ON THE SIGMA INDEX OF THE CORONA PRODUCTS OF MONOGENIC SEMIGROUP GRAPHS

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Abstract. In [1], Das et.al. considered the monogenic semi group $S_M$ with zero having $\{0, x, x^2, ..., x^n\}$. Also they defined undirected graph $\Gamma(S_M)$ associated with $S_M$ whose vertices are the non-zero elements $x, x^2, ..., x^n$ and any two different vertices $x^i$ and $x^j$ are adjacent if $i + j > n$ (for $1 \leq i, j \leq n$).

In this paper we present sigma index of corona products of any two monogenic semigroup graphs $\Gamma(S^1_M)$ and $\Gamma(S^2_M)$. Also we give forgotten index and irregularity index of monogenic semigroup graphs.

1. Introduction

Each commutative ring $R$ can be described by means of a simple graph $\Gamma(R)$. There are many studies in the literature about zero-divisor graphs [1-7]. In a recently study Das et al. [8] the graph $\Gamma(S_M)$ is defined. The authors considered the finite multiplicative monogenic semi group (with zero) $S_M = \{0, x, x^2, ..., x^n\}$. The vertices of this graph obtained $S_M$ monogenic semi-group are the non-zero elements $\{x, x^2, ..., x^n\}$ and any two distinct vertices $x^i$ and $x^j$ are adjacent with the rule $x^i.x^j = 0$ if and only if $i + j \geq n + 1$ (1 $\leq i, j \leq n$).

Many graphs of general and in particular of chemical interest arise from simple graphs via various graph operations sometimes known as graph products. Hence, it is important to understand how certain invariants of such composite graphs related to the corresponding invariants of their components. Some more properties and applications of graph products can be seen in [11-15].

In [8], Das et al. studied the Cartesian product of monogenic semi-group graphs. In [9] Akgunes defined the Strong product of monogenic semi-group graphs and gave some properties as diameter, clique number, chromatic number etc.

In [10], Nacaroglu defined the corona product of monogenic semi-group graphs. Also he examined some graph parameter of this graphs.

The corona product of graphs $G_1$ and $G_2$, denoted $G_1 \circ G_2$, is the graph obtained by taking one copy $G_1$ and $n_1$ copies of $G_2$ by joining the $i$-th copy of $G_2$ for $1 \leq i \leq n_1$. The degree of a vertex of $G_1 \circ G_2$ is defined by

$$d_{G_1 \circ G_2}(u) = \begin{cases} d_{G_1}(u) + n_2, & \text{if } u \in V(G_1) \\ d_{G_2}(u) + 1, & \text{if } u \in V(G_2) \end{cases}$$

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Theorem 1.1. [8] Let $S_M$ be a monogenic semigroups. Then the degree sequence of $\Gamma(S_M)$ is given by

$$DS(\Gamma(S_M)) = \{1, 2, 3, ..., \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 2, n - 1\}.$$ 

Theorem 1.2. Let $G = \Gamma(S^1_M) \circ \Gamma(S^2_M)$. Then the degree sequence of $\Gamma(S^1_M) \circ \Gamma(S^2_M)$ is given by

$$DS(G) = \left\{\frac{m}{2}, \frac{m}{2} + 1, \frac{m}{2} + 2, \frac{m}{2} + 3, ..., m - 1, m, m + 1, m + 2, m + 3, ..., \frac{n}{2} - 1 + m, \frac{n}{2} + m, \frac{n}{2} + m + 1, \frac{n}{2} + m + 2, ..., n - m - 2, n + m - 1\right\}.$$

In this study we present some topological indices corona products of any two monogenic semigroup graphs as sigma index, irregularity index etc.

2. MAIN RESULTS

A graph $G$ is regular if all its vertices have the same degree, otherwise it is irregular. Albertson [16] defines the irregularity of $G$ as

$$irr(G) = \sum_{uv \in E(G)} |d_G(u) - d_G(v)|$$

, where $d_G(u)$ denotes the degree of a vertex $u \in V(G)$.

Theorem 2.1. Let $G = \Gamma(S_M)$. Then

$$irr(\Gamma(S_M)) = \begin{cases} \frac{n^3 - 4n}{12}, & n \text{ is even} \\ \frac{n^3 - n}{12}, & n \text{ is odd} \end{cases}$$
Proof. Assume that \( n \) is even:

\[
\text{irr}(\Gamma(S_M)) = (d_n - d_1) + (d_n - d_2) + \ldots + (d_n - d_{n-2}) + (d_n - d_{n-1}) + \\
\text{(which is written by } I_n: \text{ Let us say } J_n \text{ to this sum)} \\
+ (d_{n-1} - d_2) + (d_{n-1} - d_3) + \ldots + (d_{n-1} - d_{n-2}) + \\
\text{(which is written by } I_{n-1}: \text{ Let us say } J_{n-1} \text{ to this sum)} \\
+ \ldots + \\
+ (d_{n+2} - d_{n-1}) + (d_{n+2} - d_2) + (d_{n+2} - d_{n+1}) \\
\text{(which is written by } I_{n+2}: \text{ Let us say } J_{n+2} \text{ to this sum)} \\
(d_{n+1} - d_2) \\
\text{(which is written by } I_{n+1}: \text{ Let us say } J_{n+1} \text{ to this sum)}
\]

