ON BOREL CONVERGENCE OF DOUBLE SEQUENCES

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ABSTRACT. In this paper, we introduce the concept of (Ber)-convergence of bounded double sequences in the Fock space $\mathcal{F}\left(\mathbb{C}^2\right)$. We show that every (Ber)-convergent double sequence is Borel convergent. Namely, we prove the following theorem by using the Berezin symbol method: If the $\{x_{ij}\}_{i,j=0}^{\infty}$ is regularly convergent to x, then

$$\lim_{k,t\to\infty} e^{-k-t} \sum_{i,j=0}^{\infty} x_{ij} \frac{k^i t^j}{i!j!} = x.$$

1. INTRODUCTION

Recall that a double sequence $\{x_{ij}\}_{i,j=0}^{\infty}$ is said to be convergent in Pringsheim's sense [7] if there exists a number x such that x_{ij} converges to x as both i and j tend to infinity independently of one another

$$\lim_{i,j\to\infty} x_{ij} = x,$$

that is, if for every $\varepsilon > 0$ there exists $N = N\left(\varepsilon\right) \in \mathbb{N}$ such that $|x_{ij} - x| < \varepsilon$ for every $i, j \geq N$ and also x is said to the Pringsheim's limit of x_{ij}. A double sequence $\{x_{ij}\}$ is said to be a Cauchy sequence in Pringsheim's sense if and only if for every $\varepsilon > 0$ there exists an integer $N = N\left(\varepsilon\right) \in \mathbb{N}$ such that $|x_{ij} - x_{mn}| < \varepsilon$ for $\min\{m,n,i,j\} \geq N$. It is obvious that a double sequence is a Cauchy sequence if and only if it is convergent. A double sequence $\{x_{ij}\}$ is bounded if there exists a positive number K such that $|x_{ij}| \leq K$ for every i and j, i.e., $\sup_{i,j} |x_{ij}| < \infty$.

A double sequence $\{x_{ij}\}$ is said to be regularly convergent if it is convergent in Pringsheim's sense and the following limits hold:

$$\lim_{i\to\infty} x_{ij} = x_j, \ (j = 1, 2, ...) \text{ and } \lim_{j\to\infty} x_{ij} = x_i, \ (i = 1, 2, ...).$$

As it is known that, a convergent double sequence in Pringsheim's sense fails in general to be bounded. The concept of regular convergence, which was introduced by Hardy in [4], lacks this advantage. In addition to the Pringsheim's convergence, the regular convergence requires the convergence of rows and columns of a double sequence. (One can find more information about several type convergence for double sequences in [11] and its references.)

A reproducing kernel Hilbert space (shorty, RKHS) $\mathcal{H} = \mathcal{H}(\Omega)$ on some set Ω is a Hilbert space of functions on Ω such that for every $\lambda \in \Omega$ the linear functional (evaluation functional) $f \to f(\lambda)$ is bounded on $\mathcal{H}$. If $\mathcal{H}$ is RKHS on set Ω, then by the classical Riesz Representation Theorem for every $\lambda \in \Omega$ there is a unique element $k_\lambda \in \mathcal{H}$ for which $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The function k_λ is called the reproducing kernel at λ. It is well known that (see, Aronzajn [1] and Saitoh [8]) if $(e_j)_{j \in J}$ is an orthonormal basis for the RKHS $\mathcal{H}$, then

$$k_\lambda = \sum_{j \in J}^{\infty} \overline{e_j(\lambda)} e_j,$$

where the convergence is in $\mathcal{H}$. In particular,

$$k_\lambda(z) = \sum_{j \in J}^{\infty} \overline{e_j(\lambda)} e_j(z), \quad z \in \Omega.$$

The function

$$\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}} = \frac{1}{\left(\sum_{j \in J} |e_j(\lambda)|^2 \right)^{1/2}} \sum_{j \in J} \overline{e_j(\lambda)} e_j(z)$$

is called the normalized reproducing kernel at λ.

The Berezin symbol $\widetilde{A}$ of a bounded linear operator A on $\mathcal{H}$ is defined by the formula (see [2])

$$\widetilde{A}(\lambda) := < A\widehat{k}_\lambda, \widehat{k}_\lambda >, \ \lambda \in \Omega.$$

The Berezin symbol is a bounded function by the norm of the operator. It is obvious that every bounded operator on the most familiar RKHS is uniquely determined by its Berezin symbol. So, by finding the corresponding Berezin symbol, the behavior of the operator can be analyzed.

