Journal of Universal Mathematics
Vol.2 No.1 pp.68-74 (2019)
ISSN-2618-5660

ON THE SIGMA INDEX OF THE CORONA PRODUCTS OF MONOGENIC SEMIGROUP GRAPHS

YAŞAR NACAROĞLU AND NİHAT AKGÜNEŞ

Abstract. In [1], Das et.al. considered the monogenic semi group S_M with zero having $\{0, x, x^2, ..., x^n\}$. Also they defined undirected graph $\Gamma(S_M)$ associated with S_M whose vertices are the non-zero elements $x, x^2, ..., x^n$ and any two different vertices x^i and x^j are adjacent if $i + j > n$ (for $1 \leq i, j \leq n$).

In this paper we present sigma index of corona products of any two monogenic semigroup graphs $\Gamma(S_M^1)$ and $\Gamma(S_M^2)$. Also we give forgotten index and irregularity index of monogenic semigroup graphs.

1. Introduction

Each commutative ring R can be described by means of a simple graph $\Gamma(R)$. There are many studies in the literature about zero-divisor graphs [1-7]. In a recently study Das et al. [8] the graph $\Gamma(S_M)$ is defined. The authors considered the finite multiplicative monogenic semi group (with zero) $S_M = \{0, x, x^2, ..., x^n\}$. The vertices of this graph obtained S_M monogenic semi-group are the non-zero elements $\{x, x^2, ..., x^n\}$ and any two distinct vertices x^i and x^j are adjacent with the rule $x^i.x^j = 0$ if and only if $i + j \geq n + 1$ $(1 \leq i, j \leq n)$.

Many graphs of general and in particular of chemical interest arise from simple graphs via various graph operations sometimes known as graph products. Hence, it is important to understand how certain invariants of such composite graphs related to the corresponding invariants of their components. Some more properties and applications of graph products can be seen in [11-15].

In [8], Das et al. studied the Cartesian product of monogenic semi-group graphs. In [9] Akgunes defined the Strong product of monogenic semi-group graphs and gave some properties as diameter,clique number, chromatic number etc.

In [10], Nacaroglu defined the corona product of monogenic semi-group graphs. Also he examined some graph parameter of this graphs.

The corona product of graphs G_1 and G_2, denoted $G_1 \circ G_2$, is the graph obtained by taking one copy G_1 and n_1 copies of G_2 by joining the $i-$th copy of G_2 for $1 \leq i \leq n_1$. The degree of a vertex of $G_1 \circ G_2$ is defined by

$$d_{G_1 \circ G_2}(u) = \begin{cases} d_{G_1}(u) + n_2, & \text{if } u \in V(G_1) \\ d_{G_{2i}}(u) + 1, & \text{if } u \in V(G_{2i}) \end{cases}$$

Date: Review November 19, 2018, accepted January 27, 2019.

2000 *Mathematics Subject Classification.* 05C10; 05C12; 06A07; 15A18; 15A36.

Key words and phrases. Monogenic semigroup, Zero-divisor graph, graph products, corona product, graph parameters.

, where $|V(G_1 \circ G_2)| = n_1(1+n_2)$(see, for instance [13,17,18]).

In here we replace G_1 by $\Gamma(S_M^1)$ and G_2 by $\Gamma(S_M^2)$, where $S_M^1 = \{x, x^2, ..., x^n\}$ with 0 and $S_M^2 = \{y, y^2, ..., y^m\}$ with 0. We have rules for monogenic semigroup graphs as follows:

$\Gamma(S_M^1) \circ \Gamma(S_M^2)$ has vertex set $V(\Gamma(S_M^1) \circ \Gamma(S_M^2)) = V(S_M^1) \cup V(S_M^{21}) \cup V(S_M^{22}) \cup ... \cup V(S_M^{2n})$ and let us take any two vertices of $\Gamma(S_M^1) \circ \Gamma(S_M^2)$ is adjacent if and only if

$$\begin{cases} x^i x^j = 0 \Leftrightarrow i+j \geq n+1, \text{where} \quad x^i, x^j \in V(\Gamma(S_M^1)) \\ \qquad or \\ y^i y^j = 0 \Leftrightarrow i+j \geq m+1, \text{where} \quad y^i, y^j \in V(\Gamma(S_M^{2i})) \\ \qquad or \\ x^i y^j = 0 \Leftrightarrow i+j \geq 2, \text{where} \quad x^i \in V(\Gamma(S_M^1)), y^j \in V(\Gamma(S_M^{2i})) \end{cases}$$