As a result, we get

\[
\text{irr}(\Gamma(S_M)) = \sum_{ij \in E(\Gamma S_M)} |d_i - d_j|
\]

\[
= \sum_{k=1}^{\frac{n}{2}} \sum_{p=1}^{\frac{n-2k}{2}} p + \sum_{k=1}^{\frac{n}{2}} k
\]

\[
= \frac{n^3 - 4n}{12}.
\]

Assume that \( n \) is odd: For \( n \) is odd, we have \( \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}. \) By taking following same steps as in \( n \) is even case, we obtain

\[
\text{irr}(\Gamma(S_M)) = \sum_{ij \in E(\Gamma S_M)} |d_i - d_j|
\]

\[
= \sum_{k=1}^{\frac{n-1}{2}} \sum_{p=1}^{\frac{n-2k}{2}} p + \sum_{k=1}^{\frac{n}{2}} k
\]

\[
= \frac{n^3 - n}{12}.
\]

Hence the result. \( \square \)

**Theorem 2.2.** Let \( G = \Gamma(S^1_M) \circ \Gamma(S^2_M) \). Then

\[
\text{irr}(\Gamma(G)) = \begin{cases} 
\frac{n^3 - 4n}{12} + \frac{m^3 - 4m}{12} + \frac{mn(n + m + 2)}{2}, & n, m \text{ even} \\
\frac{n^3 - n}{12} + \frac{m^3 - 4m}{12} + \frac{mn(n + m + 2)}{2}, & n \text{ odd, } m \text{ even} \\
\frac{n^3 - n}{12} + \frac{m^3 - m}{12} + \frac{mn(n + m + 2)}{2}, & n, m \text{ odd} \\
\frac{n^3 - 4n}{12} + \frac{m^3 - m}{12} + \frac{mn(n + m + 2)}{2}, & n \text{ even, } m \text{ odd}
\end{cases}
\]
Proof. From definition of graph irregularity we can write

\[ \text{irr}(G) = \sum_{u \neq v \in E(G)} |d_G(u) - d_G(v)| \]

\[ = \sum_{u \neq v \in E(S_{1M})} |d_G(u) - d_G(v)| + \sum_{i=1}^{n} \sum_{u \neq v \in E(F(S_{2M}^i))} |d_G(u) - d_G(v)| \]

(2.1) \[ + \sum_{u \in V(S_{1M}) \atop v \in V(S_{2M})} |d_G(u) - d_G(v)|. \]

By applying Theorem 2.1 and Theorem 1.2 we get the following results.

**Case 1**: \(n \) and \( m \) even: From (2.1) we have

\[ \text{irr}(G) = \sum_{u \neq v \in E(G)} |d_G(u) - d_G(v)| \]

\[ = \frac{n^3 - 4n}{12} + \frac{m^3 - 4m}{12} + mn(n + m + 2) \]

**Case 2**: \(n\) odd and \( m\) even: From (2.1) we have

\[ \text{irr}(G) = \sum_{u \neq v \in E(G)} |d_G(u) - d_G(v)| \]

\[ = \frac{n^3 - n}{12} + \frac{m^3 - 4m}{12} + mn(n + m + 2) \]

In a similar way, we obtain in other cases. \(\square\)

In [17] forgotten topological index \(F\) was defined as

\[ F(G) = \sum_{u \in V(G)} d_G^3(u). \]

**Theorem 2.3.** Let \( G = \Gamma(S_M) \). Then

\[ F(G) = (n - 1) \sum_{k=2}^{m} k^3 + \sum_{k=2}^{n-1+m} k^3 + n \left(\lfloor \frac{m}{2} \rfloor + 1\right)^3 + \left(\lfloor \frac{n}{2} \rfloor + m\right)^3. \]

Proof. By Theorem 1.2 we have

\[ F(G) = \sum_{u \in V(G)} d_G(u) \]

\[ = \sum_{k=2}^{m} k^3 + \sum_{k=m+1}^{n-1+m} k^3 + n \left(\lfloor \frac{m}{2} \rfloor + 1\right)^3 + \left(\lfloor \frac{n}{2} \rfloor + m\right)^3 \]

\[ = (n - 1) \sum_{k=2}^{m} k^3 + \sum_{k=2}^{n-1+m} k^3 + n \left(\lfloor \frac{m}{2} \rfloor + 1\right)^3 + \left(\lfloor \frac{n}{2} \rfloor + m\right)^3. \]
If $G$ is a graph and $d_G(u)$ the degree of its vertex $u$, then its sigma index [18,19] is defined as 

$$\sigma(G) = \sum_{uv \in E(G)} (d_G(u) - d_G(v))^2,$$

, with summation going over all pairs of adjacent vertices.

