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STATISTICAL CONVERGENCE OF WEIGHT g IN A LOCALLY SOLID RIESZ SPACE

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Abstract. In this work, we introduce the notions of statistical convergence and lacunary statistical convergence of weight g in a locally solid Riesz space and establish some inclusion relations.

1. Introduction

The Riesz space was first introduced by F. Riesz in 1928, at the International Mathematical Congress in Bologna [1]. Soon after, in the mid-thirties, H. Freudental [2] and L. V. Kantrovich [3] independently set up the axiomatic foundation and derived a number of properties dealing with the lattice structure of Riesz space. Riesz space have many applications in measure theory, operator theory and optimization. They also have some applications in economics [4], we may refer to [5, 6, 7, 8].

Recently, Balcerzak et al. [9] show that one can further extend the concept of natural or asymptotic density (as well as natural density of order α) by considering natural density of weight g where $g : N \to [0, \infty]$ is a function with $lim_{n\to\infty} g(n) = \infty$ and $\frac{n}{g(n)}$ does not go to 0 as $n \to \infty$ (Throughout the paper by N, R and C, we will denote the set of all natural, real and complex numbers, respectively). We denote by G, the set of all such functions g.

In this work, we introduce the notions of statistical convergence and lacunary statistical convergence with respect to weight g in locally solid Riesz space and establish some inclusion relations.

2. Definitions and Preliminaries

Let $E \subseteq N$. Then the natural density of E is denoted by $\delta(E)$ and defined by

$$\delta(E) = \lim_{n\to\infty} |\{k \in E : k \leq n\}|,$$

where the vertical bars denote the cardinality of the respective set [10].

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A sequence $x = (x_k)$ of real numbers is said to be statistically convergent to x_0 if for arbitrary $\epsilon > 0$, the set $A(\epsilon) = \{n \in N : |x_n - x_0| \geq \epsilon\}$ has natural density [9].

Let $g : N \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. The upper density of weight g was defined by the formula

$$\bar{d}_g(A) = \limsup_{n\to\infty} \frac{A(1,n)}{g(n)}$$

for $A \subseteq N$ where $A(1,n)$ denotes the cardinality of the set $A \cap [1,n]$. If the $\lim_{n\to\infty} A(1,n)/g(n)$ exists then we say that the density of weight g of the set A exists and we denote it by $d_g(A)$.

Let X be a real vector space and " $\leq$ " be a partial order on this space if it satisfies the following properties:

(1) $\forall x, y \in X$ and $y \leq x$ imply $y + z \leq x + z$ for each $z \in X$,
(2) $\forall x, y \in X$ and $y \leq x$ imply $\alpha y \leq \alpha x$ for each $\alpha \geq 0$.

In addition, if X is lattice with respect to the partial order, then X is said to be a Riesz space (or a vector lattice) [7].

A subset S of a Riesz space X is said to be solid if $y \in S$ and $|x| \leq |y|$ implies $x \in X$.

A topological vector space (X, τ) is a vector space X which has a linear topology τ such that the algebraic operations of additions and scalar multiplication in X are continuous.

Every linear topology τ on a vector space X has a base $\mathcal{N}_{sol}$ for the neighbourhoods of zero satisfying the following properties:

(1) Each $Y \in \mathcal{N}_{sol}$ is a balanced set, that is, $\alpha x \in Y$ holds for all $x \in Y$ and every $\alpha \in R$ with $|\alpha| \leq 1$.
(2) Each $Y \in \mathcal{N}_{sol}$ is an absorbing set, that is, for every $x \in X$ there exists $\alpha > 0$ such that $\alpha x \in Y$.
(3) For each $Y \in \mathcal{N}_{sol}$, there exists some $W \in Y$ with $W + W \subseteq Y$.

A linear topology τ on a Riesz space X is said to be locally solid Riesz space if τ has a base at zero consisting of solid sets. A LSRS (X, τ) is a Riesz space equipped with a locally solid topology τ.

Throughout the paper, the symbol $\mathcal{N}_{sol}$ will denote any base at zero consisting of solid sets and satisfying the above conditions (1), (2), (3) in a locally solid Riesz topology τ. For abbreviation, here and in where follows, we shall write a word "LSRS" instead of a locally solid Riesz space and we mean $\lim_{k\to\infty} x_k$ by $\lim x$ for brevity.