Theorem 1.1. [8] *Let S_M be a monogenic semigroups. Then the degree sequence of $\Gamma(S_M)$ is given by*

$$DS(\Gamma(S_M)) = \{1, 2, 3, ..., \lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, ..., n-2, n-1\}.$$

Theorem 1.2. *Let $G = \Gamma(S_M^1) \circ \Gamma(S_M^2))$. Then the degree sequence of $\Gamma(S_M^1) \circ \Gamma(S_M^2)$ is given by*

$$\begin{aligned} DS(G) = \{ & \overbrace{2, 3, 4, ..., \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \lfloor \frac{m}{2} \rfloor + 1, \lfloor \frac{m}{2} \rfloor + 2,}^{n \quad times} \\ & \overbrace{\lfloor \frac{m}{2} \rfloor + 3, ..., m-1, m}^{n \quad times}, m+1, m+2, m+3, ..., \lfloor \frac{n}{2} \rfloor - 1 + m, \\ & \lfloor \frac{n}{2} \rfloor + m, \lfloor \frac{n}{2} \rfloor + m, \lfloor \frac{n}{2} \rfloor + m + 1, \lfloor \frac{n}{2} \rfloor + m + 2, ..., n+m-2, \\ & n+m-1\}. \end{aligned}$$

In this study we present some topological indices corona products of any two monogenic semigroup graphs as sigma index, irregularity index etc.

2. Main Results

A graph G is regular if all its vertices have the same degree, otherwise it is irregular. Albertson [16] defines the irregularity of G as

$$irr(G) = \sum_{uv \in E(G)} \mid d_G(u) - d_G(v) \mid$$

, where $d_G(u)$ denotes the degree of a vertex $u \in V(G)$.

Theorem 2.1. *Let $G = \Gamma(S_M$. Then*

$$irr(\Gamma(S_M) = \begin{cases} \frac{n^3-4n}{12}, n \text{ is even} \\ \\ \frac{n^3-n}{12}, n \text{ is odd} \end{cases}$$

Proof. Assume that n is even:

$$\begin{aligned}
irr(\Gamma(S_M) &= (d_n - d_1) + (d_n - d_2) + ... + (d_n - d_{n-2}) + (d_n - d_{n-1}) + \\
&\quad \text{(which is written by } I_n\text{: Let us say } J_n \text{ to this sum)} \\
&\quad +(d_{n-1} - d_2) + (d_{n-1} - d_3) + ... + (d_{n-1} - d_{n-2}) + \\
&\quad \text{(which is written by } I_{n-1}\text{: Let us say } J_{n-1} \text{ to this sum)} \\
&\quad +... + \\
&\quad +(d_{\frac{n}{2}+2} - d_{\frac{n}{2}-1}) + (d_{\frac{n}{2}+2} - d_{\frac{n}{2}}) + (d_{\frac{n}{2}+2} - d_{\frac{n}{2}+1}) \\
&\quad \text{(which is written by } I_{\frac{n}{2}+2}\text{: Let us say } J_{\frac{n}{2}+2} \text{ to this sum)} \\
&\quad (d_{\frac{n}{2}+1} - d_{\frac{n}{2}}) \\
&\quad \text{(which is written by } I_{\frac{n}{2}+1}\text{: Let us say } J_{\frac{n}{2}+1} \text{ to this sum)}
\end{aligned}$$

As a result, we get

$$\begin{aligned}
irr(\Gamma(S_M)) &= \sum_{ij \in E(\Gamma S_M)} | d_i - d_j | \\
&= \sum_{k=1}^{\frac{n}{2}} \sum_{p=1}^{n-2k} p + \sum_{k=1}^{\frac{n}{2}} k \\
&= \frac{n^3 - 4n}{12}.
\end{aligned}$$

Assume that n is odd: For n is odd, we have $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. By taking following same steps as in n is even case, we obtain

$$\begin{aligned}
irr(\Gamma(S_M)) &= \sum_{ij \in E(\Gamma S_M)} | d_i - d_j | \\
&= \sum_{k=1}^{\frac{n-1}{2}} \sum_{p=1}^{n-2k} p + \sum_{k=1}^{\frac{n-1}{2}} k \\
&= \frac{n^3 - n}{12}.
\end{aligned}$$

