

GENERALIZATION MITTAG-LEFFLER FUNCTION
ASSOCIATED WITH OF THE HADAMARD AND FEJER
HADAMARD INEQUALITIES FOR $(h - m)$ -STRONGLY CONVEX
FUNCTIONS VIA FRACTIONAL INTEGRALS

SEDA KILINÇ, ABDULLAH AKKURT, AND HÜSEYİN YILDIRIM

ABSTRACT. The aim of this paper, Hadamard and Fejer Hadamard inequalities for $(h - m)$ -strongly convex functions via generalized fractional integral operators involving the generalized Mittag-Leffler function are established. In particular several known results are mentioned.

1. INTRODUCTION AND PRELIMINARES

The relationship between theory of convex functions and theory of inequalities has occurred as a result of many researches investigation of these theories. A very interesting result in this regard is due to Hermite and Hadamard independently that is Hermite-Hadamard's inequality. This remarkable result of Hermite and Hadamard can be viewed as necessary and sufficient condition for a function to be convex. The $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on an interval I of real numbers $a, b \in I$ and $a < b$, we have,

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \leq \frac{f(a)+f(b)}{2}$$

Both inequalities hold in the reversed direction if f is concave. The classical Hermite-Hadamard inequalities have attracted many researchers since 1893 see [2, 4, 5, 7, 9 – 13, 15 – 21]. Researchers investigated Hermite-Hadamard inequalities involving fractional integrals according to the associated fractional integral equalities and different types of convex functions. Also, its extensions and generalizations have been considered in various directions using novel and innovative techniques. For example $(h - m)$ strongly convexity is the generalization of convexity, $(h - m)$ convexity, m -convexity, s -convexity defined on the right half of real line including zero ([1]).

Definition 1.1. Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \rightarrow \mathbb{R}$ is a $(h - m)$ convex function, also f is non-negative, if

$$f(\alpha x + m(1 - \alpha)y) \leq h(\alpha) f(x) + mh(1 - \alpha) f(y)$$

for all $x, y \in [0, b]$, $m \in [0, 1]$ and $\alpha \in (0, 1)$.

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Definition 1.2. Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \rightarrow \mathbb{R}$ is a $(h - m)$ strongly convex function, also f is non-negative $c \geq 0$, if

$$f(\alpha x + m(1 - \alpha)y) \leq h(\alpha) f(x) + mh(1 - \alpha) f(y) - ch(\alpha) h(1 - \alpha) (mx - y)^2$$

for all $x, y \in [0, b]$, $m \in [0, 1]$ and $\alpha \in (0, 1)$.

For suitable choice of h and m , class of $(h - m)$ strongly convex functions is reduces to the different knows classes of convex functions defined on $[0, b]$. Fractional calculus is a field of mathematical study that grows out of the traditional definitions of the calculus integral and derivative exponents is an outgrowth of exponents with integer value. The Mittag- Lefler function is an important function that finds widespread use in the world of fractioanl calculus.

Definition 1.3. Let $\alpha, \beta, k, l, \gamma$ be positive real numbers and $w \in \mathbb{R}$. Then the generalized fractional integral operators containing Mittag leffer function for a real valued continuous function f is defined by

$$\begin{aligned} & \left(\epsilon_{\alpha, \beta, l, w, a^+}^{\gamma, \delta, k} f \right) (x) \\ &= \int_a^x (x - t)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k} (w(x - t)^\alpha) f(t) dt \end{aligned}$$

and

$$\begin{aligned} & \left(\epsilon_{\alpha, \beta, k, w, b^-}^{\gamma, \delta, k} f \right) (x) \\ &= \int_x^b (t - x)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k} (w(t - x)^\alpha) f(t) dt. \end{aligned}$$

it is also common to represent the Mittag-Leffer function as

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{kn} t^n}{\Gamma(\alpha n + \beta)(\delta)_{tn}},$$

where $(a)_n = a(a + 1)(a + 2) \dots (a + n - 1)$, $(a)_0 = 1$. This is the more general- ized form of Mittag-Leffer function. If $\delta = l = 1$, then integral operators $\epsilon_{\alpha, \beta, k, w, a^+}^{\gamma, \delta, k}$ reduces to an integral operator $\epsilon_{\alpha, \beta, k, w, a^+}^{\gamma, \delta, k}$ containing generalized Mittag-Leffler Function $E_{\alpha, \beta, 1}^{\gamma, 1, k}$ introduced by Srivastava and Tomovski. [6].