Let (X, τ) be a locally solid Riesz space. A sequence $x = (x_k)$ in X is said to be $S(\tau)$-convergent to an element x_0 in X if for each τ- neighbourhood V of zero

$$\delta\left(\{k \in N : x_k - x_0 \notin V\}\right) = 0$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : x_k - x_0 \notin V\}\right| = 0.$$

In this case, we write $S(\tau)\text{-}\lim x = x_0$ or $(x_k) \overset{S(\tau)}{\to} x_0$ [8].

By a lacunary sequence, we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout the paper, the intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} will be abbreviated by q_r.

Let θ be a lacunary sequence and (X,τ) be a LSRS. Then a sequence $x = (x_k)$ in X is said to be lacunary statistically τ-convergent to the element $x_0 \in X$ if for every τ-neighbourhood V of zero, $\delta(K)$, where $K = \{k \in N : x_k - x_0 \notin V\}$, i.e.,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : x_k - x_0 \notin V\}| = 0.$$

In this case we write $S_\theta(\tau)$-$\lim x = x_0$ or $S_\theta(\tau)$-$\lim x_k = x_0$ or $x_k \overset{S_\theta(\tau)}{\to} x_0$ [10].

3. Main Results

Definition 3.1. Let (X,τ) be a locally solid Riesz space and (x_n) be a sequence in X. We say that (x_n) is statistically τ-convergent of weight g to $x_0 \in X$ or $S_g(\tau)$-convergent to x_0 provided that for every $\tau-$neighbourhood U of zero,

$$\lim_{n\to\infty} \frac{1}{g\,(n)} |\{k \le n : x_k - x_0 \notin U\}| = 0$$

holds. We denote this by $S_g(\tau)$-$\lim x_n = x_0$ or ($x_k \overset{S_g(\tau)}{\to} x_0$ briefly). The class of all sequences which are statistically τ-convergent of weight g will be denoted by $S_g(\tau)$.

Remark 3.2. For $g(n) = n^\alpha$ and $X = R$ the definition given above reduces to statistical convergence of order α [11].

Definition 3.3. Let (X,τ) be a locally solid Riesz space. We say that $x = (x_k)$ in X is said to be $S_g\,(\tau)$-bounded if for every neighbourhood V of zero, there exists some $\alpha > 0$ such that,

$$\lim_{n\to\infty} \frac{1}{g\,(n)} |\{k \le n : x_k \notin V\}| = 0.$$

Theorem 3.4. Let (X,τ) be a Hausdorff LSRS and $x = (x_k)$, $y = (y_k)$ be two sequences in X. Then the followings hold:

(1) If $S_g\,(\tau) - \lim x_k = x_0$ and $S_g\,(\tau) - \lim x_k = y_0$, then $x_0 = y_0$.
(2) If $S_g\,(\tau) - \lim x_k = x_0$, then $S_g\,(\tau) - \lim \alpha x_k = \alpha x_0$, for every $\alpha \in R$,
(3) If $S_g\,(\tau) - \lim x_k = x_0$ and $S_g\,(\tau) - \lim y_k = y_0$, then $S_g\,(\tau) - \lim x_k + y_k = x_0 + y_0$.

Proof. (1) Suppose that $S_g\,(\tau) - \lim x_k = x_0$ and $S_g\,(\tau) - \lim x_k = y_0$. Let V be any τ-neighbourhood of zero. Then there exists a $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq V$. Choose $W \in \mathcal{N}_{sol}$ such that $W + W \subseteq Y$. Since $S_g\,(\tau) - \lim x_k = x_0$ and $S_g\,(\tau) - \lim x_k = y_0$, then we have $d_g\,(G_1) = d_g\,(G_2) = 1$ where

$$G_1 = |\{k \le n : x_k - x_0 \in W\}|\,,$$
$$G_2 = |\{k \le n : x_k - y_0 \in W\}|\,.$$

Now let $G = G_1 \cap G_2$. Then we have

$$x_0 - y_0 = x_0 - x_k + x_k - y_0 \in W + W \subseteq Y \subseteq V$$

for every $k \in G$. Hence for every τ-neighbourhood V of zero we have $x_0 - y_0 \in V$. Since (X,τ) is Hausdorff, the intersection of all τ-neighbourhood V of zero is the singleton set $\{\theta\}$. Thus we get $x_0 - y_0 = \theta$, i.e., $x_0 = y_0$.