Hence the result. □

Theorem 2.2. *Let $G = \Gamma(S_M^1) \circ \Gamma(S_M^2))$. Then*

$$irr(\Gamma(G)) = \begin{cases} \frac{n^3-4n}{12} + \frac{m^3-4m}{12} + \frac{mn(n+m+2)}{2}, & n,m \text{ even} \\ \frac{n^3-n}{12} + \frac{m^3-4m}{12} + \frac{mn(n+m+2)}{2}, & n \text{ odd, } m \text{ even} \\ \frac{n^3-n}{12} + \frac{m^3-m}{12} + \frac{mn(n+m+2)}{2}, & n,m \text{ odd} \\ \frac{n^3-4n}{12} + \frac{m^3-m}{12} + \frac{mn(n+m+2)}{2}, & n \text{ even, } m \text{ odd.} \end{cases}$$

Proof. From definition of graph irregularity we can write

$$\begin{aligned} irr(G) &= \sum_{uv\in E(G)} \mid d_G(u) - d_G(v) \mid \\ &= \sum_{uv\in E(\Gamma(S_M^1))} \mid d_G(u) - d_G(v) \mid + \sum_{i=1}^{n} \sum_{uv\in E(\Gamma(S_M^{2i}))} \mid d_G(u) - d_G(v) \mid \\ &\quad + \sum_{\substack{u\in V(\Gamma(S_M^1)) \\ v\in V(\Gamma(S_M^2))}} \mid d_G(u) - d_G(v) \mid . \end{aligned} \tag{2.1}$$

By applying Theorem 2.1 and Theorem 1.2 we get the following results.
Case1 : n and m even: From (2.1) we have

$$\begin{aligned} irr(G) &= \sum_{uv\in E(G)} \mid d_G(u) - d_G(v) \mid \\ &= \frac{n^3-4n}{12} + \frac{m^3-4n}{12} + m \sum_{u\in V(S_M^1)} d_{S_M^1}(u) + n \sum_{u\in V(S_M^{2i})} d_{S_M^{2i}}(u) \\ &= \frac{n^3-4n}{12} + \frac{m^3-4m}{12} + \frac{mn(n+m+2)}{2} \end{aligned}$$

Case2 : n odd and m even: From (2.1) we have

$$\begin{aligned} irr(G) &= \sum_{uv\in E(G)} \mid d_G(u) - d_G(v) \mid \\ &= \frac{n^3-n}{12} + \frac{m^3-4n}{12} + m \sum_{u\in V(S_M^1)} d_{S_M^1}(u) + n \sum_{u\in V(S_M^{2i})} d_{S_M^{2i}}(u) \\ &= \frac{n^3-n}{12} + \frac{m^3-4m}{12} + \frac{mn(n+m+2)}{2} \end{aligned}$$

In a similar way, we obtain in other cases. □

In [17] forgotten topological index F was defined as

$$F(G) = \sum_{u\in V(G)} d_G^3(u).$$

Theorem 2.3. *Let $G = \Gamma(S_M)$. Then*

$$F(G) = (n-1)\sum_{k=2}^{m} k^3 + \sum_{k=2}^{n-1+m} k^3 + n\left(\lfloor \frac{m}{2} \rfloor + 1\right)^3 + \left(\lfloor \frac{n}{2} \rfloor + m\right)^3.$$

Proof. By Theorem 1.2 we have

$$\begin{aligned} F(G) &= \sum_{u\in V(G)} d_G(u) \\ &= n\sum_{k=2}^{m} k^3 + \sum_{k=m+1}^{n-1+m} k^3 + n\left(\lfloor \frac{m}{2} \rfloor + 1\right)^3 + \left(\lfloor \frac{n}{2} \rfloor + m\right)^3 \\ &= (n-1)\sum_{k=2}^{m} k^3 + \sum_{k=2}^{n-1+m} k^3 + n\left(\lfloor \frac{m}{2} \rfloor + 1\right)^3 + \left(\lfloor \frac{n}{2} \rfloor + m\right)^3 \end{aligned}$$

□

If G is a graph and $d_G(u)$ the degree of its vertex u, then its sigma index [18,19] is defined as

$$\sigma(G) = \sum_{uv\in E(G)} (d_G(u) - d_G(v))^2.$$

, with summation going over all pairs of adjacent vertices.

