution. The following algorithm is based on this result and generates n random numbers.

Step 1. Generate $n \times 1$ random vectors $X \sim Laplace\,(1)$ and $Y \sim Laplace\,(1)$.
Step 2. Calculate $g\,(X) = (\lambda_1 X + \lambda_3 X^3)\,(1 + \lambda_2 X^2)^{-0.5}$.
Step 3. Calculate $T = [Y < g\,(X)]\,X - [Y \geq g\,(X)]\,X$.
Step 4. Calculate $Z = [\mu + \sigma T]\,(\mathrm{mod}\,2\pi)$.

Obtaining clear forms of maximum likelihood (ml) estimators is an insurmountable problem because of complex likelihood function. Therefore, likelihood function must be maximized by an iterative method. In this simulation study we used Matlab's mle function to obtain ml estimates of parameters. We ran the above algorithm 100 times for different values of n and $\mu = 0.79$, $\sigma = 1.5$, $\lambda_1 = 3$, $\lambda_2 = 3$ and $\lambda_3 = -5$. The bias and MSE(in parentheses) values of the parameters calculated with the ml estimates obtained in each step, are shown in Table 1.

Table 1. Average values of bias and MSE (in parentheses) of parameters.

n	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$
100	0.1187(0.0113)	0.0067(0.0388)	0.4875(3.0552)	0.2682(11.491)	-1.3720(19.436)
250	0.0006(0.0032)	0.0029(0.0200)	0.2574(1.0001)	0.7384(7.7087)	-0.5679(4.9759)
500	0.0070(0.0017)	-0.0134(0.0113)	0.0164(0.3399)	0.2394(5.7541)	-0.0891(1.6226)
1000	0.0034(0.0009)	-0.0030(0.0049)	0.0105(0.1427)	0.0440(1.6380)	-0.0269(0.7009)

From Table 1 it can be seen that, as the sample size increases, the bias and MSE values of parameters decrease to zero.

4. Application to Real Data

In order to demonstrate the modelling behavior of the $WFSL$ distribution, we will analyze two popular data sets from the literature. Both data sets in this section have been discussed in many studies and used for fitting the distributions proposed by the authors. Table 2 shows estimates of the parameters, estimated average direction and resultant length. We also provide maximized log likelihood value (L), Akaike information criterion (AIC), Bayesian information criterion (BIC) and Watson's U^2 (W^2) value in the same table.

Table 2. Summary of fits for the turtle data and ant data.

Turtle Data				Ant Data			
Estimates				Estimates			
$\widehat{\mu}$	1.7104	-L	107.7552	$\widehat{\mu}$	3.67	-L	128.936
$\widehat{\sigma}$	1.1761	AIC	225.5104	$\widehat{\sigma}$	0.92	AIC	267.8722
$\widehat{\lambda}_1$	-3.029	BIC	237.1641	$\widehat{\lambda}_1$	-2033	BIC	280.8980
$\widehat{\lambda}_2$	0.0127	W^2	0.0375	$\widehat{\lambda}_2$	$2.86E6$	W^2	0.2050
$\widehat{\lambda}_3$	1.1393			$\widehat{\lambda}_3$	280.5		
Mean Direction			$1.09(\sim 62.6^{\circ})$	Mean Direction			$3.15(\sim 180.63^{\circ})$
Res. Lenght			0.5146	Res. Lenght			0.6209

Turtle Data: The first dataset in this section refers to the orientations of 76 turtles after laying eggs [7]. Left panel of Figure 4 represents the circular data plot over rose diagram and fitted $WFSL$ distribution. The arrow and its length represents the sample mean resultant vector m_1 and resultant length r_1, respectively. Calculated sample statistics are $a_1 = 0.2166$, $b_1 = 0.4474$, $m_1 = 1.12$ ($\sim 64.2^{\circ}$) and $r_1 = 0.4971$. Maximum likelihood estimation of parameters are obtained by maximizing the likelihood function in Matlab via mle function. In order to avoid localmaxima, parameter intervals have been kept as wide as possible. The maximum likelihood estimates are seen

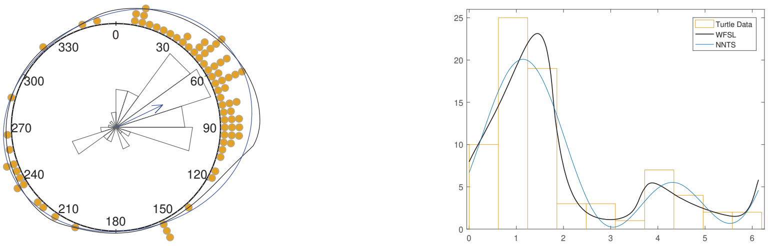

Figure 4. Plots for turtle data. Circular data plot, fitted circular pdf and rose diagram (left), linear histogram and fitted pdf (right).