(2) Let $S_g(\tau)-\lim x_k = x_0$ and let V be an arbitrary $\tau-$neighbourhood of zero. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq V$. Since $S_g(\tau) - \lim x_k = x_0$, we have

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : x_k - x_0 \in Y\}| = 1.$$

Let $|\alpha| \leq 1$. Since Y is balanced, $x_n - x_0 \in Y$ implies that $\alpha(x_n - x_0) \in Y$ for every $\alpha \in R$ with $|\alpha| \leq 1$. Hence we have

$$\begin{aligned}\{k \leq n : x_k - x_0 \in Y\} &\subseteq \{k \leq n : \alpha(x_n - x_0) \in Y\} \\ &\subseteq \{k \leq n : \alpha(x_k - x_0) \in V\}.\end{aligned}$$

Thus we get

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : \alpha(x_k - x_0) \in Y\}| = 1,$$

for each τ-neighbourhood V of zero. Now let $|\alpha| > 1$ and $[\![\alpha]\!]$ be the smallest integer greater than or equal to α. There exists a $W \in \mathcal{N}_{sol}$ such that $[\![\alpha]\!] W \subseteq Y$. Since $S_g(\tau) - \lim x_n = x_0$, we have $d(K) = 1$, where

$$K = |\{k \leq n : x_k - x_0 \in W\}|.$$

Then we have

$$|\alpha x_n - \alpha y_0| = |\alpha| |x_n - x_0| \leq [\![\alpha]\!] |x_n - x_0| \in W \subseteq Y \subseteq V$$

for each $n \in K$. Since the set V is solid, we have $\alpha x_n - \alpha x_0 \in Y$ and so $\alpha x_n - \alpha x_0 \in V$ for each $n \in K$. Thus we get

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : x_k - x_0 \in V\}| = 1$$

for each τ-neighbourhood V of zero. Hence $S_g(\tau) - \lim \alpha x_k = \alpha x_0$ for every $\alpha \in R$.

(3) Let V be an arbitrary $\tau-$neighbourhood of zero. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq V$. We choose $W \in \mathcal{N}_{sol}$ such that $W + W \subseteq Y$. Since $S_g(\tau) - \lim x_n = \alpha x_0$ and $S^I_{(\bar{N},\theta)}(\tau) - \lim y = y_0$, we have $d(B_1) = d(B_2) = 1$ where

$$B_1 = \{k \leq n : x_k - x_0 \in W\}$$

and

$$B_2 = \{k \leq n : x_k - x_0 \in W\}.$$

Now let $B = B_1 \cap B_2$. Hence we have $d(B) = 1$ and

$$(x_n + y_n) - (x_0 - y_0) = (x_n - x_0) + (y_n - y_0) \in W + W \subseteq Y \subseteq V$$

for each $n \in B$. Thus we get

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : (x_n - x_0) + (y_n - y_0) \in V\}| = 1.$$

Since V is arbitrary, we have $S_g(\tau) - \lim x_k + y_k = x_0 + y_0$.

□

Theorem 3.5. Let (X, τ) be a locally solid Riesz space. If a sequence $x = (x_k)$ is $S_g(\tau)$-convergent, then $x = (x_k)$ is $S_g(\tau)$-bounded.