Following Nordgren and Rosenthal [6], we say that RKHS $\mathcal{H}(\Omega)$ is standard if the underlying set Ω is a subset of a topological space and the boundary $\partial\Omega$ is non-empty and has the property that $(k_{\mathcal{H},\lambda_n})_n$ weakly converges to 0, whenever $(\lambda_n)_n$ is a sequence in Ω that converges to a point in $\partial\Omega$. The prototypical standard RKHSs are, for example, Hardy-Hilbert space $H^2(\mathbb{D})$ over the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, Fock-Hilbert spaces $\mathcal{F}(\mathbb{C})$ and $\mathcal{F}(\mathbb{C}^2) = \mathcal{F}(\mathbb{C} \times \mathbb{C})$.

For any compact operator K on the standard RKHS $\mathcal{H}$, it is clear that

$$\lim_{n \to \infty} \widetilde{K}(\lambda_n) = 0,$$

whenever $(\lambda_n)_n \subset \Omega$ converges to a point of $\partial\Omega$. In this case, the Berezin symbol of a compact operator on a standard RKHS vanishes on the boundary.

The normalized reproducing kernel of the Fock space $\mathcal{F}(\mathbb{C}^2)$ on the $\mathbb{C}^2$ is the following:

$$\widehat{k}_{\lambda,\mu}(z,w) = e^{-|\lambda|^2/2 - |\mu|^2/2} \sum_{m,n=0}^{\infty} \frac{\overline{\lambda}^m \overline{\mu}^n z^m w^n}{(m!2^m)(n!2^n)}.$$

The diagonal operator on the Fock space $\mathcal{F}(\mathbb{C}^2)$ is defined by

$$D_{\{x_{ij}\}} z^i w^j = x_{ij} z^i w^j \quad (i,j = 0,1,2,...).$$

Definition 1. *The bounded sequence $\{x_{ij}\}_{i,j=0}^{\infty}$ is (Ber)-convergent to the number x, if the Berezin symbol $\widetilde{D}_{\{x_{ij}-x\}}(\lambda,\mu)$ of the corresponding diagonal operator $D_{\{x_{ij}-x\}}$ in the Fock space $\mathcal{F}(\mathbb{C}^2)$ tends to zero as (λ,μ) tends to the boundary of $\mathbb{C}^2$.*

Since $\mathcal{F}(\mathbb{C}^2)$ is a standard RKHS, it is obvious that every bounded and convergent double sequence in Pringsheim's sense is (Ber)-convergent; in particular, every regularly convergent double sequence is (Ber)-convergent.

It was shown in [3, 10] that for the single sequence (Ber)-convergence implies Borel convergence. Also, Karaev and Zelster in [5] showed that for the double sequence (Ber)-convergence implies Abel convergence.

The double sequence $\{x_{ij}\}_{i,j=0}^{\infty}$ is Borel convergent to x if $\displaystyle\sum_{i,j=0}^{\infty} x_{ij} \frac{k^i t^j}{i!j!}$ converges for all $k,t \in \mathbb{R}^+$ and

$$\lim_{k,t \to \infty} e^{-k-t} \sum_{i,j=0}^{\infty} x_{ij} \frac{k^i t^j}{i!j!} = x$$

(for more information about Borel summability, see [9]).

In this article, we will prove Borel type theorem for the double sequences (x_{ij}) of complex numbers by using Berezin symbols technique. Namely, we will show that every (Ber)-convergent double sequence is Borel convergent.

2. Main results

The main goal is to prove the following result.