Theorem 2.4. *Let $G = \Gamma(S_M^1) \circ \Gamma(S_M^2))$. Then*

$$\sigma(G) = \begin{cases} \frac{n^4-n^3+2n^2+4n}{24} + \frac{m^4-m^3+2m^2+4m}{24} + \frac{m^2n-2m^2+mn^2-3mn+3m+4n-4}{2}, n,m \text{ even} \\ \frac{n^4-n^3-n^2+n}{24} + \frac{m^4-m^3+2m^2+4m}{24} + \frac{m^2n-2m^2+mn^2-3mn+3m+4n-4}{2}, n \text{ odd, } m \text{ even} \\ \frac{n^4-n^3-n^2+n}{24} + \frac{m^4-m^3-m^2+m}{24} + \frac{(n-1)(mn+m^2-2n-5)}{2}, n,m \text{ odd} \\ \frac{n^4-n^3+2n^2+4n}{24} + \frac{m^4-m^3-m^2+m}{24} + \frac{(n-1)(mn+m^2-2n-5)}{2}., n \text{ even, } m \text{ odd.} \end{cases} \quad (2.2)$$

Proof. From definition of sigma index and Theorem 2.2 we have

$$\begin{aligned} \sigma(G) &= \sum_{uv\in E(S_M^1)} (d_G(u) - d_G(v))^2 + \sum_{i=1}^{n} \sum_{uv\in E(S_M^{2i})} (d_G(u) - d_G(v))^2 + \sum_{\substack{u\in V(S_M^1)\\ v\in V(S_M^2)}} (d_G(u) - d_G(v))^2 \\ &= \sum_{k=1}^{\lfloor \frac{n}{2}\rfloor} \sum_{p=1}^{n-2k} p^2 + \sum_{k=1}^{\lfloor \frac{n}{2}\rfloor} k^2 + n\left(\sum_{k=1}^{\lfloor \frac{m}{2}\rfloor} \sum_{p=1}^{m-2k} p^2 + \sum_{k=1}^{\lfloor \frac{m}{2}\rfloor} k^2\right) + \sum_{k=1}^{n-1}\sum_{p=2}^{m}(n+m-k-p) \\ &\quad + \sum_{k=1}^{n-1}(n+m-k-\lfloor\frac{m}{2}\rfloor+1). \end{aligned}$$

Case1: n and m are even.

$$\begin{aligned} \sigma(G) &= \sum_{k=1}^{\frac{n}{2}} \sum_{p=1}^{n-2k} p^2 + \sum_{k=1}^{\frac{n}{2}} k^2 + n\left(\sum_{k=1}^{\frac{m}{2}} \sum_{p=1}^{m-2k} p^2 + \sum_{k=1}^{\frac{m}{2}} k^2\right) + \sum_{k=1}^{n-1}\sum_{p=2}^{m}(n+m-k-p) \\ &\quad + \sum_{k=1}^{n-1}(n+m-k-\frac{m}{2}+1) \\ &= \frac{n^4-n^3+2n^2+4n}{24} + \frac{m^4-m^3+2m^2+4m}{24} \\ &\quad + \frac{m^2n-2m^2+mn^2-3mn+3m+4n-4}{2} \end{aligned}$$

Case2: n is odd and m is even.

$$\begin{aligned} \sigma(G) &= \sum_{k=1}^{\frac{n-1}{2}} \sum_{p=1}^{n-2k} p^2 + \sum_{k=1}^{\frac{n-1}{2}} k^2 + n\left(\sum_{k=1}^{\frac{m}{2}} \sum_{p=1}^{m-2k} p^2 + \sum_{k=1}^{\frac{m}{2}} k^2\right) + \sum_{k=1}^{n-1}\sum_{p=2}^{m}(n+m-k-p) \\ &\quad + \sum_{k=1}^{n-1}(n+m-k-\frac{m}{2}+1) \\ &= \frac{n^4-n^3-n^2+n}{24} + \frac{m^4-m^3+2m^2+4m}{24} \\ &\quad + \frac{m^2n-2m^2+mn^2-3mn+3m+4n-4}{2} \end{aligned}$$

Case3: n and m are odd.

$$\begin{aligned}\sigma(G) &= \sum_{k=1}^{\frac{n-1}{2}}\sum_{p=1}^{n-2k} p^2+\sum_{k=1}^{\frac{n-1}{2}} k^2+n\left(\sum_{k=1}^{\frac{m-1}{2}}\sum_{p=1}^{m-2k} p^2+\sum_{k=1}^{\frac{m-1}{2}} k^2\right)+\sum_{k=1}^{n-1}\sum_{p=2}^{m}(n+m-k-p)\\ &\quad+\sum_{k=1}^{n-1}(n+m-k-\frac{m}{2}+1)\\ &= \frac{n^4-n^3-n^2+n}{24}+\frac{m^4-m^3-m^2+m}{24}\\ &\quad+\frac{(n-1)(mn+m^2-2n-5)}{2}\end{aligned}$$