Proof. Let $S_g(\tau) - \lim x_k = x_0$. Let V be an arbitrary τ-neighbourhood of zero. Then there exists a $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq V$. Let us choose $W \in \mathcal{N}_{sol}$ such that $W + W \subseteq Y$. Since $x_k \overset{S_g(\tau)}{\to} x_0$, we have $d(K) = 0$, where

$$K = |\{k \leq n : x_k - x_0 \in V\}| .$$

Since W is absorbing, there exists a $a > 0$ such that $ax_0 \in W$. Let b be such that $b \leq 1$ and $b \leq a$. Since W is solid and $|bx_0| \leq |ax_0|$, we have $bx_0 \in W$. Since W is balanced, $x_k - x_0 \in W$ implies that $b(x_k - x_0) \in W$. Then we have

$$bx_n = b(x_n - x_0) + bx_0 \in W + W \subseteq Y \subseteq V$$

for each $n \in N \backslash K$ and thus we get

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : bx_k \notin V\}| = 0.$$

Consequently, (x_n) is $S_g(\tau)$-bounded. □

Theorem 3.6. Let (X, τ) be a locally solid Riesz space. If $(x_k), (y_k), (z_k)$ are sequences such that;

(1) $x_k \leq y_k \leq z_k$, for all $k \in N$,
(2) $S_g(\tau) - \lim x_k = x_0 = S_g(\tau) - \lim z_k$, then $S_g(\tau) - \lim x_k = x_0$.

Proof. Let V be an arbitrary $\tau-$neighbourhood of zero. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq V$. Choose $W \in \mathcal{N}_{sol}$ such that $W + W \subseteq Y$. From the condition (2), we have $S_g(\tau)(E_1) = 1 = S_g(\tau)(E_2)$ where

$$E_1 = \{k \leq n : x_k - x_0 \in W\}$$

and

$$E_2 = \{k \leq n : z_k - x_0 \in W\} .$$

Also, we get $S_g(\tau)(E_1 \cap E_2) = 1$ and from (1) we have

$$x_k - x_0 \leq y_k - x_0 \leq z_k - x_0$$

for all $k \in N$. This implies that for all $k \in E_1 \cap E_2$, we get

$$|y_k - x_0| \leq |z_k - x_0| + |x_k - x_0| \in W + W \subseteq Y \subseteq V.$$

Since Y is solid, we have $y_k - y_0 \in V$. Thus

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : y_k - x_0 \in V\}| = 1$$

for each τ-neighbourhood V of zero. Hence $S_g(\tau) - \lim x_k = x_0$. □

4. Lacunary Statistical Convergence of Weight g in a Locally Solid Riesz Space

In this section, we define lacunary statistical convergence of weight g in a locally solid Riesz space and we examine some inclusion relations.

Definition 4.1. Let θ be a lacunary sequence. A sequence $x = (x_k)$ is said to be lacunary statistically convergent of weight g to x_0 or $S_g^\theta(\tau)$-convergent to x_0 if for every τ-neighbourhood U of zero,

$$\lim_{n\to\infty} \frac{1}{g(h_r)} |\{k \in I_r : x_k - x_0 \notin U\}| = 0$$

holds. We denote this by $S_g^{\theta}(\tau)$-$\lim x_n = x_0$ (or $x_k \overset{S_g^{\theta}(\tau)}{\to} x_0$ briefly). The class of all sequences which are lacunary statistically τ-convergent of weight g will be denoted by $S_g^{\theta}(\tau)$.

Definition 4.2. Let (X,τ) be a locally solid Riesz space and θ be a lacunary sequence. We say that $x=(x_k)$ in X is said to be $S_g^{\theta}(\tau)$-bounded if for every neighbourhood V of zero, there exists some $\alpha > 0$ such that,

$$\lim_{r\to\infty} \frac{1}{g(h_r)} |\{k \le I_r : x_k \notin V\}| = 0$$

We leave the proofs of the following two theorems to the reader. They can be done in a similar manner as the proofs of Theorem 3.4 and Theorem 3.5.

Theorem 4.3. Let (X,τ) be a Hausdorff locally solid Riesz space and θ be a lacunary sequence and $x=(x_k)$, $y=(y_k)$ be two sequences in X. Then the followings hold:

(1) If $S_g^{\theta}(\tau) - \lim x_k = x_0$ and $S_g^{\theta}(\tau) - \lim x_k = y_0$, then $x_0 = y_0$.
(2) If $S_g^{\theta}(\tau) - \lim x_k = x_0$, then $S_g^{\theta}(\tau) - \lim \alpha x_k = \alpha x_0$, for every $\alpha \in R$,
(3) If $S_g^{\theta}(\tau) - \lim x_k = x_0$ and $S_g^{\theta}(\tau) - \lim y_k = y_0$, then $S_g^{\theta}(\tau) - \lim x_k + y_k = x_0 + y_0$.

