Available online: January 28, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 2, Pages 1273–1288 (2019)
DOI: 10.31801/cfsuasmas.419098
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

ON THE INVERSE PROBLEM FOR FINITE DISSIPATIVE JACOBI MATRICES WITH A RANK-ONE IMAGINARY PART

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Abstract. This paper deals with the inverse spectral problem consisting in the reconstruction of a finite dissipative Jacobi matrix with a rank-one imaginary part from its eigenvalues. Necessary and sufficient conditions are formulated for a prescribed collection of complex numbers to be the spectrum of a finite dissipative Jacobi matrix with a rank-one imaginary part. Uniqueness of the matrix having prescribed eigenvalues is shown and an algorithm for reconstruction of the matrix from prescribed eigenvalues is given.

1. Introduction

An $N \times N$ (real) Jacobi matrix is a tri-diagonal symmetric matrix of the form

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}, \tag{1}$$

where for each n, a_n and b_n are arbitrary real numbers such that a_n is positive:

$$a_n > 0, \quad b_n \in \mathbb{R}. \tag{2}$$

Quantities connected with the eigenvalues and eigenvectors of the matrix are called the spectral characteristics of the matrix. The general inverse spectral problem is to reconstruct the matrix given some of its spectral characteristics (spectral data). Many versions of the inverse spectral problem for finite and infinite Jacobi matrices have been investigated in the literature and many procedures and algorithms for

Received by the editors: April 27, 2018; Accepted: August 07, 2018.

2010 *Mathematics Subject Classification.* 15A29.

Key words and phrases. Jacobi matrix, eigenvalue, normalizing number, dissipative, inverse spectral problem.

their solution have been proposed (for some of them see [12, 13, 11, 7, 3, 2, 6, 15, 8, 9, 10, 14, 16]).

Note that, in general, one spectrum consisting of the eigenvalues of the Jacobi matrix does not determine this matrix. It turns out that the eigenvalues together with the normalizing numbers (a spectral measure) or the so-called two-spectra are enough to determine the Jacobi matrix uniquely.

Now let us along with the matrix J given by (1), (2) consider also the matrix

$$\widetilde{J} = \begin{bmatrix} \widetilde{b}_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix} \quad (3)$$

in which all a_n and b_n are the same as in J, except b_0 which is replaced by $\widetilde{b}_0$ and we will assume that

$$\widetilde{b}_0 = b_0 + i\omega, \quad \omega > 0. \quad (4)$$

Therefore $\widetilde{J}$ is a dissipative matrix with a rank-one imaginary part. It turns out that the matrix $\widetilde{J}$ has N, counting algebraic multiplicity, nonreal eigenvalues $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_N$ with positive imaginary parts. These eigenvalues are determined by their N free, in general, real parts and N free positive imaginary parts. The matrix $\widetilde{J}$ also contains N free real parameters $b_0, b_1, \dots, b_{N-1}$ and N free positive parameters $a_0, a_1, \dots, a_{N-2}$ and ω. Therefore one may expect that the inverse problem from the eigenvalues $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_N$ to the matrix $\widetilde{J}$ is uniquely solvable. Such an inverse problem was recently investigated and solved in the paper [1] by using the Livsic characteristic function of the matrix $\widetilde{J}$ and expansion into continuous fractions of the Weyl-Titchmarsh function of $\widetilde{J}$, expressed in terms of the Livsic characteristic function.

In the present paper, we revisit the reconstruction problem of a dissipative Jacobi matrix with a rank-one imaginary part from its eigenvalues. Our approach to this problem in this paper differs from that in the paper [1] and is based on a known fact on polynomials with roots in the open upper half-plane (Theorem 2 below in Section 2) and on the reduction of the inverse problem for the dissipative Jacobi matrix $\widetilde{J}$ of the form (3), (4) to the inverse problem from two-spectra for the associated real Jacobi matrix J defined by (1), (2). Besides, the latter problem we solve by the discrete Gelfand-Levitan method instead of the continuous fraction expansion. We hope that the approach suggested in the present paper may contribute some additional insights to the theory of inverse problems for dissipative Jacobi matrices and may be useful in related problems.

The paper consists, besides this introductory section, of four sections. In Section 2, we formulate a solution procedure of the inverse problem for real Jacobi matrices from the eigenvalues and normalizing numbers and also from the two-spectra, and present a known theorem on the polynomials with roots in the open upper half-plane. In Section 3, we deal with the inverse problem for dissipative Jacobi matrices with a rank-one imaginary part from its eigenvalues and establish necessary and sufficient conditions for solvability of the inverse problem. In Section 4, we describe the reconstruction procedure and give an example. Finally, in Section 5, we make some concluding remarks on the reconstruction procedure.

2. Auxiliary facts

In this section, we present briefly known solutions of the inverse problems with respect to eigenvalues and normalizing numbers and with respect to two-spectra, for real finite Jacobi matrices. Also, we present a known theorem on complex polynomials with zeros in the open upper half-plane. This knowledge is given here for easy reference and they will be used in the subsequent sections.