Definition 1.4. Let $f \in L_1[a, b]$. Then Riemann-Liouville integrals $J_{a^+}^k f$ and $J_{b^-}^k f$ of order $k > 0$, $a < b$, $a \geq 0$ with are defined by

$$(1.2) \quad J_{a^+}^k f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{k-1} f(t) dt, \quad x > a,$$

and

$$(1.3) \quad J_{b^-}^k f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{k-1} f(t) dt, \quad x < b.$$

These integrals are called right-sided Riemann-Liouville fractional integral and left-sided Riemann-Liouville fractional integral respectively. Where

$$(1.4) \quad \Gamma(\alpha) = \int_0^{\infty} e^{-t} x^{\alpha-1} dx,$$

is the well known Gamma function.

2. MAIN RESULTS

In this section, we obtain our main results.

Theorem 2.1. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an integrable and $(h - m)$ -strongly convex function with $m \in (0, 1]$, $c \geq 0$. Then the following inequality for generalized fractional integral holds*

$$\begin{aligned}
& f\left(\frac{bm+a}{2}\right) \epsilon\left(\gamma, \delta, k, w^\circ, a+1\right)(mb) \\
& \leq h\left(\frac{1}{2}\right) \left[m^{\beta+1} \epsilon\left(\gamma, \delta, k, w^\circ, b-f\right)\left(\frac{\alpha}{m}\right) + \epsilon\left(\gamma, \delta, k, w^\circ, a+f\right)(mb) \right] \\
& - 4ch^2\left(\frac{1}{2}\right) (a-mb)^2 \left[\frac{1}{4} \epsilon\left(\gamma, \delta, k, w^\circ, a+1\right)(mb) - \epsilon\left(\gamma, \delta, k, w^\circ, a+1\right)(mb) \right. \\
& \left. + \epsilon\left(\gamma, \delta, k, w^\circ, a+1\right)(mb) \right] \\
& \leq h\left(\frac{1}{2}\right) \left[m^2 f\left(\frac{\alpha}{m^2}\right) + mf(b) \right] \int_0^1 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) h(1-t) dt \\
& + h\left(\frac{1}{2}\right) [mf(b) + f(a)] \int_0^1 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) h(t) dt \\
& - 2c(a-mb)^2 \int_0^1 (1-2t)^2 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) h(t) h(1-t) dt
\end{aligned}$$

where $w^\circ = \frac{w}{(bm-a)^\alpha}$.

Proof. Using that f is $(h - m)$ -strongly convex function, we have

$$f\left(\frac{xm+y}{2}\right) \leq h\left(\frac{1}{2}\right) (mf(x) + f(y)) - ch^2\left(\frac{1}{2}\right) (mx - y)^2$$

By setting $x = (1-t)\frac{\alpha}{m} + tb$ and $y = m(1-t)b + ta$, then integrating over $[0, 1]$ after multiply with $t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha)$, we have

$$\begin{aligned}
& f\left(\frac{bm+a}{2}\right) \int_0^1 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) dt \\
& \leq h\left(\frac{1}{2}\right) \left[\int_0^1 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) mf\left((1-t)\frac{\alpha}{m} + tb\right) dt \right. \\
& \left. + \int_0^1 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) f(m(1-t)b + ta) dt \right] \\
& - ch^2\left(\frac{1}{2}\right) (a-mb)^2 \int_0^1 (1-2t)^2 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) dt
\end{aligned}$$

By substituting $w = (1-t)\frac{\alpha}{m} + tb$, and $z = m(1-t)b + ta$ one can have,

