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Energy Decay of Damped System of Wave Equations in $\mathbb{R}^n$ via Fourier Transform

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Abstract: The wave equation with fading memory, in bounded domain, has been deeply studied by several authors. Here, we establish a decay results for a class of weak-viscoelastic wave system in $\mathbb{R}^n$. We are going to construct an appropriate Lyapunov function associate with our main system by taking the Fourier transform. The main question here is: If the dissipation given by the weak-viscoelasticity is strong enough to drag a rate of decay for the whole system, what type of rate of decay can we expect?

1. Introduction and Related Results

In recent years, Fourier transform method have become central to the study of theoretical and applied mathematical problems in any space dimension. An advantage of such an approach is its generality and its potential unifying effect of particular results and techniques.
In this article, we consider a linear evolution problem in the weak-viscoelastic as follows

$$\begin{cases} u'' - \Delta\Big(u + u' - \alpha_1(t)\int_0^t g(t-s)u(s,x)ds\Big) + \gamma v = 0, \\ v'' - \Delta\Big(v + v' - \alpha_2(t)\int_0^t h(t-s)v(s,x)ds\Big) + \gamma u = 0, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^n), u'(0,x) = u_1(x) \in L^2(\mathbb{R}^n), \\ v(0,x) = v_0(x) \in H^1(\mathbb{R}^n), v'(0,x) = v_1(x) \in L^2(\mathbb{R}^n), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n, t \in \mathbb{R}_*^+, n \geq 2, \gamma \neq 0$.
It is well known that, as in one equation, the coupled system with the presence of a viscoelastic terms with and without the functions α_1, α_2 does not preclude the question of existence, but its effects are on the stability of the existing solution, we refer the reader to works in [9, 11–13, 17–19].
The energy of (u,v) at time t is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\Big(\|u'\|_2^2 + \|v'\|_2^2\Big) + \frac{1}{2}\left(1 - \alpha_1(t)\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}\alpha_1(t)(g \circ \nabla u) \\ &+ \frac{1}{2}\left(1 - \alpha_2(t)\int_0^t h(s)ds\right)\|\nabla v\|_2^2 + \frac{1}{2}\alpha_2(t)(h \circ \nabla v) + \gamma\int_{\mathbb{R}^n} uv dx. \end{aligned} \tag{2}$$

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and

$$\begin{aligned} E'(t) \quad \leq \quad & \frac{1}{2}\alpha_1(t)(g'\circ\nabla u) - \frac{1}{2}\alpha_1(t)g(t)\|\nabla u\|_2^2 + \frac{1}{2}\alpha_1'(t)(g\circ\nabla u) - \frac{1}{2}\alpha_1'(t)\int_0^t g(s)ds\|\nabla u\|_2^2 \\ + \quad & \frac{1}{2}\alpha_2(t)(h'\circ\nabla v) - \frac{1}{2}\alpha_2(t)h(t)\|\nabla v\|_2^2 + \frac{1}{2}\alpha_2'(t)(h\circ\nabla v) - \frac{1}{2}\alpha_2'(t)\int_0^t h(s)ds\|\nabla v\|_2^2. \end{aligned} \tag{3}$$

Noting by

$$(f\circ\Psi) = \int_0^t f(t-\tau)\,\|\Psi(t)-\Psi(\tau)\|_2^2\,d\tau, \text{ for any } \Psi\in L^\infty(0,T;L^2(\mathbb{R}^n)) \tag{4}$$

This type of problems is usually encountered in viscoelasticity in various areas of mathematical physics. For the literature, in $\mathbb{R}^n$ we quote the results of [1, 2, 5–8, 10, 18, 19].
In [1], the authors considered the following Petrowsky-Petrowsky system

$$\begin{cases} u_{tt} + \phi(x)\left(\Delta^2 u - \int_{-\infty}^t \mu(t-s)\Delta^2 u(s)ds\right) + \alpha v = 0 & \text{in } \mathbb{R}^n\times\mathbb{R}^+ \\ v_{tt} + \phi(x)\Delta^2 v + \alpha u = 0 & \text{in } \mathbb{R}^n\times\mathbb{R}^+ \\ (u_0,v_0)\in D^{2,2}(\mathbb{R}^n), \quad (u_1,v_1)\in L_g^2(\mathbb{R}^n) \end{cases} \tag{5}$$

This research described a polynomial decay rate of solution for a coupled system of Petrowsky equations in $\mathbb{R}^n$ with infinite memory acting in the first equation. The main contributions was to show that the infinite memory lets the problem (5) still dissipative and that the system is not exponentially stable in spite of the kernel in the memory term is sub-exponential.
In [18], in order to compensate the lack of Poincare's inequality in $\mathbb{R}^n$ and for wider class of relaxation functions, the author looked into a following linear equation

$$\rho(x)\left(|u'|^{q-2}u'\right)' - M(\|\nabla_x u\|_2^2)\Delta_x u + \int_0^t g(t-s)\Delta_x u(s)ds = 0, x\in\mathbb{R}^n, t>0 \tag{6}$$

where $q,n\geq 2$ and M is a positive C^1 function satisfying for $s\geq 0, m_0>0, m_1\geq 0, \gamma\geq 1, M(s)=m_0+m_1 s^\gamma$. The author used weighted spaces to establish a very general decay rate of solutions of (6). The same results obtained later by [19], where a semi-linear viscoelastic wave equation in any spaces dimension was considered in

$$u'' - \phi(x)\left(\Delta_x u - \int_0^t g(t-s)\Delta_x u(s)ds\right) + au' = u|u|^{p-1} \tag{7}$$