1. It is well known that any real Jacobi matrix of the form (1), (2) has precisely N real and distinct eigenvalues $\lambda_1, \ldots, \lambda_N$, so

$$\det(\lambda I - J) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_N). \tag{5}$$

The normalizing numbers of the matrix J can be introduced as follows. Let $R(\lambda) = (J - \lambda I)^{-1}$ be the resolvent of the matrix J (by I we denote the identity matrix of needed dimension) and e_0 be the N-dimensional column vector with the components $1, 0, \ldots, 0$. The rational function

$$w(\lambda) = -\langle R(\lambda)e_0, e_0\rangle = \left\langle (\lambda I - J)^{-1} e_0, e_0 \right\rangle, \tag{6}$$

introduced earlier in [13], is called the *resolvent function* of the matrix J, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{C}^N$. This function is known also as the Weyl or Weyl-Titchmarsh function of J. The resolvent function $w(\lambda)$ admits, by (6) and (5), the decomposition into partial fractions,

$$w(\lambda) = \sum_{k=1}^{N} \frac{\beta_k}{\lambda - \lambda_k}, \tag{7}$$

where β_k's are some positive real numbers uniquely determined by the matrix J and such that

$$\sum_{k=1}^{N} \beta_k = 1. \tag{8}$$

The number β_k is called the *normalizing number* of the matrix J, associated with the eigenvalue λ_k (it is related to the norm of the eigenvector of J, corresponding to the eigenvalue λ_k).

The collection of the eigenvalues and normalizing numbers

$$\{\lambda_k,\ \beta_k\ (k = 1, \ldots, N)\} \tag{9}$$

of the matrix J of the form (1), (2) is called the *spectral data* of this matrix.

The matrix J given in (1) contains N free real parameters $b_0, b_1, \ldots, b_{N-1}$ and $N-1$ free real positive parameters $a_0, a_1, \ldots, a_{N-2}$. The spectral data in (9) of the matrix J contain N real parameters $\lambda_1, \ldots, \lambda_N$ and N real positive parameters $\beta_1, \ldots, \beta_N$ but (8) eliminates one parameter. Therefore it is reasonable to consider the inverse problem from the spectral data to the matrix J.

The *inverse spectral problem* is stated as follows:

(i) To see if it is possible to reconstruct the matrix J, given its spectral data (9). If it is possible, to describe the reconstruction procedure.

(ii) To find the necessary and sufficient conditions for a given collection (9) to be spectral data for some matrix J of the form (1) with entries belonging to the class (2).

The solution of this problem is well known and can be formulated as follows.

Given a collection (9), where $\lambda_1, \ldots, \lambda_N$ and $\beta_1, \ldots, \beta_N$ are arbitrary numbers, define the numbers

$$s_l = \sum_{k=1}^{N} \beta_k \lambda_k^l, \quad l = 0, 1, 2, \ldots, \tag{10}$$

and using these numbers introduce the Hankel determinants

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}, \quad n = 0, 1, 2, \ldots . \tag{11}$$

Theorem 1. *Let an arbitrary collection (9) of numbers be given. In order for this collection to be the spectral data for a Jacobi matrix J of the form (1) with entries belonging to the class (2), it is necessary and sufficient that the following two conditions are satisfied:*

(i) *The numbers $\lambda_1, \ldots, \lambda_N$ are real and distinct.*

(ii) *The numbers $\beta_1, \ldots, \beta_N$ are positive and such that $\beta_1 + \ldots + \beta_N = 1$.*

Under the conditions (i) and (ii) we have $D_n > 0$ for $n \in \{0, 1, \ldots, N-1\}$ and the entries a_n and b_n of the unique Jacobi matrix J for which the collection (9) is spectral data, are recovered by the formulas

$$a_n = \frac{\sqrt{D_{n-1} D_{n+1}}}{D_n}, \quad n \in \{0, 1, \ldots, N-2\},\ D_{-1} = 1, \tag{12}$$

$$b_n = \frac{\Delta_n}{D_n} - \frac{\Delta_{n-1}}{D_{n-1}}, \quad n \in \{0, 1, \ldots, N-1\},\ \Delta_{-1} = 0,\ \Delta_0 = s_1, \tag{13}$$

where D_n is defined by (11) and (10), and Δ_n is the determinant obtained from the determinant D_n by replacing in D_n the last column by the column with the components $s_{n+1}, s_{n+2}, \ldots, s_{2n+1}$.

For a detailed proof of Theorem 1 see, for example, [9, Section 2], where a discrete version of the Gelfand-Levitan [5] procedure for the reconstruction of a differential equation from its spectral function is carried out. Note that the Gelfand-Levitan procedure of solution of the inverse problem is based on a Fredholm type linear integral equation (the so-called Gelfand-Levitan equation). In the discrete case of Jacobi matrices the Gelfand-Levitan equation becomes an inhomogeneous linear system of algebraic equations and solution of this system by Cramer's rule in terms of determinants yields formulas (12) and (13).