$$\begin{aligned}
& f\left(\frac{bm+a}{2}\right) \int_{mb}^a \left(\frac{y-mb}{a-mb}\right)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}\left(w\left(\frac{y-mb}{a-mb}\right)^\alpha\right) \frac{dy}{a-mb} \\
& \leq h\left(\frac{1}{2}\right) \left[\int_{\frac{\alpha}{m}}^b \left(\frac{mx-a}{mb-a}\right)^{\beta-1} m E_{\alpha, \beta, l}^{\gamma, \delta, k}\left(w\left(\frac{mx-a}{mb-a}\right)^\alpha\right) f(x) \frac{mdx}{mb-a} \right. \\
& \left. + \int_{mb}^a \left(\frac{y-mb}{a-mb}\right)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}\left(w\left(\frac{y-mb}{a-mb}\right)^\alpha\right) f(y) \frac{dy}{a-mb} \right] \\
& - ch^2\left(\frac{1}{2}\right) (a-mb)^2 \int_{mb}^a \left(\frac{y-mb}{a-mb}\right)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}\left(w\left(\frac{y-mb}{a-mb}\right)^\alpha\right) \frac{dy}{a-mb} \\
& + 4ch^2\left(\frac{1}{2}\right) (a-mb)^2 \int_{mb}^a \left(\frac{y-mb}{a-mb}\right)^\beta E_{\alpha, \beta, l}^{\gamma, \delta, k}\left(w\left(\frac{y-mb}{a-mb}\right)^\alpha\right) \frac{dy}{a-mb} \\
& - 4ch^2\left(\frac{1}{2}\right) (a-mb)^2 \int_{mb}^a \left(\frac{y-mb}{a-mb}\right)^{\beta+1} E_{\alpha, \beta, l}^{\gamma, \delta, k}\left(w\left(\frac{y-mb}{a-mb}\right)^\alpha\right) \frac{dy}{a-mb}
\end{aligned}$$

namely

$$\begin{aligned}
 & f\left(\frac{bm+a}{2}\right) \epsilon\left(\begin{matrix} \gamma, \delta, k \\ \alpha, \beta, l, w^\circ, a+1 \end{matrix}\right)(mb) \\
 & \leq h\left(\frac{1}{2}\right) \left[m^{\beta+1} \epsilon\left(\begin{matrix} \gamma, \delta, k \\ \alpha, \beta, k, w^\circ, b- \end{matrix}\right) f\left(\frac{\alpha}{m}\right) + \epsilon\left(\begin{matrix} \gamma, \delta, k \\ \alpha, \beta, k, w^\circ, a+ \end{matrix}\right) f\right](mb) \\
 & - ch^2\left(\frac{1}{2}\right) (a-mb)^2 \epsilon\left(\begin{matrix} \gamma, \delta, k \\ \alpha, \beta, l, w^\circ, a+1 \end{matrix}\right)(mb) \\
 & + 4ch^2\left(\frac{1}{2}\right) (a-mb)^2 \epsilon\left(\begin{matrix} \gamma, \delta, k \\ \alpha, \beta+1, l, w^\circ, a+1 \end{matrix}\right)(mb) \\
 & - 4ch^2\left(\frac{1}{2}\right) (a-mb)^2 \epsilon\left(\begin{matrix} \gamma, \delta, k \\ \alpha, \beta+2, l, w^\circ, a+1 \end{matrix}\right)(mb).
 \end{aligned}$$

This completes the proof of first inequality in above . For second inequality $(h-m)$ -strongly convexity of f also gives

$$\begin{aligned}
 & mf\left(\left(1-t\right)\frac{\alpha}{m}+tb\right)+f\left(m\left(1-t\right)b+ta\right)-ch\left(\frac{1}{2}\right)(a-mb)^2(1-2t)^2 \\
 & \leq m^{\frac{\beta}{2}}h(1-t)f\left(\frac{\alpha}{m^2}\right)+mh(t)f(b)+mh(1-t)f(b) \\
 & +h(t)f(a)-2ch(t)h(1-t)(a-mb)^2(1-2t)^2,
 \end{aligned}$$

Multiplying both sides of above inequality with $h\left(\frac{1}{2}\right)t^{\beta-1}E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha)$, and interating over $[0, 1]$, we have

$$\begin{aligned}
 & h\left(\frac{1}{2}\right) \left[m^{\beta+1} \epsilon\left(\begin{matrix} \gamma, \delta, k \\ \alpha, \beta, k, w^\circ, b- \end{matrix}\right) f\left(\frac{\alpha}{m}\right) + \epsilon\left(\begin{matrix} \gamma, \delta, k \\ \alpha, \beta, k, w^\circ, a+ \end{matrix}\right) f\right](mb) \\
 & - ch^2\left(\frac{1}{2}\right) (a-mb)^2 \epsilon\left(\begin{matrix} \gamma, \delta, k \\ \alpha, \beta, l, w^\circ, a+1 \end{matrix}\right)(mb) \\
 & + 4ch^2\left(\frac{1}{2}\right) (a-mb)^2 \epsilon\left(\begin{matrix} \gamma, \delta, k \\ \alpha, \beta+1, l, w^\circ, a+1 \end{matrix}\right)(mb) \\
 & - 4ch^2\left(\frac{1}{2}\right) (a-mb)^2 \epsilon\left(\begin{matrix} \gamma, \delta, k \\ \alpha, \beta+2, l, w^\circ, a+1 \end{matrix}\right)(mb) \\
 & \leq h\left(\frac{1}{2}\right) \left[m^2f\left(\frac{\alpha}{m^2}\right) + mf(b) \right] \int_0^1 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) h(1-t) dt \\
 & h\left(\frac{1}{2}\right) [mf(b) + f(a)] \int_0^1 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) h(t) dt \\
 & - 2c(a-mb)^2 \int_0^1 (1-2t)^2 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) h(t) h(1-t) dt
 \end{aligned}$$