Theorem 2. *If $\{x_{ij}\}_{i,j=0}^{\infty}$ is (Ber)-convergent to x, then $\{x_{ij}\}_{i,j=0}^{\infty}$ is Borel convergent to x.*

Proof. Since $\{x_{ij}\}$ (Ber)-converges to x, $\{x_{ij}\}$ is a bounded double sequence. So, the diagonal operator $D_{\{x_{ij}\}}$ on $\mathcal{F}(\mathbb{C}^2)$ is a bounded operator on the Fock space $\mathcal{F}(\mathbb{C}^2)$ over the $\mathbb{C}^2$. Now, we calculate the Berezin symbol of an operator $D_{\{x_{ij}\}}$:

$$
\begin{aligned}
\widetilde{D}_{\{x_{ij}\}}(\lambda,\mu) &= \left\langle D_{\{x_{ij}\}}\widehat{k}_{\lambda,\mu}, \widehat{k}_{\lambda,\mu}\right\rangle \\
&= e^{-|\lambda|^2/2-|\mu|^2/2}\left\langle D_{\{x_{ij}\}}\sum_{m,n=0}^{\infty}\frac{\overline{\lambda}^m\overline{\mu}^n z^m w^n}{(m!2^m)(n!2^n)}, \sum_{m,n=0}^{\infty}\frac{\overline{\lambda}^m\overline{\mu}^n z^m w^n}{(m!2^m)(n!2^n)}\right\rangle \\
&= e^{-|\lambda|^2/2-|\mu|^2/2}\left\langle \sum_{m,n=0}^{\infty}x_{mn}\frac{\overline{\lambda}^m\overline{\mu}^n z^m w^n}{(m!2^m)(n!2^n)}, \sum_{m,n=0}^{\infty}\frac{\overline{\lambda}^m\overline{\mu}^n z^m w^n}{(m!2^m)(n!2^n)}\right\rangle \\
&= e^{-|\lambda|^2/2-|\mu|^2/2}\sum_{m,n=0}^{\infty}x_{mn}\frac{\left(|\lambda|^2/2\right)^m}{m!}\frac{\left(|\mu|^2/2\right)^n}{n!},
\end{aligned}
$$

and hence,

$$
\widetilde{D}_{\{x_{ij}\}}(\lambda,\mu) = e^{-|\lambda|^2/2-|\mu|^2/2}\sum_{m,n=0}^{\infty}x_{mn}\frac{\left(|\lambda|^2/2\right)^m}{m!}\frac{\left(|\mu|^2/2\right)^n}{n!}.
$$

Thus $\widetilde{D}_{\{x_{ij}\}}$ is a radial function, that is, $\widetilde{D}_{\{x_{ij}\}}(\lambda,\mu) = \widetilde{D}_{\{x_{ij}\}}(|\lambda|,|\mu|)$.

By setting $k = \dfrac{|\lambda|^2}{2}$ and $t = \dfrac{|\mu|^2}{2}$, we get

$$
\widetilde{D}_{\{x_{ij}\}}\left(\sqrt{2k},\sqrt{2t}\right) = e^{-k-t}\sum_{m,n=0}^{\infty}x_{mn}\frac{k^m t^n}{m!n!}. \tag{1}
$$

Since $\{x_{ij}\}$ is a bounded sequence, it is obvious that $\sum_{m,n=0}^{\infty}x_{mn}\dfrac{k^m t^n}{m!n!}$ converges for all $k,t \in \mathbb{R}^+$. Then again, it follows from (1) that

$$
\begin{aligned}
e^{-k-t}\sum_{m,n=0}^{\infty}x_{mn}\frac{k^m t^n}{m!n!} &= e^{-k-t}\sum_{m,n=0}^{\infty}(x_{mn}-x+x)\frac{k^m t^n}{m!n!} \\
&= e^{-k-t}\sum_{m,n=0}^{\infty}(x_{mn}-x)\frac{k^m t^n}{m!n!} + x \\
&= \widetilde{D}_{\{x_{ij}-x\}}(k,t) + x.
\end{aligned}
$$

Since $\{x_{ij}\}_{i,j=0}^{\infty}$ is (Ber)-convergent, $\widetilde{D}_{\{x_{ij}-x\}}$ vanishes on the boundary, hence we have

$$
\lim_{k,t\to\infty}e^{-k-t}\sum_{i,j=0}^{\infty}x_{ij}\frac{k^i t^j}{i!j!} = x.
$$

Therefore, the proof is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 3. *If $\{x_{ij}\}_{i,j=0}^{\infty}$ is regularly convergent to x, then $\{x_{ij}\}_{i,j=0}^{\infty}$ is Borel convergent to x.*

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