2. There are different versions of the inverse two-spectra problem for Jacobi matrices. One of them was introduced by Hochstadt in [12, 13] as follows. Let J_1 be the $(N-1)\times(N-1)$ matrix obtained from J defined by (1), (2) by deleting its first row and first column:

$$J_1 = \begin{bmatrix} b_1 & a_1 & \cdots & 0 & 0 & 0 \\ a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}. \tag{14}$$

The matrix J_1 is called the *first truncated matrix* (with respect to the matrix J). The eigenvalues $\{\lambda_k\}_{k=1}^N$ and $\{\mu_k\}_{k=1}^{N-1}$ of J and J_1, respectively, interlace:

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \lambda_3 < \ldots < \mu_{N-1} < \lambda_N. \tag{15}$$

The author showed in [13] that if $\{\lambda_k\}$ are the eigenvalues of some Jacobi matrix of the form (1) with the entries (2) and if $\{\mu_k\}$ are the eigenvalues of the corresponding J_1, then there is precisely one such matrix with these $\{\lambda_k, \mu_k\}$, and gave a constructive method for calculating the entries of J in terms of the given eigenvalues. The collections $\{\lambda_k\}_{k=1}^N$ and $\{\mu_k\}_{k=1}^{N-1}$ are called the *two-spectra* of the matrix J. It turns out that condition (15) is not only necessary but also sufficient for two collections of real numbers $\{\lambda_k\}_{k=1}^N$ and $\{\mu_k\}_{k=1}^{N-1}$ to be two-spectra for a Jacobi matrix J of the form (1) with entries belonging to the class (2).

Let us show that the inverse problem about two-spectra can be solved by reducing it to the inverse problem about spectral data consisting of eigenvalues and normalizing numbers of the matrix and solved in Theorem 1.

For the resolvent function $w(\lambda)$ of the matrix J, given by (6), the formula

$$w(\lambda) = \frac{\det(\lambda I - J_1)}{\det(\lambda I - J)}$$

holds. Substituting here (7) and

$$\det(\lambda I - J) = \prod_{j=1}^{N}(\lambda - \lambda_j), \quad \det(\lambda I - J_1) = \prod_{j=1}^{N-1}(\lambda - \mu_j),$$

we can write

$$\sum_{j=1}^{N} \frac{\beta_j}{\lambda - \lambda_j} = \frac{\prod_{j=1}^{N-1} (\lambda - \mu_j)}{\prod_{j=1}^{N} (\lambda - \lambda_j)}.$$

Multiplying both sides of the last equation by $\lambda - \lambda_k$ and passing then to the limit as $\lambda \to \lambda_k$, we find that

$$\beta_k = \frac{\prod_{j=1}^{N-1} (\lambda_k - \mu_j)}{\prod_{j=1, j\neq k}^{N} (\lambda_k - \lambda_j)}, \quad k = 1, \ldots, N. \tag{16}$$

The formula (16) expresses the normalizing numbers β_k of the matrix J in terms of its two-spectra $\{\lambda_j\}_{j=1}^N$ and $\{\mu_j\}_{j=1}^{N-1}$. Since the normalizing numbers $\{\beta_k\}_{k=1}^N$ together with the eigenvalues $\{\lambda_k\}_{k=1}^N$ determine the matrix J uniquely, we get that the two-spectra determine the matrix J uniquely.

The formula (16) allows also to prove the sufficiency of the condition (15) for two given collections of real numbers $\{\lambda_k\}_{k=1}^N$ and $\{\mu_k\}_{k=1}^{N-1}$ to be two-spectra for a Jacobi matrix J of the form (1) with entries belonging to the class (2).

Indeed, suppose that two collections of real numbers $\{\lambda_k\}_{k=1}^N$ and $\{\mu_k\}_{k=1}^{N-1}$ are given which satisfy the condition (15). Using these collections we construct the numbers β_k $(k = 1, \ldots, N)$ by Eq. (16). The condition (15) guarantees that such defined numbers β_k are positive. Let us show that

$$\sum_{k=1}^{N} \beta_k = 1. \tag{17}$$

For sufficiently large positive number R such that $\lambda_1, \ldots, \lambda_N$ are inside the circle $\{\lambda \in \mathbb{C} : |\lambda| = R\}$, we have

$$\sum_{k=1}^{N} \beta_k = \sum_{k=1}^{N} \frac{\prod_{j=1}^{N-1} (\lambda_k - \mu_j)}{\prod_{j=1, j\neq k}^{N} (\lambda_k - \lambda_j)}$$

$$= \sum_{k=1}^{N} \operatorname{Res}_{\lambda=\lambda_k} \frac{(\lambda - \mu_1) \cdots (\lambda - \mu_{N-1})}{(\lambda - \lambda_1) \cdots (\lambda - \lambda_N)}$$

$$= \frac{1}{2\pi i} \oint_{|\lambda|=R} \frac{(\lambda - \mu_1) \cdots (\lambda - \mu_{N-1})}{(\lambda - \lambda_1) \cdots (\lambda - \lambda_N)} d\lambda = \frac{1}{2\pi i} \oint_{|\lambda|=R} \frac{\lambda^{N-1} + \ldots}{\lambda^N + \ldots} d\lambda$$

$$= \frac{1}{2\pi i} \oint\limits_{|\lambda|=R} \left[\frac{1}{\lambda} + O\left(\frac{1}{|\lambda|^2} \right) \right] d\lambda = 1 + \frac{1}{2\pi i} \oint\limits_{|\lambda|=R} O\left(\frac{1}{|\lambda|^2} \right) d\lambda.$$

Passing here to the limit as $R \to \infty$ and noting that

$$\lim_{R\to\infty} \oint\limits_{|\lambda|=R} O\left(\frac{1}{|\lambda|^2} \right) d\lambda = 0,$$

we arrive at (17).