combininig it with (2.3) we get (2.1) which was required to prove. \square

Several known results are special cases of the above generalized fractional Hadamard inequality comprise in the following remark. i) If we take $h(t) = t$, $c = 0$ and $m = 1$ in above theorem, then we get [18] ii) If we take $h(t) = t$, $c = 0$ in above theorem, then we get [10] iii) If we take $h(t) = t$, $c = 0$ and $w = 0$ in above theorem, then we get [11] iv) If we take $h(t) = t$, $c = 0, w = 0$ and $m = 1$ in above theorem, then we get [19] v) If we take $h(t) = t$, $c = 0, \beta = 1, w = 0$ and $m = 1$ in above theorem, then we get Hadamard inequality. vi) If we take $c = 0$, then we get [21].

Theorem 2.2. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an integrable and $(h-m)$ -strongly convex function with $m \in (0, 1]$. Then the following inequality for generalized fractional*

integral holds

$$\begin{aligned}
& f\left(\frac{bm+a}{2}\right) \in \left(\gamma, \delta, k\right)_{\alpha, \beta, l, w^\circ, a+1}(mb) \\
& \leq h\left(\frac{1}{2}\right) \left[\epsilon\left(\gamma, \delta, k\right)_{\alpha, \beta, l, w^\circ, \left(\frac{a+bm}{2}\right)} + f\right](mb) + m^{\beta+1} \epsilon\left(\gamma, \delta, k\right)_{\alpha, \beta, l, w^\circ, \left(\frac{a+bm}{2m}\right)} - f\left(\frac{\alpha}{m}\right) \\
& - 2ch^2\left(\frac{1}{2}\right) (mb-a)^2 \left[\frac{1}{2} \epsilon\left(\gamma, \delta, k\right)_{\alpha, \beta, l, w^\circ, a+1}(mb) - \epsilon\left(\gamma, \delta, k\right)_{\alpha, \beta+1, l, w^\circ, a+1}(mb) \right. \\
& \left. + \epsilon\left(\gamma, \delta, k\right)_{\alpha, \beta+2, l, w^\circ, a+1}(mb) \right] \\
& \leq h\left(\frac{1}{2}\right) \left\{ \left(m^2 f\left(\frac{a}{m^2}\right) + mf(b)\right) \int_0^1 h\left(\frac{2-t}{2}\right) t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) dt \right. \\
& \left. + [mf(b) + f(a)] \int_0^1 h\left(\frac{t}{2}\right) t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) dt \right\} \\
& - 2ch\left(\frac{2-t}{2}\right) (mb-a)^2 \int_0^1 h\left(\frac{t}{2}\right) (t-1)^2 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) dt.
\end{aligned}$$

Proof. By putting $x = \frac{t}{2}b + \frac{(2-t)}{2}\frac{\alpha}{m}$ and $y = \frac{t}{2}a + m\frac{(2-t)}{2}b$ in 2.2 where $t \in [0, 1]$, we have

$$\begin{aligned}
f\left(\frac{mx+y}{2}\right) & \leq h\left(\frac{1}{2}\right) (mf(x) + f(y)) - ch^2\left(\frac{1}{2}\right) (mx-y)^2 \\
f\left(\frac{a+mb}{2}\right) & \leq h\left(\frac{1}{2}\right) \left(mf\left(\frac{t}{2}b + \frac{(2-t)}{2}\frac{\alpha}{m}\right) + f\left(\frac{t}{2}a + m\frac{(2-t)}{2}b\right) \right) \\
& - ch^2\left(\frac{1}{2}\right) (t-1)^2 (mb-a)^2
\end{aligned}$$