This dataset was recently used by Joshi and Jose 2018 as an application of the wrapped Lindley ($\mathcal{WL}$) distribution. The authors reported the AIC value for $\mathcal{WL}$ distribution is 243.29, BIC value is 243.75 and maximized log likelihood value is 119.71. In the same study, the AIC value for the wrapped exponential distribution is 243.29, the BIC is 245.63 and maximized log likelihood value is 120.65. Yilmaz and Biçer 2018 modeled this data set using the transmuted version of wrapped exponential (TWE) distribution and they obtained the AIC, W^2 and maximized log likelihood value values as 239.89, 0.25 and 117.95, respectively. Also, Fernandez-Duran 2004 used this data set as an application for non-negative trigonometric sums ($NNTS$) distribution and obtained the lowest AIC value is 225.94. Based on the AIC, BIC, maximized log likelihood and W^2 statistics values reported by these authors, the $WFSL$ distribution gives better fit to turtle data than the mentioned alternatives.

Ant Data: The second data set consist of the directions chosen by 100 ants which have been analyzed by Fisher 1995 with the aim of fitting a von Mises distribution. Ants were placed into an arena one by one, and the directions they chose relative to an evenly illuminated black light source placed at 180° were recorded. Calculated sample statistics are $a_1 = -0.6091$, $b_1 = -0.0334$, $m_1 = 3.1964$ ($\sim 183.14°$) and $r_1 = 0.6101$. The maximum likelihood estimates obtained via Matlab are seen in Table 2

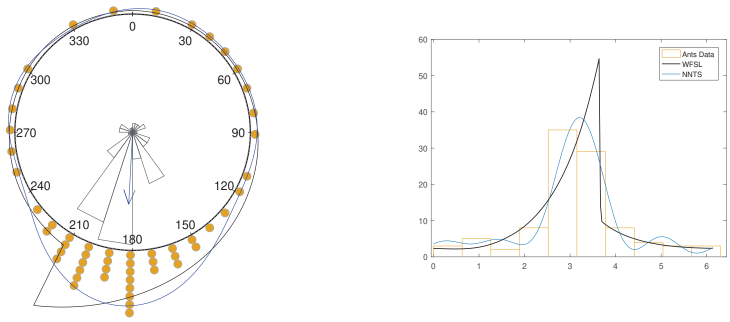

Figure 5. Plots for ant data. Circular data plot, fitted circular pdf and rose diagram (left), linear histogram and fitted pdf (right).

Fisher concludes that the von Mises distribution is not a suitable model for this data. The AIC and BIC values for this model was equal to 288.24 and 293.4, respectively. Fernandez-Duran 2004 reported the AIC values as 276.64 for the $NNTS$ distribution, 276.84 for the skewed wrapped Laplace and 275.74 for the symmetric wrapped Laplace distribution. Based on the AIC and BIC values reported by these authors, the $WFSL$ distribution gives better fit to ant data than the mentioned alternatives.

5. Conclusion

In this study, wrapped version of the flexible skew Laplace (FSL) distribution is introduced. The proposed distribution inherits the flexibility properties of FSL distribution. We also discussed characteristic function, trigonometric moments, location, dispersion and coefficients of skewness and kurtosis of proposed distribution. As we mentioned about, it is not possible to find explicit forms of maximum likelihood (ML) estimators of parameters. However, as can also be seen from many studies in recent years, this problem can be overcome with the help of computer softwares. Therefore, in last section, the mle function of Matlab used for obtaining the estimation of the parameters. Based on the AIC, BIC, maximized log likelihood and W^2 statistics values, the results showed that the proposed model is better fits to these datasets than the recently published wrapped Lindley distribution [8], transmuted wrapped exponential distribution [15] and non-negative trigonometric sums distribution [3].

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