Thus, the collection $\{\lambda_k, \beta_k \ (k = 1, \dots, N)\}$ satisfies all the conditions of Theorem 1 and hence there exist a unique Jacobi matrix J of the form (1) with entries from the class (2) such that λ_k are the eigenvalues and β_k are the corresponding normalizing numbers for J. The entries $a_n, \ b_n$ of the matrix J are found by formulas (12), (13) in which the numbers β_k are calculated by (16). It remains to show that μ_j are the eigenvalues of J_1, where J_1 is the first truncated matrix (with respect to the constructed matrix J). To do this we denote the eigenvalues of J_1 by $\mu'_1, \dots, \mu'_{N-1}$. We have to show that $\mu_j = \mu'_j \ (j = 1, \dots, N-1)$ with a possible reorder of μ'_j's. Let us set

$$f(\lambda) = \prod_{j=1}^{N-1} (\lambda - \mu_j), \quad g(\lambda) = \prod_{j=1}^{N-1} (\lambda - \mu'_j). \tag{18}$$

By the direct problem for the constructed matrix J we have (formula (16) in which μ_j should be replaced by μ'_j)

$$\beta_k = \frac{\prod\limits_{j=1}^{N-1} (\lambda_k - \mu'_j)}{\prod\limits_{j=1, j\neq k}^{N} (\lambda_k - \lambda_j)}, \quad k = 1, \dots, N.$$

On the other hand, by our construction of β_k we have (16). Hence

$$\prod_{j=1}^{N-1} (\lambda_k - \mu'_j) = \prod_{j=1}^{N-1} (\lambda_k - \mu_j), \quad k = 1, \dots, N.$$

This means that the polynomials $f(\lambda)$ and $g(\lambda)$ of degree $N-1$, defined in (18), coincide at N different points $\lambda_1, \dots, \lambda_N$. Then $f(\lambda) \equiv g(\lambda)$ and consequently $\mu_j = \mu'_j \ (j = 1, \dots, N-1)$ with a possible reorder of μ'_j's.

3. Given a complex polynomial $P(\lambda)$, we define

$$P^*(\lambda) = \overline{P(\overline{\lambda})} \tag{19}$$

so that the polynomial $P^*(\lambda)$ is obtained from the polynomial $P(\lambda)$ by replacing the coefficients of $P(\lambda)$ by their complex conjugates. Define the real and imaginary parts of $P(\lambda)$ by

$$P^{re}(\lambda) = \frac{P(\lambda) + P^*(\lambda)}{2}, \quad P^{im}(\lambda) = \frac{P(\lambda) - P^*(\lambda)}{2i}. \tag{20}$$

Therefore $P^{re}(\lambda)$ and $P^{im}(\lambda)$ are polynomials with real coefficients and

$$P(\lambda) = P^{re}(\lambda) + iP^{im}(\lambda).$$

Theorem 2. *(*[4, Theorem 9.9]*) Let $P(\lambda)$ be a complex polynomial and let $P^{re}(\lambda)$ and $P^{im}(\lambda)$ be its real and imaginary parts, defined by (20), (19). Then all the zeros of $P(\lambda) = P^{re}(\lambda) + iP^{im}(\lambda)$ are in the open upper half-plane if and only if all the zeros of $P^{re}(\lambda)$ and $P^{im}(\lambda)$ are real, simple, and separate each other (interlace).*

3. The inverse problem for dissipative Jacobi matrices

Let E_0 be the $N \times N$ matrix whose elements are all zero except the first main diagonal element which is equal to 1. Then the matrix $\widetilde{J}$ defined by (3) with (2) and (4) can be written in the form

$$\widetilde{J} = J + i\omega E_0,$$

where J is defined by (1). Because the matrices J and E_0 are selfadjoint, it follows that the adjoint $\widetilde{J}^*$ of $\widetilde{J}$ is

$$\widetilde{J}^* = J - i\omega E_0.$$

Therefore, the Hermitian components of $\widetilde{J}$ are

$$\mathrm{Re}\widetilde{J} = \frac{\widetilde{J} + \widetilde{J}^*}{2} = J, \quad \mathrm{Im}\widetilde{J} = \frac{\widetilde{J} - \widetilde{J}^*}{2\dot{I}} = \omega E_0.$$

Next, let the linear space $\mathbb{C}^N$ of columns be equipped by the usual inner product

$$\langle x, y \rangle = \sum_{n=0}^{N-1} x_n \overline{y_n}.$$

Then for any $x \in \mathbb{C}^N$ we have

$$\left\langle \widetilde{J}x, x \right\rangle = \langle (J + i\omega E_0)x, x \rangle$$

$$= \langle Jx, x \rangle + i\omega \langle E_0 x, x \rangle = \langle Jx, x \rangle + i\omega \left| x_0 \right|^2$$

so that

$$\mathrm{Im} \left\langle \widetilde{J}x, x \right\rangle = \omega \left| x_0 \right|^2, \tag{21}$$

and

$$\left(\widetilde{J} - \widetilde{J}^* \right) x = 2i\omega E_0 x = 2i\omega x_0 e_0,$$

where x_0 is the first component of the vector x and e_0 is the column vector with the components $1, 0, \ldots, 0$.