Multiplying both sides of above inequality with $t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha)$, and interating over $[0, 1]$, we have

$$\begin{aligned}
& f\left(\frac{a+mb}{2}\right) \int_0^1 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) dt \\
& \leq h\left(\frac{1}{2}\right) \left[\int_0^1 mf\left(\frac{t}{2}b + \frac{(2-t)}{2}\frac{\alpha}{m}\right) t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) dt \right. \\
& \left. \int_0^1 f\left(\frac{t}{2}a + m\frac{(2-t)}{2}b\right) t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) dt \right] \\
& - ch^2\left(\frac{1}{2}\right) (mb-a)^2 \int_0^1 (t-1)^2 t^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(wt^\alpha) dt
\end{aligned}$$

By change of variables one can have,

$$\begin{aligned}
& f\left(\frac{bm+a}{2}\right) \in \left(\gamma, \delta, k\right)_{\alpha, \beta, l, w^\circ, a+1}(mb) \\
& \leq h\left(\frac{1}{2}\right) \left[\epsilon\left(\gamma, \delta, k\right)_{\alpha, \beta, l, w^\circ, \left(\frac{a+bm}{2}\right)} + f\right](mb) + m^{\beta+1} \epsilon\left(\gamma, \delta, k\right)_{\alpha, \beta, l, w^\circ, \left(\frac{a+bm}{2m}\right)} - f\left(\frac{\alpha}{m}\right) \\
& - ch^2\left(\frac{1}{2}\right) (mb-a)^2 \epsilon\left(\gamma, \delta, k\right)_{\alpha, \beta, l, w^\circ, a+1}(mb) \\
& + 2ch^2\left(\frac{1}{2}\right) (mb-a)^2 \epsilon\left(\gamma, \delta, k\right)_{\alpha, \beta+1, l, w^\circ, a+1}(mb) \\
& - ch^2\left(\frac{1}{2}\right) (mb-a)^2 \epsilon\left(\gamma, \delta, k\right)_{\alpha, \beta+2, l, w^\circ, a+1}(mb)
\end{aligned}$$

Now by using the $(h-m)$ -strongly convexity of f we have,

$$\begin{aligned}
& f\left(\frac{t}{2}a + m\frac{(2-t)}{2}b\right) + mf\left(\frac{t}{2}b + \frac{(2-t)}{2}\frac{\alpha}{m}\right) - ch\left(\frac{1}{2}\right) (t-1)^2 (mb-a)^2 \\
& \leq h\left(\frac{t}{2}\right) f(a) + mh\left(\frac{2-t}{2}\right) f(b) + mh\left(\frac{t}{2}\right) f(b) + m^2 h\left(\frac{2-t}{2}\right) f\left(\frac{a}{m^2}\right) \\
& - 2ch\left(\frac{2-t}{2}\right) h\left(\frac{t}{2}\right) (t-1)^2 (mb-a)^2
\end{aligned}$$

Multiplying both sides of above inequality with $h\left(\frac{1}{2}\right) t^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha)$, and interating over $[0, 1]$, we have

$$\begin{aligned}
 & h\left(\frac{1}{2}\right) \left[\epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta,l,w^\circ, \left(\frac{a+bm}{2}\right) \end{matrix} \right) + f \right] (mb) + m^{\beta+1} \epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta,l,w^\circ, \left(\frac{a+bm}{2m}\right) \end{matrix} \right) - f \left] \left(\frac{\alpha}{m} \right) \right] \\
 & - ch^2 \left(\frac{1}{2} \right) (mb-a)^2 \epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta,l,w^\circ, a+1 \end{matrix} \right) (mb) \\
 & + 2ch^2 \left(\frac{1}{2} \right) (mb-a)^2 \epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta+1,l,w^\circ, a+1 \end{matrix} \right) (mb) \\
 & - ch^2 \left(\frac{1}{2} \right) (mb-a)^2 \epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta+2,l,w^\circ, a+1 \end{matrix} \right) (mb) \\
 & \leq h\left(\frac{1}{2}\right) \left\{ \left(m^2 f \left(\frac{a}{m^2} \right) + mf(b) \right) \int_0^1 h\left(\frac{2-t}{2}\right) t^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) dt \right. \\
 & \left. + [mf(b) + f(a)] \int_0^1 h\left(\frac{t}{2}\right) t^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) dt \right\} \\
 & - 2ch\left(\frac{2-t}{2}\right) (mb-a)^2 \int_0^1 h\left(\frac{t}{2}\right) (t-1)^2 t^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) dt.
 \end{aligned}$$