Our main contribution here is to establish a new decay results for a class of coupled system of weak-viscoelastic wave equations in $\mathbb{R}^n$, by introducing a suitable Lyapunov function.

2. Assumptions

We need the following assumption on the relaxation functions:
$g,h,\alpha_1,\alpha_2 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are non-increasing differentiable functions of class C^1 satisfying:

$$1-\alpha_1(t)\int_0^t g(s)ds \geq k_1 > 0, \quad g(0)=g_0>0, \infty > \int_0^\infty g(t)dt, \tag{8}$$

$$1-\alpha_2(t)\int_0^t h(s)ds \geq k_2 > 0, \quad h(0)=h_0>0, \infty > \int_0^\infty h(t)dt, \tag{9}$$

In addition, there exists two positive nonincreasing differentiable functions β_1,β_2 satisfying

$$g'(t)+\beta_1(t)g(t)\leq 0, \quad \forall t\geq 0, \quad \lim_{t\to\infty} = \frac{\alpha_1'(t)}{\beta_1(t)\alpha_1(t)} = 0. \tag{10}$$

$$h'(t)+\beta_2(t)h(t)\leq 0, \quad \forall t\geq 0, \quad \lim_{t\to\infty} = \frac{\alpha_2'(t)}{\beta_2(t)\alpha_2(t)} = 0. \tag{11}$$

There are many functions satisfying (8)-(11), for example

$$\begin{aligned} h_1(t) &= \frac{a}{(1+t)^{\nu}}, \nu > 1, \\ h_2(t) &= aExp[-b(1+t)^q], 0<q<1, \\ h_2(t) &= \frac{a}{(1+t)[ln(1+t)]^{\nu}}, \nu > 1. \end{aligned}$$

We give some notations to be used below. Let F denote the Fourier transform in $L^2(\mathbb{R}^n)$ defined as follows:

$$F[f](\xi) = \widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(-ix.\xi) f(x) dx. \tag{12}$$

Here ξ is the variable associated with the Fourier transform, where $i = \sqrt{-1}$, $\quad x.\xi = \sum_{i=1}^{n} x_i \xi_i$ and denote its inverse transform by F^{-1}. The operator $-\Delta$ is defined by

$$-\Delta v(x) = F^{-1}\left(|\xi|^2 F(v)(\xi)\right)(x), v \in H^2(\mathbb{R}^n), x \in \mathbb{R}^n.$$

For $1 \leq p \leq \infty$, we denote by $L^p(\mathbb{R}^n)$ the usual Lebesgue space on $\mathbb{R}^n$ with the norm $\|.\|_{L^p}$. For a nonnegative integer $m, H^m(\mathbb{R}^n)$ denotes the Sobolev space of $L^2(\mathbb{R}^n)$ functions on $\mathbb{R}^n$, equipped with the norm $\|.\|_{H^m}$.

Lemma 2.1. *([17], Lemma 2.1) For any two functions $g \in C^1(\mathbb{R}), v \in W^{1,2}(0,T)$, it holds that*

$$\begin{aligned} Re\left\{ \alpha(t) \int_0^t g(t-s)v(s)ds v'(t) \right\} &= -\frac{1}{2}\alpha(t)g(t)|v(t)|^2 + \frac{1}{2}\alpha(t)(g' \circ v)(t) \\ &- \frac{1}{2}\frac{d}{dt}\alpha(t)(g\circ v)(t) + \frac{1}{2}\frac{d}{dt}\alpha(t)\int_0^t g(s)ds|v(t)|^2 \\ &+ \frac{1}{2}\alpha'(t)(g\circ v)(t) - \frac{1}{2}\alpha'(t)\int_0^t g(s)ds|v|^2 \end{aligned} \tag{13}$$

and

$$|\int_0^t g(t-s)(v(s)-v(t))ds|^2 \leq \int_0^t |g(s)|ds \int_0^t |g|(t-s)|v(t)-v(s)|^2 ds$$

Finally, we give the definition of *weak solutions* for the problem (1).