Consequently,

$$\mathrm{Im} \left\langle \widetilde{J}x, x \right\rangle \geq 0 \quad \text{for all} \quad x \in \mathbb{C}^N,$$

$$\operatorname{ran}\left(\widetilde{J}-\widetilde{J}^*\right)=\{\alpha e_0:\alpha\in\mathbb{C}\},$$

so that, $\widetilde{J}$ is a dissipative Jacobi matrix with a rank-one imaginary part.

Lemma 3. *The eigenvalues of the matrix $\widetilde{J}$ belong to the open upper half-plane.*

Proof. Let $\lambda\in\mathbb{C}$ be an eigenvalue of the matrix $\widetilde{J}$ and $y\in\mathbb{C}^N,\ y\neq 0$ be a corresponding eigenvector:

$$\widetilde{J}y=\lambda y. \tag{22}$$

Hence

$$\left\langle \widetilde{J}y,y\right\rangle=\langle\lambda y,y\rangle=\lambda\langle y,y\rangle=\lambda\|y\|^2$$

and

$$\operatorname{Im}\left\langle \widetilde{J}y,y\right\rangle=(\operatorname{Im}\lambda)\|y\|^2.$$

On the other hand, by (21) applied to the vector y, we have

$$\operatorname{Im}\left\langle \widetilde{J}y,y\right\rangle=\omega|y_0|^2,$$

where y_0 is the first component of the vector y. Therefore

$$\operatorname{Im}\lambda=\omega\frac{|y_0|^2}{\|y\|^2}. \tag{23}$$

Further, it is not difficult to see that for the eigenvector y its first component y_0 is different from zero. Indeed, Eq. (22) in coordinates has the form

$$\widetilde{b}_0y_0+a_0y_1=\lambda y_0, \tag{24}$$

$$a_{n-1}y_{n-1}+b_ny_n+a_ny_{n+1}=\lambda y_n,\quad n=1,\ldots,N-2, \tag{25}$$

$$a_{N-2}y_{N-2}+b_{N-1}y_{N-1}=\lambda y_{N-1}. \tag{26}$$

Therefore, if $y_0=0$, then we find recurrently from equations (24), (25) (using the condition $a_n\neq 0$) that $y_1=\ldots=y_{N-1}=0$ which contradict to the fact that $y\neq 0$ as an eigenvector. Thus, $y_0\neq 0$ and (23) together with the condition $\omega>0$ implies that $\operatorname{Im}\lambda>0$. □

Note that to each eigenvalue of $\widetilde{J}$ there corresponds only one linearly independent eigenvector (this can easily be seen from equations (24)–(26)) so that the geometric multiplicity of each eigenvalue of $\widetilde{J}$ is 1. However, there may exist the so-called associated vectors attached to the eigenvectors. The algebraic multiplicity of an eigenvalue of the matrix $\widetilde{J}$ is its multiplicity as the root of the characteristic polynomial $\det\left(\lambda I-\widetilde{J}\right)$.

Denote all the (not necessarily distinct) eigenvalues of the matrix $\widetilde{J}$ by $\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_N$ counting their algebraic multiplicities. By Lemma 3 the numbers $\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_N$ are nonreal and lie in the open upper half-plane.

Lemma 4. *The equality*

$$\sum_{j=1}^{N} \operatorname{Im}\widetilde{\lambda}_j = \omega \tag{27}$$

holds, where $\omega > 0$ is taken from (4).

Proof. For any matrix $A = [a_{jk}]_{j,k=1}^N$ the spectral trace of A equals its matrix trace: If $z_1, \ldots, z_N$ are all the eigenvalues (counting their algebraic multiplicities) of A, then

$$\sum_{j=1}^{N} z_j = \sum_{j=1}^{N} a_{jj}.$$

Therefore we can write, for the matrix $\widetilde{J}$,

$$\sum_{j=1}^{N} \widetilde{\lambda}_j = \widetilde{b}_0 + b_1 + \ldots + b_{N-1},$$

where $\widetilde{b}_0$ has the form (4). Taking here the imaginary part we get (27). □

The following simple lemma is crucial in our solving the inverse spectral problem for the matrix $\widetilde{J}$.

Lemma 5. *The identity*

$$\det\left(\lambda I - \widetilde{J}\right) = \det(\lambda I - J) - i\omega \det(\lambda I - J_1) \tag{28}$$

holds, where J, $\widetilde{J}$, and ω are defined by (1), (3), and (4), J_1 is the matrix in (14) obtained from J by deleting its first row and first column, and by I we denote identity matrices of needed dimension.