combining it with (2.5),(2.4) we get which was the required inequality. \square

Corollary 2.3. *if we take $h(t) = t$, and $m = 1$, $c = 0$ in above theorem, then we get the following inequality analogue to hadamard inequality [18, Theorem 2.1] for convex functions via generalized fractional integrals*

$$\begin{aligned}
 & f\left(\frac{b+a}{2}\right) \epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta,l,w^\circ, a+1 \end{matrix} \right) (b) \leq \frac{1}{2} \left[\epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta,l,w^\circ, \left(\frac{a+b}{2}\right) \end{matrix} \right) + f \right] (b) + \epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta,l,w^\circ, \left(\frac{a+b}{2}\right) \end{matrix} \right) - f \right] (a) \\
 & \leq \frac{1}{2} [f(a) + f(b)] \epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta,l,w^\circ, b-1 \end{matrix} \right) (a).
 \end{aligned}$$

Theorem 2.4. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable and $(h-m)$ -strongly convex function with $0 \leq a < b$, and $f \in L_1[a, b]$. Also let $g : [a, b] \rightarrow \mathbb{R}$, be a function with which is non-negative, integrable and symmetric about $\frac{a+mb}{2}$. If $f(mb+a-mx) = f(x)$. Then the following inequality for generalized fractional integrals hold*

$$\begin{aligned}
 & f\left(\frac{bm+a}{2}\right) \epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta,l,w^\circ, b-g \end{matrix} \right) \left(\frac{a}{m} \right) \\
 & \leq h\left(\frac{1}{2}\right) (m+1) \epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta,l,w^\circ, b-gf \end{matrix} \right) \left(\frac{a}{m} \right) \\
 & - 4ch^2 \left(\frac{1}{2} \right) (a-mb)^2 \left[\frac{1}{4} \epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta,l,w^\circ, b-g \end{matrix} \right) \left(\frac{a}{m} \right) - \epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta+1,l,w^\circ, b-g \end{matrix} \right) \left(\frac{a}{m} \right) \right. \\
 & \left. + \epsilon \left(\begin{matrix} \gamma,\delta,k \\ \alpha,\beta+2,l,w^\circ, b-g \end{matrix} \right) \left(\frac{a}{m} \right) \right] \\
 & \leq h\left(\frac{1}{2}\right) \left\{ \left(m^2 f \left(\frac{a}{m^2} \right) + mf(b) \right) \int_0^1 t^{\beta-1} h(1-t) g\left(tb + (1-t)\frac{a}{m}\right) E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) dt \right. \\
 & \left. + [mf(b) + f(a)] \int_0^1 t^{\beta-1} h(t) g\left(tb + (1-t)\frac{a}{m}\right) E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) dt \right\} \\
 & - 2c(mb-a)^2 \int_0^1 h(1-t) h(t) (1-2t)^2 t^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) dt
 \end{aligned}$$

Proof. Using that f is $(h-m)$ -strongly convex function, therefore $x = (1-t)\frac{a}{m} + tb$ and $y = m(1-t)b + ta$, we have

$$\begin{aligned}
 & f\left(\frac{bm+a}{2}\right) \leq h\left(\frac{1}{2}\right) \left(mf\left((1-t)\frac{a}{m} + tb\right) + f\left(m(1-t)b + ta\right) \right) \\
 & - ch^2\left(\frac{1}{2}\right) (2t-1)^2 (mb-a)^2
 \end{aligned}$$

Then integrating over $[0, 1]$ after multiply with $t^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) g\left(tb + (1-t)\frac{a}{m}\right)$, we have

$$\begin{aligned} & f\left(\frac{bm+a}{2}\right) \int_0^1 t^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) g\left(tb + (1-t)\frac{a}{m}\right) dt \\ & \leq h\left(\frac{1}{2}\right) \left[\int_0^1 t^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) mg\left(tb + (1-t)\frac{a}{m}\right) f\left((1-t)\frac{a}{m} + tb\right) dt \right. \\ & \quad \left. + \int_0^1 t^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) g\left(tb + (1-t)\frac{a}{m}\right) f\left(m(1-t)b + ta\right) dt \right] \\ & \quad - ch^2\left(\frac{1}{2}\right) (a - mb)^2 \int_0^1 (1-2t)^2 g\left(tb + (1-t)\frac{a}{m}\right) t^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) dt \end{aligned}$$