Definition 2.2. A weak solution of (1) is (u,v) such that

- $(u,v) \in \left(C([0,T);H^1(\mathbb{R}^n))\right)^2, \quad (u',v') \in \left(C^1([0,T);L^2(\mathbb{R}^n))\right)^2$
- For all $w_1, w_2 \in C_0^\infty([0,T]\times\mathbb{R}^n)$, (u,v) satisfies the generalized formula:

$$\begin{aligned} 0 &= \int_0^T (u'',w_1)_{L^2} ds + \int_0^T \int_{\mathbb{R}^n} \nabla u \nabla w_1 dxds + \int_0^T \int_{\mathbb{R}^n} \nabla u' \nabla w_1 dxds \\ &- \int_0^T \int_{\mathbb{R}^n} \alpha_1(t) \int_0^s g(s-\tau)\nabla u(\tau) d\tau \nabla w_1(s) dxds + \gamma \int_0^T \int_{\mathbb{R}^n} v w_1 dxds, \end{aligned}$$

$$\begin{aligned} 0 &= \int_0^T (v'',w_2)_{L^2} ds + \int_0^T \int_{\mathbb{R}^n} \nabla v \nabla w_2 dxds + \int_0^T \int_{\mathbb{R}^n} \nabla v' \nabla w_2 dxds \\ &- \int_0^T \int_{\mathbb{R}^n} \alpha_2(t) \int_0^s h(s-\tau)\nabla v(\tau) d\tau \nabla w_2(s) dxds + \gamma \int_0^T \int_{\mathbb{R}^n} u w_2 dxds, \end{aligned} \tag{14}$$

- (u,v) satisfies the initial conditions
  $(u_0(x), v_0(x)) \in \left(H^1(\mathbb{R}^n)\right)^2, \quad (u_1(x), v_1(x)) \in \left(L^2(\mathbb{R}^n)\right)^2.$

We can now state and prove the asymptotic behavior of the solution of (1). Throughout this paper, let us set $\widehat{u}(t,\xi) = F(u(t,.))(\xi), \quad \widehat{v}(t,\xi) = F(v(t,.))(\xi)$.

3. Main Result

We shall show that our solution decays time asymptotically to zero and the decay rate of solution is fast and similar to both α and g,h, where $\alpha(t)=\min\{\alpha_1(t),\alpha_2(t)\}$

Theorem 3.1. *Assume (u,v) is the solution of (1), then the next general exponential estimate satisfies in the Fourier space*

$$E(\xi,t)\leq W\exp\left(-\omega\int_0^t \alpha(s)\beta(s)ds\right)E(\xi,0),\qquad \forall t\geq 0. \tag{15}$$

for some positive constants W,ω, where $\beta(t)=\min\{\beta_1(t),\beta_2(t)\}$.

Proof. We take the Fourier transform of both sides of (1). Then one has the reduced equation for $\xi\in\mathbb{R}^n, t\in\mathbb{R}^+_*$:

$$\begin{cases} \widehat{u}''(t,\xi)+|\xi|^2\Big(\widehat{u}(t,\xi)-\alpha_1(t)\int_0^t g(t-s)\widehat{u}(s,\xi)ds+\widehat{u}'(t,\xi)\Big)+\gamma\widehat{v}(t,\xi)=0\\ \widehat{v}''(t,\xi)+|\xi|^2\Big(\widehat{v}(t,\xi)-\alpha_2(t)\int_0^t h(t-s)\widehat{v}(s,\xi)ds+\widehat{v}'(t,\xi)\Big)+\gamma\widehat{u}(t,\xi)=0\\ (\widehat{u}(0,\xi),\widehat{v}(0,\xi))=(\widehat{u}_0(\xi),\widehat{v}_0(\xi))\in(H^1(\mathbb{R}^n))^2,\\ (\widehat{u}'(0,\xi),\widehat{v}'(0,\xi))=(\widehat{u}_1(\xi),\widehat{v}_1(\xi))\in(L^2(\mathbb{R}^n))^2. \end{cases} \tag{16}$$

We apply the multiplier techniques in Fourier transform, for this purpose, we shall also need to proceed in three steps. First, to derive the equality for the physical energy, we multiply the first equation of (16) by $\overline{\widehat{u}'}$ and the second by $\overline{\widehat{v}'}$. Then, summing ant taking the real part of the resulting identities, we obtain

$$\begin{aligned} E_1(t) &= \frac{1}{2}\Big(|\widehat{u}'|^2+|\widehat{v}'|^2\Big)+\frac{1}{2}|\xi|^2\Big((1-\alpha_1(t)\int_0^t g(s)ds)|\widehat{u}|^2+\alpha_1(t)(g\circ\widehat{u})(t)\Big)\\ &+ \frac{1}{2}|\xi|^2\Big((1-\alpha_2(t)\int_0^t h(s)ds)|\widehat{u}|^2+\alpha_2(t)(h\circ\widehat{u})(t)\Big)+\gamma|\widehat{u}\widehat{v}| \end{aligned}$$

and

$$\begin{aligned} e_1(t) &= \frac{1}{2}|\xi|^2\Big(\alpha_1(t)g(t)|\widehat{u}|^2-\alpha_1(t)(g'\circ\widehat{u})(t)+2|\widehat{u}'|^2+\alpha_1'(t)(g\circ\widehat{u})(t)-\alpha_1'(t)\int_0^t g(s)ds|\widehat{u}|^2\Big)\\ &+ \frac{1}{2}|\xi|^2\Big(\alpha_2(t)h(t)|\widehat{u}|^2-\alpha_2(t)(h'\circ\widehat{u})(t)+2|\widehat{u}'|^2+\alpha_2'(t)(h\circ\widehat{u})(t)-\alpha_2'(t)\int_0^t h(s)ds|\widehat{u}|^2\Big). \end{aligned}$$