Proof. We have

$$\lambda I - \widetilde{J} = \begin{bmatrix} \lambda - \widetilde{b}_0 & -a_0 & 0 & \cdots & 0 & 0 & 0 \\ -a_0 & \lambda - b_1 & -a_1 & \cdots & 0 & 0 & 0 \\ 0 & -a_1 & \lambda - b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda - b_{N-3} & -a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & -a_{N-3} & \lambda - b_{N-2} & -a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & -a_{N-2} & \lambda - b_{N-1} \end{bmatrix}.$$

Therefore expanding the determinant $\det\left(\lambda I - \widetilde{J}\right)$ by elements of its first row, we can write

$$\det\left(\lambda I - \widetilde{J}\right) = \left(\lambda - \widetilde{b}_0\right) \det(\lambda I - J_1) + a_0 \det D(\lambda), \tag{29}$$

where $D(\lambda)$ is the matrix of order $N-1$ obtained from $\lambda I - \widetilde{J}$ by eliminating its 1st row and 2nd column.

Similarly, expanding the determinant $\det(\lambda I - J)$ by elements of its first row, we have

$$\det(\lambda I - J) = (\lambda - b_0)\det(\lambda I - J_1) + a_0 \det D(\lambda), \tag{30}$$

where $D(\lambda)$ is the same matrix as in (29). Now subtracting (29) and (30) side-by-side we arrive at (28). □

Because the polynomials $\det(\lambda I - J)$ and $\det(\lambda I - J_1)$ have real coefficients, it follows from (28) that

$$\left[\det\left(\lambda I - \widetilde{J}\right)\right]^{re} = \det(\lambda I - J), \tag{31}$$

$$\left[\det\left(\lambda I - \widetilde{J}\right)\right]^{im} = -\omega \det(\lambda I - J_1). \tag{32}$$

The characteristic polynomial $\det\left(\lambda I - \widetilde{J}\right)$ of $\widetilde{J}$ is uniquely determined by the eigenvalues $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ of $\widetilde{J}$. Next, equations (31) and (32) show that the polynomial $\det\left(\lambda I - \widetilde{J}\right)$ uniquely determine the polynomials $\det(\lambda I - J)$ and $-\omega\det(\lambda I - J_1)$. Since the roots of the last two polynomials give the two-spectra of the matrix J and the matrix J is determined uniquely from its two-spectra (see Section 2 above), we conclude that the matrix J (the real part of $\widetilde{J}$) is determined from the eigenvalues $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ of $\widetilde{J}$ uniquely. Besides, the number ω in (4) is also determined uniquely from $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ by (27). Thus, we have established the following uniqueness result.

Theorem 6. *The eigenvalues $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ of the matrix $\widetilde{J}$ of the form (3) determine this matrix uniquely in the class of entries (2), (4).*

The following theorem states the existence result for solution of the inverse spectral problem for $\widetilde{J}$. Its proof given below contains also an algorithm for the construction of the finite dissipative Jacobi matrix with a rank-one imaginary part from the prescribed eigenvalues.

Theorem 7. *In order for given not necessarily distinct complex numbers $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ to be the eigenvalues counting algebraic multiplicity for a Jacobi matrix $\widetilde{J}$ of the form (3) with the entries in the class (2), (4), it is necessary and sufficient that the numbers $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ belong to the open upper half-plane, i.e. that these numbers have positive imaginary parts:*

$$\mathrm{Im}\widetilde{\lambda}_j > 0 \quad (j = 1, \ldots, N). \tag{33}$$

Proof. The necessity of the condition (33) has been proved above in Lemma 3. To prove the sufficiency suppose that we are given the numbers $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ satisfying the condition (33). Using these numbers we form the positive number

$$\omega = \sum_{j=1}^{N} \mathrm{Im}\widetilde{\lambda}_j \tag{34}$$

and the polynomial

$$P(\lambda) = (\lambda - \widetilde{\lambda}_1) \cdots (\lambda - \widetilde{\lambda}_N) = \lambda^N + c_1 \lambda^{N-1} + \ldots + c_{N-1}\lambda + c_N. \qquad (35)$$

Let us set

$$A(\lambda) = P^{re}(\lambda) = \lambda^N + (\mathrm{Re} c_1)\, \lambda^{N-1} + \ldots + (\mathrm{Re} c_{N-1})\lambda + \mathrm{Re} c_N,$$

$$B(\lambda) = -\frac{1}{\omega} P^{im}(\lambda) = \lambda^{N-1} + \left(-\frac{\mathrm{Im} c_2}{\omega}\right) \lambda^{N-2} + \ldots + \left(-\frac{\mathrm{Im} c_N}{\omega}\right),$$

where in writing the second equation we have used the fact that $c_1 = -(\widetilde{\lambda}_1 + \ldots + \widetilde{\lambda}_N)$ and therefore $\mathrm{Im} c_1 = -\omega$ by (34). Then

$$P(\lambda) = A(\lambda) - i\omega B(\lambda) \qquad (36)$$

and by Theorem 2 all the zeros of $A(\lambda)$ and $B(\lambda)$ are real, simple, and separate each other (interlace). Therefore if we denote the zeros of $A(\lambda)$ by $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ and the zeros of $B(\lambda)$ by $\mu_1 < \mu_2 < \ldots < \mu_{N-1}$, then

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \lambda_3 < \ldots < \mu_{N-1} < \lambda_N.$$