Case4 : n is even and m is odd.

$$\begin{aligned}\sigma(G) &= \sum_{k=1}^{\frac{n-1}{2}}\sum_{p=1}^{n-2k} p^2+\sum_{k=1}^{\frac{n-1}{2}} k^2+n\left(\sum_{k=1}^{\frac{m-1}{2}}\sum_{p=1}^{m-2k} p^2+\sum_{k=1}^{\frac{m-1}{2}} k^2\right)+\sum_{k=1}^{n-1}\sum_{p=2}^{m}(n+m-k-p)\\ &\quad+\sum_{k=1}^{n-1}(n+m-k-\frac{m}{2}+1)\\ &= \frac{n^4-n^3+2n^2+4n}{24}+\frac{m^4-m^3-m^2+m}{24}\\ &\quad+\frac{(n-1)(mn+m^2-2n-5)}{2}.\end{aligned}$$

Hence the result. □

References

[1] D.F. Anderson and P.S. Livingston, The zero-divisor graph of commutative ring, Journal of Algebra, Vol. 217, pp. 434-447 (1999).

[2] D.F. Anderson and A. Badawi, On the zero-divisor graph of a ring, Comm. Algebra, Vol. 36(8), pp. 3073-3092 (2008).

[3] D.D. Anderson and M. Naseer, Becks coloring of a commutative ring, Journal of Algebra, Vol. 159, pp. 500-514 (1991).

[4] I. Beck, Coloring of commutating ring, Journal of Algebra, Vol. 116, pp. 208-226 (1988).

[5] F.R. DeMeyer, T. McKenzie and K. Schneider, The zero-divisor graph of a commutative semigroup, Semigroup Forum, Vol. 65, pp. 206-214 (2002).

[6] F.R. DeMeyer and L. DeMeyer, Zero-divisor graphs of semigroups, Journal of Algebra, Vol. 283, pp. 190-198 (2005).

[7] R. Frucht and F. Harary, On the corona of two graphs, Aequationes Math., Vol. 4, pp. 322-325 (1970).

[8] K.C. Das, N. Akgunes, A.S. Cevik, AS, On a graph of monogenic semigroup. J. Inequal. Appl. 2013, 44 (2013).

[9] N. Akgunes, Some graph parameters on the strong product of monogenic semigroup graphs, J. Baun Inst. Sci. Technol., XX(X), pp. 1-9 (2018).

[10] Y.Nacaroglu, On the corona product of monogenic semigroup graphs, Adv. and Appl. in Discrete Math., in press (2018)

[11] J.L. Gross and J. Yellen, Handbook of graph theory, Chapman Hall, CRC Press (2004).

[12] W. Imrich and S. Klavar, Product graphs. structure and recognition, Wiley-Interscience Series in Discrete Math.and Optim., Wiley- Interscience, New York, (2000).

[13] S. Klavar, Coloring graph products - a survey, Discrete Math., Vol. 155, pp. 135-145 (1996).

[14] Y. Nacaroglu and A.D. Maden, The Upper Bounds for Multiplicative Sum Zagreb Index of Some Graph Operations, J. of Math. Ineq., Vol. 11(3), pp. 749-761 (2017).

[15] Y. Nacaroglu and A.D. Maden, The Multiplicative Zagreb Coindices of Graph Operations, Util. Math., Vol. 102, pp. 19-38 (2017).
[16] M. O. Albertson, The irregularity of a graph, Ars Combin., 46, pp. 219225 (1997).
[17] B. Furtula and I. Gutman. A forgotten topological index. J. Math. Chem., Vol. 53(4), pp. 1184-1190 (2015).
[18] H. Abdo, D. Dimitrov, The total irregularity of graphs under graph operations, Miskolc Math. Notes 15, pp. 317 (2014).
[19] I. Gutman, M. Togan, A. Yurttas, A.S. Cevik, I.N. Cangul, Inverse Problem for Sigma Index, MATCH Commun. Math. Comput. Chem., 79, pp. 491-508 (2018).

(author one) Kahramanmaras Sutcu Imam university, Mathematics department, 46100, Kahramanmaras, Turkey

E-mail address, author one: yasarnacaroglu@ksu.edu.tr

(author two) Necmettin Erbakan university, Department of Mathematics-Computer science, 42090, Konya, Turkey

E-mail address, author two: nakgunes@konya.edu.tr