Then,

$$\frac{d}{dt}E_1(t)+e_1(t)=0. \tag{17}$$

Second, the existence of the memory terms forces us to make the first modification of the energy by multiplying the first equation of (16) by $\Big(-\frac{d}{dt}\Big(\alpha_1(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\Big)$ and the second by $\Big(-\frac{d}{dt}\Big(\alpha_2(t)\int_0^t h(t-s)\overline{\widehat{v}}(s)ds\Big)\Big)$. Summing and taking the real part, we have that

$$\begin{aligned} 0 &= -Re\left\{\widehat{u}''\frac{d}{dt}\Big(\alpha_1(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\ &- Re\left\{\widehat{v}''\frac{d}{dt}\Big(\alpha_2(t)\int_0^t h(t-s)\overline{\widehat{v}}(s)ds\Big)\right\}\\ &- Re\left\{|\xi|^2\widehat{u}\frac{d}{dt}\Big(\alpha_1(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)+|\xi|^2\widehat{v}\frac{d}{dt}\Big(\alpha_2(t)\int_0^t h(t-s)\overline{\widehat{v}}(s)ds\Big)\right\}\\ &+ \frac{1}{2}|\xi|^2\frac{d}{dt}\left(\left(\Big|\alpha_1(t)\int_0^t g(t-s)\widehat{u}(s)ds\Big|^2\right)+\left(\Big|\alpha_2(t)\int_0^t h(t-s)\widehat{v}(s)ds\Big|^2\right)\right)\\ &- |\xi|^2 Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha_1(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\widehat{v}'\frac{d}{dt}\Big(\alpha_2(t)\int_0^t h(t-s)\overline{\widehat{v}}(s)ds\Big)\right\}\\ &+ \gamma Re\left\{\widehat{v}\frac{d}{dt}\Big(\alpha_1(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)+\widehat{u}\frac{d}{dt}\Big(\alpha_2(t)\int_0^t h(t-s)\overline{\widehat{v}}(s)ds\Big)\right\} \end{aligned} \tag{18}$$

Since, for $\alpha=\alpha_1,\alpha_2, f=g,h$

$$\begin{aligned}\frac{d}{dt}\left(\alpha(t)\int_0^t f(t-s)\bar{\hat{u}}(s)ds\right) &= \alpha'(t)\int_0^t f(t-s)\bar{\hat{u}}(s)ds+\alpha(t)\frac{d}{dt}\left(\int_0^t f(t-s)\bar{\hat{u}}(s)ds\right)\\ &= \alpha'(t)\int_0^t f(t-s)\bar{\hat{u}}(s)ds+\alpha(t)f_0\bar{\hat{u}}+\alpha(t)\int_0^t f'(t-s)\bar{\hat{u}}(s)ds. \end{aligned} \tag{19}$$

The first two terms in (18) take the form

$$\begin{aligned} &- Re\left\{\hat{u}''\frac{d}{dt}\left(\alpha_1(t)\int_0^t g(t-s)\bar{\hat{u}}(s)ds\right)\right\}\\ &= -Re\left\{\hat{u}'\frac{d}{dt}\left(\alpha_1(t)\int_0^t g(t-s)\bar{\hat{u}}(s)ds\right)\right\}'\\ &+ Re\left\{\hat{u}'\frac{d^2}{dt^2}\left(\alpha_1(t)\int_0^t g(t-s)\bar{\hat{u}}(s)ds\right)\right\}\\ &= -Re\left\{\hat{u}'\frac{d}{dt}\left(\alpha_1(t)\int_0^t g(t-s)\bar{\hat{u}}(s)ds\right)\right\}'+\alpha_1(t)g_0|\hat{u}'|^2\\ &+ Re\left\{\hat{u}'\left(\alpha_1(t)\frac{d}{dt}\left(\int_0^t g'(t-s)\bar{\hat{u}}(s)ds\right)+\alpha_1'(t)\int_0^t g(t-s)\bar{\hat{u}}(s)ds\right)\right\}. \end{aligned}$$

Similar to obtain

$$\begin{aligned} &- Re\left\{\hat{v}''\frac{d}{dt}\left(\alpha_2(t)\int_0^t h(t-s)\bar{\hat{v}}(s)ds\right)\right\}\\ &= -Re\left\{\hat{v}'\frac{d}{dt}\left(\alpha_2(t)\int_0^t h(t-s)\bar{\hat{v}}(s)ds\right)\right\}'+\alpha_2(t)h_0|\hat{v}'|^2\\ &+ Re\left\{\hat{v}'\left(\alpha_2(t)\frac{d}{dt}\left(\int_0^t h'(t-s)\bar{\hat{v}}(s)ds\right)+\alpha_2'(t)\int_0^t h(t-s)\bar{\hat{v}}(s)ds\right)\right\}. \end{aligned}$$