Theorem 4.4. Let (X,τ) be a locally solid Riesz space and θ be a lacunary sequence. If a sequence $x=(x_k)$ is $S_g^{\theta}(\tau)$-convergent, then $x=(x_k)$ is $S_g^{\theta}(\tau)$-bounded.

Theorem 4.5. Let (X,τ) be a locally solid Riesz space and θ be a lacunary sequence. If $\liminf_r \frac{g(h_r)}{g(k_r)} > 1$, then $S_g(\tau) \subset S_g^{\theta}(\tau)$

Proof. Since $\liminf_r \frac{g(h_r)}{g(k_r)} > 1$, so we can find a $\delta > 0$ such that $\frac{g(h_r)}{g(k_r)} \ge 1+\delta$ for sufficiently large values of r. Assume that $x_k \overset{S_g(\tau)}{\to} x_0$, hence for every U neighbourhood of zero and for sufficiently large values of r we have

$$\begin{aligned} \tfrac{1}{g(k_r)} |\{k \le k_r : x_k - x_0 \notin U\}| &\ge \tfrac{1}{g(k_r)} |\{k \in I_r : x_k - x_0 \notin U\}| \\ &= \tfrac{g(h_r)}{g(k_r)} \tfrac{1}{g(h_r)} |\{k \in I_r : x_k - x_0 \notin U\}| \\ &\ge (1+\delta) \tfrac{1}{g(h_r)} |\{k \in I_r : x_k - x_0 \notin U\}| . \end{aligned}$$

Hence we have $x_k \overset{S_g^{\theta}(\tau)}{\to} x_0$ while taking limit as $n \to \infty$. □

5. Conclusion

In this work, we introduce the notions of statistical convergence and lacunary statistical convergence with respect to weight g in a locally solid Riesz space and establish some inclusion relations. One can see that if $\limsup_r \frac{g(h_r)}{g(k_r)} < \infty$ the inverse of the Theorem 4.5 holds.

References

[1] F. Riesz, Sur la decomposition des operations fonctionnelles lineaires, In Atti del Congresso Internazionale dei Matematici: Bologna del 3 al 10 de settembre di 1928, pp. 143-148 (1929).
[2] H. Freudenthal, Teilweise geordnete Moduln, K. Akademie van Wetenschappen, Afdeeling Natuurkunde, Proceedings of the Section of Sciences, 39, 647–657 (1936).
[3] L. V. Kantorovich, Concerning the general theory of operations in partially ordered spaces, Dok. Akad. Nauk. SSSR 1, 271-274 (1936).
[4] C. D. Aliprantis, O. Burkinshaw, Locally solid Riesz spaces with applications to economics, American Mathematical Society, Second Edition, USA, 105, (2003).
[5] L. V. Kantorovich, Lineare halbgeordnete Raume, Recueil Mathmatique, 2(44), 121-168 (1937).
[6] W. A. J. Luxemburg and A. C. Zaanen, Riesz spaces, American Elsevier Publishing Company, New York, (1971).
[7] A. C. Zannen, Introduction to Operator Theory in Riesz Spaces, Springer-Verlag, (1997).
[8] H. Albayrak and S. Pehlivan, Statistical convergence and statistical continuity on locally solid Riesz spaces, Topology and its Applications, 159(7), 1887-1893 (2012).
[9] M. Balcerzak, P. Das, M. Filipczak and J. Swaczyna Generalized kinds of density and the associated ideals, Acta Mathematica Hungarica, 147(1), 97-115 (2015).
[10] B. Hazarika, S. A. Mohiuddine and M. Mursaleen. Some inclusion results for lacunary statistical convergence in locally solid Riesz spaces, Iranian Journal of Science and Technology, 38(A1), 61-68 (2014).
[11] R. Çolak and Ç A. Bektaş. λ-statistical convergence of order α, Acta Mathematica Scientia 31(3), 953-959 (2011).

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