Thus, the two sequences $\{\lambda_j\}_{j=1}^N$ and $\{\mu_j\}_{j=1}^{N-1}$ satisfy the necessary and sufficient condition of solvability of the inverse problem from two-spectra (see previous Section 2). Therefore there exists a unique Jacobi matrix J of the form (1) with entries in the class (2) such that λ_j $(j = 1, \ldots, N)$ are the eigenvalues of J and μ_j $(j = 1, \ldots, N-1)$ are the eigenvalues of J_1 of the form (14) obtained from J by deleting its first row and first column, so that

$$A(\lambda) = \det(\lambda I - J), \quad B(\lambda) = \det(\lambda I - J_1). \qquad (37)$$

Next, using the constructed matrix J and the number ω defined by (34) we construct the matrix $\widetilde{J}$ of the form (3) with (4). It remains to show that $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ are eigenvalues of the constructed matrix $\widetilde{J}$ counting algebraic multiplicity.

By the direct spectral problem for the constructed matrix $\widetilde{J}$, we have (Lemma 5), taking into account (37),

$$\det\left(\lambda I - \widetilde{J}\right) = A(\lambda) - i\omega B(\lambda).$$

Comparing this with (36), we get that

$$\det\left(\lambda I - \widetilde{J}\right) = P(\lambda) = (\lambda - \widetilde{\lambda}_1) \cdots (\lambda - \widetilde{\lambda}_N)$$

which shows that $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ are the eigenvalues of $\widetilde{J}$ counting algebraic multiplicity. $\square$

4. Reconstruction procedure

The proof of Theorem 7 gives the following algorithm for reconstruction of the dissipative Jacobi matrix $\widetilde{J}$ from its eigenvalues $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$.

If we are given the N not necessarily distinct complex numbers $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ in the open upper half-plane, then using these numbers we form the number $\omega > 0$ by (34) and the polynomial $P(\lambda)$ by (35). Next, we find the roots $\lambda_1, \ldots, \lambda_N$ of the polynomial $P^{re}(\lambda)$ and the roots $\mu_1, \ldots, \mu_{N-1}$ of the polynomial $P^{im}(\lambda)$ and solve the inverse problem from the two-spectra $\{\lambda_j\}_{j=1}^N$, $\{\mu_j\}_{j=1}^{N-1}$ to get a unique real Jacobi matrix J of the form (1) with entries in the class (2). To do so, using $\{\lambda_j\}_{j=1}^N$ and $\{\mu_j\}_{j=1}^{N-1}$ we construct the numbers β_k $(k = 1, \ldots, N)$ by (16) and then the numbers s_l $(l = 0, 1, \ldots)$ and the determinants D_n $(n = 0, 1, \ldots)$ by (10) and (11), respectively. Then we define the numbers a_n $(n = 0, 1, \ldots, N-2)$ and b_n $(n = 0, 1, \ldots, N-1)$ by (12) and (13), respectively. Using these a_n's, b_n's, and the number ω, the required dissipative Jacobi matrix $\widetilde{J}$ for which given $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ are the eigenvalues is obtained by formula (3) with (4).

Let us demonstrate this procedure of reconstruction of $\widetilde{J}$ by the following example (this example has been considered before in [1]).

Example 8. *Find the matrix $\widetilde{J}$ of the form (3) with $N = 3$ and entries in the class (2), (4) if the eigenvalues of $\widetilde{J}$ are $\widetilde{\lambda}_1 = \widetilde{\lambda}_2 = i$, $\widetilde{\lambda}_3 = 2i$.*

First we find by (34) that $\omega = 4$. Next we construct the polynomial

$$P(\lambda) = (\lambda - i)^2(\lambda - 2i) = \lambda^3 - 4i\lambda^2 - 5\lambda + 2i$$

and find its real and imaginary parts:

$$P^{re}(\lambda) = \lambda^3 - 5\lambda, \quad P^{im}(\lambda) = -4\lambda^2 + 2.$$

The roots of $P^{re}(\lambda)$ are $\lambda_1 = -\sqrt{5},\ \lambda_2 = 0,\ \lambda_3 = \sqrt{5}$ and the roots of $P^{im}(\lambda)$ are $\mu_1 = -(\sqrt{2})^{-1},\ \mu_2 = (\sqrt{2})^{-1}$. Therefore

$$\beta_1 = \frac{(\lambda_1 - \mu_1)(\lambda_1 - \mu_2)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} = \frac{9}{20},$$

$$\beta_2 = \frac{(\lambda_2 - \mu_1)(\lambda_2 - \mu_2)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} = \frac{1}{10}, \quad \beta_3 = \frac{(\lambda_3 - \mu_1)(\lambda_3 - \mu_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} = \frac{9}{20}.$$

Next, from

$$s_0 = \beta_1 + \beta_2 + \beta_3 = 1,$$

$$s_l = \beta_1\lambda_1^l + \beta_2\lambda_2^l + \beta_3\lambda_3^l = \frac{9}{20}\left[\left(-\sqrt{5}\right)^l + \left(\sqrt{5}\right)^l\right], \quad l = 1, 2, \ldots,$$

we find that

$$s_0 = 1, \quad s_1 = 0, \quad s_2 = \frac{9}{2}, \quad s_3 = 0, \quad s_4 = \frac{45}{2}, \quad s_5 = 0.$$