Denote by

$$\begin{aligned} E_2(t) &= \frac{1}{2}|\xi|^2\left(\left|\alpha_1(t)\int_0^t g(t-s)\hat{u}(s)ds\right|^2+\left|\alpha_2(t)\int_0^t h(t-s)\hat{v}(s)ds\right|^2\right)\\ &- Re\left\{\hat{u}'\frac{d}{dt}\left(\alpha_1(t)\int_0^t g(t-s)\bar{\hat{u}}(s)ds\right)\right\}\\ &- Re\left\{\hat{v}'\frac{d}{dt}\left(\alpha_2(t)\int_0^t h(t-s)\bar{\hat{v}}(s)ds\right)\right\}, \end{aligned}$$

and

$$\begin{aligned} e_2(t) &= \alpha_1(t)g_0|\hat{u}'|^2+\alpha_2(t)h_0|\hat{v}'|^2\\ &- |\xi|^2 Re\left\{\hat{u}'\frac{d}{dt}\left(\alpha_1(t)\int_0^t g(t-s)\bar{\hat{u}}(s)ds\right)+\hat{v}'\frac{d}{dt}\left(\alpha_2(t)\int_0^t h(t-s)\bar{\hat{v}}(s)ds\right)\right\}\\ &+ Re\left\{\alpha_1'(t)\hat{u}'\int_0^t g(t-s)\bar{\hat{u}}(s)ds+\alpha_2'(t)\hat{v}'\int_0^t h(t-s)\bar{\hat{v}}(s)ds\right\} \end{aligned}$$

and

$$\begin{aligned} R_2(t) &= -|\xi|^2 Re\left\{\hat{u}\frac{d}{dt}\left(\alpha_1(t)\int_0^t g(t-s)\bar{\hat{u}}(s)ds\right)+\hat{v}\frac{d}{dt}\left(\alpha_2(t)\int_0^t h(t-s)\bar{\hat{v}}(s)ds\right)\right\}\\ &+ Re\left\{\hat{u}'\frac{d}{dt}\left(\alpha_1(t)\int_0^t g'(t-s)\bar{\hat{u}}(s)ds\right)\right\}\\ &+ Re\left\{\hat{v}'\frac{d}{dt}\left(\alpha_2(t)\int_0^t h'(t-s)\bar{\hat{v}}(s)ds\right)\right\}\\ &+ \gamma Re\left\{\hat{v}\frac{d}{dt}\left(\alpha_1(t)\int_0^t g(t-s)\bar{\hat{u}}(s)ds\right)+\hat{u}\frac{d}{dt}\left(\alpha_2(t)\int_0^t h(t-s)\bar{\hat{v}}(s)ds\right)\right\}. \end{aligned}$$

Then,

$$\frac{d}{dt}E_2(t)+e_2(t)+R_2(t)=0. \tag{20}$$

Next, to make the second modification of the energy which corresponds to the strong damping, we multiply the first equation of (16) by $\bar{\widehat{u}}$, the second by $\bar{\widehat{v}}$. Summing and taking the real part, we have

$$\begin{aligned}
0 &= (Re\{\widehat{u}'\bar{\widehat{u}}\})'+(Re\{\widehat{v}'\bar{\widehat{v}}\})'-|\widehat{u}'|^2-|\widehat{v}'|^2+|\xi|^2|\widehat{u}|^2+|\xi|^2|\widehat{v}|^2\\
&- |\xi|^2 Re\left\{\alpha_1(t)\int_0^t g(t-s)\widehat{u}(s)\bar{\widehat{u}}(t)ds\right\}-|\xi|^2 Re\left\{\alpha_2(t)\int_0^t h(t-s)\widehat{v}(s)\bar{\widehat{v}}(t)ds\right\}\\
&+ \frac{1}{2}|\xi|^2(|\widehat{u}|^2)'+\frac{1}{2}|\xi|^2(|\widehat{v}|^2)'+\gamma(\widehat{v}\bar{\widehat{u}}+\widehat{u}\bar{\widehat{v}}),
\end{aligned}$$

using results in Lemma 2.1, we get

$$\begin{aligned}
0 &= (Re\{\widehat{u}'\bar{\widehat{u}}\})'+(Re\{\widehat{v}'\bar{\widehat{v}}\})'+\frac{1}{2}|\xi|^2(|\widehat{u}|^2)'+\frac{1}{2}|\xi|^2(|\widehat{v}|^2)'\\
&- |\widehat{u}'|^2-|\widehat{v}'|^2+|\xi|^2\left(1-\alpha_1(t)\int_0^t g(s)ds\right)|\widehat{u}|^2+|\xi|^2\left(1-\alpha_2(t)\int_0^t h(s)ds\right)|\widehat{v}|^2\\
&- |\xi|^2\left(Re\left\{\alpha_1(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\bar{\widehat{u}}(s)ds\right\}-Re\left\{\alpha_2(t)\int_0^t h(t-s)(\widehat{v}(s)-\widehat{v}(t))\bar{\widehat{v}}(s)ds\right\}\right).
\end{aligned}$$