**Theorem 2.4.** Let $G = \Gamma(S_{3M}^1) \circ \Gamma(S_{3M}^2)$. Then

$$\sigma(G) = \begin{cases} 
\frac{n^4-n^3+2n^2+4n}{24} + \frac{m^4-m^3+2m^2+4m}{24} + \frac{m^2n-2m^2+mn^2-3mn+3m+4n-4}{2}, & n,m \text{ even} \\
\frac{n^4-n^3+2n^2+4n}{24} + \frac{m^4-m^3+2m^2+4m}{24} + \frac{(n-1)(n+m^2-2n-5)}{2}, & n,m \text{ odd, } m \text{ even} \\
\frac{n^4-n^3+2n^2+4n}{24} + \frac{m^4-m^3+2m^2+4m}{24} + \frac{(n-1)(n+m^2-2n-5)}{2}, & n \text{ even, } m \text{ odd}
\end{cases}$$

**Proof.** From definition of sigma index and Theorem 2.2 we have

$$\sigma(G) = \sum_{uv \in E(S_{3M}^1)} (d_G(u) - d_G(v))^2 + \sum_{i=1}^n \sum_{uv \in E(S_{3M}^2)} (d_G(u) - d_G(v))^2 + \sum_{uv \in V(S_{3M}^1)} (d_G(u) - d_G(v))^2$$

$$= \sum_{k=1}^{n-1} \sum_{p=1}^{n-2k} p^2 + \sum_{k=1}^{n-1} k^2 + n \left( \sum_{k=1}^{n-1} \sum_{p=1}^{n-2k} p^2 + \sum_{k=1}^{n-1} k^2 \right) + \sum_{k=1}^{n-1} \sum_{p=2}^{m} (n + m - k - p)$$

$$+ \sum_{k=1}^{n-1} (n + m - k - \left\lfloor \frac{m}{2} \right\rfloor + 1).$$

**Case1:** $n$ and $m$ are even.

$$\sigma(G) = \sum_{k=1}^{n-1} \sum_{p=1}^{n-2k} p^2 + \sum_{k=1}^{n-1} k^2 + n \left( \sum_{k=1}^{n-1} \sum_{p=1}^{n-2k} p^2 + \sum_{k=1}^{n-1} k^2 \right) + \sum_{k=1}^{n-1} \sum_{p=2}^{m} (n + m - k - p)$$

$$+ \frac{n^4-n^3+2n^2+4n}{24} + \frac{m^4-m^3+2m^2+4m}{24} + \frac{m^2n-2m^2+mn^2-3mn+3m+4n-4}{2}$$

**Case2:** $n$ is odd and $m$ is even.

$$\sigma(G) = \sum_{k=1}^{n-1} \sum_{p=1}^{n-2k} p^2 + \sum_{k=1}^{n-1} k^2 + n \left( \sum_{k=1}^{n-1} \sum_{p=1}^{n-2k} p^2 + \sum_{k=1}^{n-1} k^2 \right) + \sum_{k=1}^{n-1} \sum_{p=2}^{m} (n + m - k - p)$$

$$+ \frac{n^4-n^3-n^2+n}{24} + \frac{m^4-m^3+2m^2+4m}{24} + \frac{m^2n-2m^2+mn^2-3mn+3m+4n-4}{2}$$
Case 3: $n$ and $m$ are odd.

$$\sigma(G) = \sum_{k=1}^{n-1} \sum_{p=1}^{n-2k} p^2 + \sum_{k=1}^{n-1} k^2 + n \left( \sum_{k=1}^{m-1} \sum_{p=1}^{m-2k} p^2 + \sum_{k=1}^{m-1} k^2 \right) + \sum_{k=1}^{n-1} m \left( n + m - k - p \right)$$

$$+ \sum_{k=1}^{n-1} \left( n + m - k - \frac{m}{2} + 1 \right)$$

$$= \frac{n^4 - n^3 - n^2 + n}{24} + \frac{m^4 - m^3 - m^2 + m}{24}$$

$$+ \frac{(n - 1)(mn + m^2 - 2n - 5)}{2}.$$

Case 4: $n$ is even and $m$ is odd.

$$\sigma(G) = \sum_{k=1}^{n-1} \sum_{p=1}^{n-2k} p^2 + \sum_{k=1}^{n-1} k^2 + n \left( \sum_{k=1}^{m-1} \sum_{p=1}^{m-2k} p^2 + \sum_{k=1}^{m-1} k^2 \right) + \sum_{k=1}^{n-1} m \left( n + m - k - p \right)$$

$$+ \sum_{k=1}^{n-1} \left( n + m - k - \frac{m}{2} + 1 \right)$$

$$= \frac{n^4 - n^3 + 2n^2 + 4n}{24} + \frac{m^4 - m^3 - m^2 + m}{24}$$

$$+ \frac{(n - 1)(mn + m^2 - 2n - 5)}{2}.$$

Hence the result. □

References


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