If we set $x = tb + (1-t)\frac{a}{m}$, and use the given condition $f(mb + a - mx) = f(x)$, we have

$$\begin{aligned} & f\left(\frac{bm+a}{2}\right) \in \left(\gamma, \delta, k, \alpha, \beta, l, w^\circ, b-g\right)\left(\frac{a}{m}\right) \\ & \leq h\left(\frac{1}{2}\right) (m+1) \in \left(\gamma, \delta, k, \alpha, \beta, l, w^\circ, b-gf\right)\left(\frac{a}{m}\right) \\ & \quad - ch^2\left(\frac{1}{2}\right) (a - mb)^2 \in \left(\gamma, \delta, k, \alpha, \beta, l, w^\circ, b-g\right)\left(\frac{a}{m}\right) \\ & \quad + 4ch^2\left(\frac{1}{2}\right) (a - mb)^2 \in \left(\gamma, \delta, k, \alpha, \beta+1, l, w^\circ, b-g\right)\left(\frac{a}{m}\right) \\ & \quad - 4ch^2\left(\frac{1}{2}\right) (a - mb)^2 \in \left(\gamma, \delta, k, \alpha, \beta+2, l, w^\circ, b-g\right)\left(\frac{a}{m}\right) \end{aligned}$$

This completes the proof of first inequality (26.=*) .For the second inequality using $(h - m)$ -strongly convex function of f we have,

$$\begin{aligned} & mf\left((1-t)\frac{a}{m} + tb\right) + f\left(m(1-t)b + ta\right) - ch\left(\frac{1}{2}\right) (a - mb)^2 (1-2t)^2 \\ & \leq m^2 h(1-t) f\left(\frac{a}{m^2}\right) + mh(t) f(b) + mh(1-t) f(b) + h(t) f(a) \\ & \quad - 2ch(t) h(1-t) (1-2t)^2 (mb - a)^2 \end{aligned}$$

Multiplying both sides of above inequality with $h\left(\frac{1}{2}\right) t^{\beta-1} g\left(tb + (1-t)\frac{a}{m}\right) E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha)$, and interating over $[0, 1]$, we have

$$\begin{aligned} & h\left(\frac{1}{2}\right) (m+1) \in \left(\gamma, \delta, k, \alpha, \beta, l, w^\circ, b-gf\right)\left(\frac{a}{m}\right) \\ & \quad - ch^2\left(\frac{1}{2}\right) (a - mb)^2 \in \left(\gamma, \delta, k, \alpha, \beta, l, w^\circ, b-g\right)\left(\frac{a}{m}\right) \\ & \quad + 4ch^2\left(\frac{1}{2}\right) (a - mb)^2 \in \left(\gamma, \delta, k, \alpha, \beta+1, l, w^\circ, b-g\right)\left(\frac{a}{m}\right) \\ & \quad - 4ch^2\left(\frac{1}{2}\right) (a - mb)^2 \in \left(\gamma, \delta, k, \alpha, \beta+2, l, w^\circ, b-g\right)\left(\frac{a}{m}\right) \\ & \leq h\left(\frac{1}{2}\right) \left\{ (m^2 f\left(\frac{a}{m^2}\right) + mf(b)) \int_0^1 t^{\beta-1} h(1-t) g\left(tb + (1-t)\frac{a}{m}\right) E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) dt \right. \\ & \quad \left. + [mf(b) + f(a)] \int_0^1 t^{\beta-1} h(t) g\left(tb + (1-t)\frac{a}{m}\right) E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) dt \right\} \\ & \quad - 2c(mb - a)^2 \int_0^1 h(1-t) h(t) (1-2t)^2 t^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(wt^\alpha) dt \end{aligned}$$

combining it with (2.7),(2.6) we get which was the required inequality. \square

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[DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, UNIVERSITY OF KAHRAMANMARAŞ SÜTÇÜ İMAM, 46000, KAHRAMANMARAŞ, TURKEY

E-mail address: sedaaa_kilinc@hotmail.com

E-mail address: abdullahmat@gmail.com

E-mail address: hyildir@ksu.edu.tr