Denote

$$E_3(t)=Re\{\widehat{u}'\bar{\widehat{u}}\}+Re\{\widehat{v}'\bar{\widehat{v}}\}+\frac{1}{2}|\xi|^2|\widehat{u}|^2+\frac{1}{2}|\xi|^2|\widehat{v}|^2,$$

and

$$e_3(t) = |\xi|^2\left(1-\alpha_1(t)\int_0^t g(s)ds\right)|\widehat{u}|^2+|\xi|^2\left(1-\alpha_2(t)\int_0^t h(s)ds\right)|\widehat{v}|^2.$$

$$\begin{aligned}
R_3(t) &= -|\widehat{u}'|^2-Re\left\{\alpha_1(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\bar{\widehat{u}}(s)ds\right\}\\
&- |\widehat{v}'|^2-Re\left\{\alpha_2(t)\int_0^t h(t-s)(\widehat{v}(s)-\widehat{v}(t))\bar{\widehat{v}}(s)ds\right\}\\
&+ \gamma(\widehat{v}\bar{\widehat{u}}+\widehat{u}\bar{\widehat{v}})
\end{aligned}$$

Then,

$$\frac{d}{dt}E_3(t)+e_3(t)+R_3(t)=0. \tag{21}$$

Let us define for some constants $\varepsilon_1,\varepsilon_2>0$ to be chosen later

$$\begin{aligned}
E_4(t) &= E_1(t)+\varepsilon_1\alpha(t)E_2(t)+\varepsilon_2\alpha(t)E_3(t)\\
&= \frac{1}{2}\left(|\widehat{u}'|^2+|\widehat{v}'|^2\right)+\frac{1}{2}|\xi|^2(1-\alpha_1(t)\int_0^t g(s)ds)|\widehat{u}|^2+\frac{1}{2}|\xi|^2\alpha_1(t)(g\circ\widehat{u})(t)\\
&+ \frac{1}{2}|\xi|^2(1-\alpha_2(t)\int_0^t h(s)ds)|\widehat{u}|^2+\frac{1}{2}|\xi|^2\alpha_2(t)(h\circ\widehat{u})(t)+\gamma|\widehat{u}\widehat{v}|\\
&+ \frac{\varepsilon_1\alpha(t)}{2}|\xi|^2\left(\left|\alpha_1(t)\int_0^t g(t-s)\widehat{u}(s)ds\right|^2+\left|\alpha_2(t)\int_0^t h(t-s)\widehat{v}(s)ds\right|^2\right)\\
&- \frac{\varepsilon_1\alpha(t)}{2}Re\left\{\widehat{u}'\frac{d}{dt}\left(\alpha_1(t)\int_0^t g(t-s)\bar{\widehat{u}}(s)ds\right)\right\}\\
&- \frac{\varepsilon_1\alpha(t)}{2}Re\left\{\widehat{v}'\frac{d}{dt}\left(\alpha_2(t)\int_0^t h(t-s)\bar{\widehat{v}}(s)ds\right)\right\},\\
&+ \varepsilon_2\alpha(t)\left(Re\{\widehat{u}'\bar{\widehat{u}}\}+Re\{\widehat{v}'\bar{\widehat{v}}\}+\frac{1}{2}|\xi|^2|\widehat{u}|^2+\frac{1}{2}|\xi|^2|\widehat{v}|^2\right)
\end{aligned}$$

and

$$\begin{aligned}
e_4(t) &= e_1(t)+\varepsilon_1\alpha(t)e_2(t)+\varepsilon_2\alpha(t)e_3(t)\\
&= \frac{1}{2}|\xi|^2\Big(\alpha_1(t)g(t)|\widehat{u}|^2-\alpha_1(t)(g'\circ\widehat{u})(t)+2|\widehat{u}'|^2+\alpha_1'(t)(g\circ\widehat{u})(t)-\alpha_1'(t)\int_0^t g(s)ds|\widehat{u}|^2\Big)\\
&+ \frac{1}{2}|\xi|^2\Big(\alpha_2(t)h(t)|\widehat{u}|^2-\alpha_2(t)(h'\circ\widehat{u})(t)+2|\widehat{u}'|^2+\alpha_2'(t)(h\circ\widehat{u})(t)-\alpha_2'(t)\int_0^t h(s)ds|\widehat{u}|^2\Big).\\
&+ \varepsilon_1\alpha(t)\left(\alpha_1(t)g_0|\widehat{u}'|^2+\alpha_2(t)h_0|\widehat{v}'|^2\right)\\
&- \varepsilon_1\alpha(t)|\xi|^2Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha_1(t)\int_0^t g(t-s)\bar{\widehat{u}}(s)ds\Big)+\widehat{v}'\frac{d}{dt}\Big(\alpha_2(t)\int_0^t h(t-s)\bar{\widehat{v}}(s)ds\Big)\right\}\\
&+ \varepsilon_1\alpha(t)Re\left\{\alpha_1'(t)\widehat{u}'\int_0^t g(t-s)\bar{\widehat{u}}(s)ds\right\}\\
&+ \varepsilon_1\alpha(t)Re\left\{\alpha_2'(t)\widehat{v}'\int_0^t h(t-s)\bar{\widehat{v}}(s)ds\right\} \qquad (22)\\
&+ \varepsilon_2|\xi|^2\alpha(t)\Big(\Big(1-\alpha_1(t)\int_0^t g(s)ds\Big)|\widehat{u}|^2+\Big(1-\alpha_2(t)\int_0^t h(s)ds\Big)|\widehat{v}|^2\Big)
\end{aligned}$$