Hence

$$D_{-1}=1,\quad D_0=s_0=1,\quad D_1=\begin{vmatrix} s_0 & s_1\\ s_1 & s_2\end{vmatrix}=\begin{vmatrix} s_0 & 0\\ 0 & s_2\end{vmatrix}=\frac{9}{2},$$

$$D_2=\begin{vmatrix} s_0 & s_1 & s_2\\ s_1 & s_2 & s_3\\ s_2 & s_3 & s_4\end{vmatrix}=\begin{vmatrix} s_0 & 0 & s_2\\ 0 & s_2 & 0\\ s_2 & 0 & s_4\end{vmatrix}=s_0s_2s_4-s_2^3=\frac{81}{8},$$

$$\Delta_{-1}=0,\quad \Delta_0=s_1=0,\quad \Delta_1=\begin{vmatrix} s_0 & s_2\\ s_1 & s_3\end{vmatrix}=\begin{vmatrix} s_0 & s_2\\ 0 & 0\end{vmatrix}=0,$$

$$\Delta_2=\begin{vmatrix} s_0 & s_1 & s_3\\ s_1 & s_2 & s_4\\ s_2 & s_3 & s_5\end{vmatrix}=\begin{vmatrix} s_0 & 0 & 0\\ 0 & s_2 & s_4\\ s_2 & 0 & 0\end{vmatrix}=0$$

and, therefore,

$$a_0=\frac{\sqrt{D_{-1}D_1}}{D_0}=\sqrt{D_1}=\frac{3}{\sqrt{2}},\quad a_1=\frac{\sqrt{D_0D_2}}{D_1}=\frac{\sqrt{D_2}}{D_1}=\frac{1}{\sqrt{2}},$$

$$b_0=\frac{\Delta_0}{D_0}-\frac{\Delta_{-1}}{D_{-1}}=0,\quad b_1=\frac{\Delta_1}{D_1}-\frac{\Delta_0}{D_0}=0,\quad b_2=\frac{\Delta_2}{D_2}-\frac{\Delta_1}{D_1}=0.$$

Thus, we have found

$$\widetilde{J}=\begin{bmatrix} b_0+i\omega & a_0 & 0\\ a_0 & b_1 & a_1\\ 0 & a_1 & b_2\end{bmatrix}=\begin{bmatrix} 4i & \frac{3}{\sqrt{2}} & 0\\ \frac{3}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & 0\end{bmatrix}.$$

5. Effectivization of the reconstruction procedure

In the reconstruction procedure given in Section 4, we need to find the zeros of the polynomials $P^{re}(\lambda)$ and $P^{im}(\lambda)$ for

$$P(\lambda)=(\lambda-\widetilde{\lambda}_1)\cdots(\lambda-\widetilde{\lambda}_N)=(\lambda-\widetilde{\lambda}_1)^{m_1}\cdots(\lambda-\widetilde{\lambda}_p)^{m_p}, \tag{38}$$

where in the end expression in (38) $\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_p$ denote all the distinct zeros of the polynomial $P(\lambda)$ and $m_1,\ldots,m_p$ denote their multiplicities, respectively, so $m_1+\ldots+m_p=N$. Since, in general, it is impossible to find explicitly the zeros of a polynomial of large degree, in this section, we offer another reconstruction procedure which is free of this difficulty. For this aim we can use the reconstruction procedure in the inverse problem from eigenvalues and normalizing numbers for finite complex Jacobi matrices, given in [8, Section 2], as follows.

Because for the matrix $\widetilde{J}$ we have (32), the Weyl-Titchmarsh function $w(\lambda)$ of the matrix $\widetilde{J}$ is expressed in the form

$$w(\lambda)=\frac{\left[\det\left(\lambda I-\widetilde{J}\right)\right]^{im}}{-\omega\det\left(\lambda I-\widetilde{J}\right)}.$$

Consequently, we get the following algorithm for reconstruction of the dissipative Jacobi matrix $\widetilde{J}$ from its eigenvalues $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$.

If we are given the N not necessarily distinct complex numbers $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ in the open upper half-plane, then using these numbers we form the number $\omega > 0$ by

$$\omega = \sum_{j=1}^{N} \mathrm{Im}\widetilde{\lambda}_j$$

and the polynomial $P(\lambda)$ by (38). Then we form the function

$$w(\lambda) = \frac{P^{im}(\lambda)}{-\omega P(\lambda)}.$$

Decomposing this function into partial fractions of the form

$$w(\lambda) = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{\beta_{kj}}{(\lambda - \lambda_k)^j},$$

we find the numbers β_{kj}. Then we define the numbers a_n and b_n by (12) and (13), respectively, where, however, now the numbers s_l are defined by the formula

$$s_l = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \binom{l}{j-1} \beta_{kj} \lambda_k^{l-j+1}, \quad l = 0, 1, 2, \ldots,$$

instead of (10), where $\binom{l}{j-1}$ is a binomial coefficient and we put $\binom{l}{j-1} = 0$ if $j-1 > l$. Using the obtained a_n's, b_n's, and the number ω, the required dissipative Jacobi matrix $\widetilde{J}$ for which given $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_N$ are the eigenvalues is obtained by formula (3) with (4).

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