and

$$\begin{aligned}
R_4(t) &= \varepsilon_1\alpha(t)R_2(t)+\varepsilon_2\alpha(t)R_3(t)\\
&= -\varepsilon_1\alpha(t)Re\left\{|\xi|^2\widehat{u}\frac{d}{dt}\Big(\alpha_1(t)\int_0^t g(t-s)\bar{\widehat{u}}(s)ds\Big)\right\}\\
&- \varepsilon_1\alpha(t)Re\left\{|\xi|^2\widehat{v}\frac{d}{dt}\Big(\alpha_2(t)\int_0^t h(t-s)\bar{\widehat{v}}(s)ds\Big)\right\}\\
&+ \varepsilon_1\alpha(t)Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha_1(t)\int_0^t g'(t-s)\bar{\widehat{u}}(s)ds\Big)\right\}\\
&+ \varepsilon_1\alpha(t)Re\left\{\widehat{v}'\frac{d}{dt}\Big(\alpha_2(t)\int_0^t h'(t-s)\bar{\widehat{v}}(s)ds\Big)\right\}\\
&+ \gamma\varepsilon_1\alpha(t)Re\left\{\widehat{v}\frac{d}{dt}\Big(\alpha_1(t)\int_0^t g(t-s)\bar{\widehat{u}}(s)ds\Big)+\widehat{u}\frac{d}{dt}\Big(\alpha_2(t)\int_0^t h(t-s)\bar{\widehat{v}}(s)ds\Big)\right\}\\
&- \varepsilon_2\alpha(t)|\widehat{u}'|^2-\varepsilon_2\alpha(t)Re\left\{\alpha_1(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\bar{\widehat{u}}(s)ds\right\}\\
&- \varepsilon_2\alpha(t)|\widehat{v}'|^2-\varepsilon_2\alpha(t)Re\left\{\alpha_2(t)\int_0^t h(t-s)(\widehat{v}(s)-\widehat{v}(t))\bar{\widehat{v}}(s)ds\right\}\\
&+ \gamma\varepsilon_2\alpha(t)(\widehat{v}\bar{\widehat{u}}+\widehat{u}\bar{\widehat{v}}) \qquad (23)
\end{aligned}$$

At this point, we introduce the Lyapunov functions as

$$L_1(t)=\left\{|\widehat{u}'|^2+|\widehat{v}'|^2+|\xi|^2\Big(k_1|\widehat{u}|^2+k_2|\widehat{v}|^2+\alpha(t)((g\circ\widehat{u})(t)+(h\circ\widehat{v})(t))\Big)\right\} \qquad (24)$$

and

$$L_2(t)=\alpha(t)\Big(g(t)|\widehat{u}|^2+h(t)|\widehat{v}|^2\Big)+\alpha(t)\beta(t)\Big((g\circ\widehat{u})(t)+(h\circ\widehat{v})(t)\Big). \qquad (25)$$

It is easy to verify that there exists positive constants c_1, c_2 such that

$$c_1L_1(t)\le E_1(t)\le c_2L_1(t), \forall t>0. \qquad (26)$$

Thanks to Holder, Young's inequalities, one gets for some constant c_3

$$|\varepsilon_1E_2(t)+\varepsilon_2E_3(t)|\le c_3L_1(t),$$

which means that $L_1(t) \sim E(t)$. Since $l \geq 2$, using again (10) and (11), Holder and Young's inequalities and assumptions on g,h to obtain

$$
\begin{aligned}
|R_4(t)| \leq\; & \varepsilon_1 \alpha(t) Re\left\{|\xi|^2 \widehat{u}\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right)\right\} \\
+\; & \varepsilon_1 \alpha(t) Re\left\{|\xi|^2 \widehat{v}\frac{d}{dt}\left(\alpha(t)\int_0^t h(t-s)\overline{\widehat{v}}(s)ds\right)\right\} \\
+\; & \varepsilon_1 \alpha(t) Re\left\{\widehat{u}'\frac{d}{dt}\left(\alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\right)\right\} \\
+\; & \varepsilon_1 \alpha(t) Re\left\{\widehat{v}'\frac{d}{dt}\left(\alpha(t)\int_0^t h'(t-s)\overline{\widehat{v}}(s)ds\right)\right\} \\
+\; & \gamma\varepsilon_1 \alpha(t) Re\left\{\widehat{v}\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right) + \widehat{u}\frac{d}{dt}\left(\alpha(t)\int_0^t h(t-s)\overline{\widehat{v}}(s)ds\right)\right\} \\
+\; & \varepsilon_2\alpha(t)|\widehat{u}'|^2 + \varepsilon_2\alpha(t) Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\} \\
+\; & \varepsilon_2\alpha(t)|\widehat{v}'|^2 + \varepsilon_2\alpha(t) Re\left\{\alpha(t)\int_0^t h(t-s)(\widehat{v}(s)-\widehat{v}(t))\overline{\widehat{v}}(s)ds\right\} \\
+\; & \gamma\varepsilon_2\alpha(t)(\widehat{v}\overline{\widehat{u}} + \widehat{u}\overline{\widehat{v}}) \\
\leq\; & \lambda(\varepsilon_1+\varepsilon_2)\alpha(t)\left(|\widehat{u}'|^2 + |\widehat{v}'|^2\right) + \gamma c_\lambda(\varepsilon_1+\varepsilon_2)|\xi|^2 L_2(t).
\end{aligned}
$$

As in [8], there exists positive constants $\varepsilon_1, \varepsilon_2, \lambda, c_\lambda, \gamma$ such that

$$|R_4(t)| \leq ce_4(t), c > 0. \tag{27}$$

By (17), (20) and (21), we get

$$\frac{d}{dt}E_4(t) = \frac{d}{dt}E_1(t) + \varepsilon_1\alpha(t)\frac{d}{dt}E_2(t) + \varepsilon_2\alpha(t)\frac{d}{dt}E_3(t) + \varepsilon_1\alpha'(t)E_2(t) + \varepsilon_2\alpha'(t)E_3(t).$$

We use $\lim\limits_{t\to\infty}\frac{\alpha'(t)}{\alpha(t)} = 0$, by (8)-(11) to choose $t_1 > 0$ and since $e_4(t) \geq cE_4(t)$, then (27) gives for some positive constant N

$$\frac{d}{dt}E_4(t) \leq -N\alpha(t)E_4(t) + c\alpha(t)\Big((g\circ\widehat{u})(t) + (h\circ\widehat{v})(t)\Big). \tag{28}$$

Multiplying (28) by $\beta(t)$ and using (10), (11), (25), we obtain

$$
\begin{aligned}
\beta(t)\frac{d}{dt}E_4(t) \leq\; & -N\beta(t)\alpha(t)E_4(t) + c\beta(t)\alpha(t)\Big((g\circ\widehat{u})(t) + (h\circ\widehat{v})(t)\Big) \\
\leq\; & -N\beta(t)\alpha(t)E_4(t) - c\alpha(t)\Big((g'\circ\widehat{u})(t) + (h'\circ\widehat{v})(t)\Big) \\
\leq\; & -N\beta(t)\alpha(t)E_4(t) - c|\xi|^2\alpha'(t)\int_0^t g(s)ds|\widehat{u}|^2 \\
-\; & c|\xi|^2\alpha'(t)\int_0^t h(s)ds|\widehat{v}|^2 - 2c\frac{d}{dt}E_4(t), \quad \forall t > t_1.
\end{aligned} \tag{29}
$$

Since $\beta'(t) \leq 0$, we set $L(s) = (\beta(s)+2c)E_4(s)$ which is equivalent to $E_4(t)$, then

$$
\begin{aligned}
\frac{d}{dt}L(t) \leq\; & -N\beta(t)\alpha(t)E_4(t) - c|\xi|^2\alpha'(t)\Big(\int_0^t g(s)ds|\widehat{u}|^2 + \int_0^t h(s)ds|\widehat{v}|^2\Big) \\
\leq\; & -\beta(t)\alpha(t)\left[N - \frac{2\alpha'(t)}{k\beta(t)\alpha(t)}\int_0^t f(s)ds\right]E_4(t), \quad \forall t > t_1,
\end{aligned} \tag{30}
$$

where

$$f(t) = \max\Big\{g(t), h(t)\Big\}.$$

By (10) and (11), we have

$$\lim_{t\to\infty} = \frac{\alpha'(t)}{\beta(t)\alpha(t)} = 0,$$

then we can choose $t_2 > t_1$ such that

$$\begin{aligned} \frac{d}{dt}L(t) &\leq -c\beta(t)\alpha(t)E_4(t) \\ &\leq -c\beta(t)\alpha(t)L(t), \quad \forall t > t_2. \end{aligned} \tag{31}$$

Integrating (31) over $[t_2,t]$ using equivalence between Lyapunov function and the energy function, it yields that

$$E(\xi,t) \leq W \exp(-\omega \int_0^t \alpha(s)\beta(s)ds)E(\xi,0), W, \omega > 0.$$

□

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