New Trends in Mathematical Sciences

Numerical simulation for SI model with variable-order fractional

Hanaa Abdel Hameed Asfour[1] *and Mohamed Gamal M. Ibrahim*[2]

[1]Doctor of mathematics, Faculty of Science, Ein Shams University, Egypt
[2]Mathematics Department, Faculty of Science, Zagazig University, Egypt

Abstract: In this paper numerical studies for the variable-order fractional delay differential equations are presented. Adams-Bashforth-Moulton algorithm has been extended to study this problem, where the derivative is defined in the Caputo variable-order fractional sense. Numerical test examples are presented to demonstrate utility of the method. Chaotic behaviors are observed in variable-order one dimensional delayed systems.

Keywords: Variable order fractional derivatives, Caputo's derivative, Adams Bashforth-Moulton method, SI model and its Existence and Uniqueness, Solution of SI Model by Using Adams Bashforth-Moulton method.

1 Introduction

Variable-order fractional calculus i.e., the fractional differentiation and integration of variable order is the generalization of classical calculus and fractional calculus, which were invented by Newton and Leibnitz hundreds of years ago. Now the study on it becomes a hotpot in recent ten years. It has turned out that many problems in physics, biology, engineering, and finance can be described excellently by models using mathematical tools from variable-order fractional calculus.

In real world systems, delay is very often encountered in many practical systems, such as control systems [1], lasers, traffic models [2], metal cutting, epidemiology, neuro science, population dynamics [3], chemical kinetics [4] etc. Delayed fractional differential equations FDEs are correspondingly used to describe such dynamical systems. In recent years, delayed FDEs begin to arouse the attention of many researchers [5,2]. Simulating these equations is an important technique in the research; accordingly, finding effective numerical methods for the delayed FDEs is a necessary process. The effective methods and their development for numerically solving fractional differential equations (FDEs) have received increasing attention over the last few years. Several methods based on Caputo or Riemann-Liouville definitions [9] have been proposed and analyzed. For instance, based on the predictor-corrector scheme, Diethelm et al. introduced Adams-Bashforth-Moulton algorithm [6,7], and mean while some error analysis presented to improve the numerical accuracy. In recent years, the application of the method is extended to more concrete physical and mathematical models [11]. Variable order differential equations, i.e., differential equations where the order of the derivative changes with respect to either the dependent or the independent variables, have not received as much attention as fractional order systems, despite of the ability of variable order formulations to model continuous spectral behavior in complex dynamics. Many authors have introduced different definitions of variable order differential operators, each of these with a specific meaning to suit desired goals. These definitions such as Riemann-Liouville, Grunwald, Caputo, Riesz ([9],[10]), and some notes as Coimbra definition [8]. The main aim of this paper is to study numerically the variable-order fractional delay differential equations (FDDEs) by using the Adams- Bashforth-Moulton method.

* Corresponding author e-mail: nadahnaa@hotmail.com

This paper is organized as follows. In Section 1, we give some definitions and mathematical tools of variable-order fractional calculus. In Section 2, we introduce the known Adams- Bashforth-Moulton method; moreover, the effectiveness of the Adams-Bashforth-Moulton method for solving variable-order fractional differential equation is illustrated. In Section 3, we give an introduction of SI Model and we present the Existence and Uniqueness of SI Model. In section 5 we present solution of SI Model by using Adams-Bashforth-Moulton method , the conclusion is given in Section 6.

2 Some mathematical tools

In this part, we give some definitions of fractional derivative and variable-order derivative ([12],[13]).

Definition 1. *let $f \in C_\alpha$ and $\alpha \geq 0$ then the (left-sided) Riemann-Liouville integral of order μ, $\mu > 0$ is given by*

$$I_t^\mu f(t) = \frac{1}{\Gamma(\mu)} \int\limits_0^t (t-\tau) f(\tau) d\tau, \ t > 0.$$

Definition 2. *The (left sided) Caputo fractional derivative of $f, f \in C_{-1}^m, m \in N \cup \{0\}$, is defined as*

$$D_t^\mu f(t) = \begin{cases} \frac{d^m}{dt^m} f(t), & \mu = m \\ I_t^{m-\mu} \frac{d^m f(t)}{dt^m}, & m-1 \leq \mu \leq m, \ m \in N \end{cases}$$

Note that for $m-1 \leq \mu \leq m, \ m \in N$,

$$I_t^\mu D_t^\mu f(t) = f(t) - \sum_{k=0}^{m-1} \frac{d^k f(0)}{dt^k} \frac{t^k}{k!}$$

$$I_t^\mu t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}.$$

Definition 3. *Let $\alpha(t)$ be a positive real number, $f \in C^m[0,T], t \leq T$ and $m = [\max_{0 \leq t \leq T} T\{\alpha(t)\}] + 1$. Then*

$$D^{\alpha(t)} f(t) = \lim_{h_N \to 0} \frac{1}{h_N^{\alpha(t)}} \sum_{k=0}^N (-1)^k \binom{\alpha(t)}{k} f(t-khN). \tag{1}$$

With $h_N = (t-0)/N$ being called the Grunwald-Letnikov variable-order fractional derivative of order $\alpha(t)$ of the function f.

Definition 4. *The Riemann-Liouville variable order derivative is defined as follows:*

$$D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(m-\alpha(t))} \left(\frac{d^m}{dt^m}\right) \int\limits_0^t (t-\tau)^{m-1-\alpha(\tau)} f(\tau) d\tau \tag{2}$$

Where $m = [\max_{0 \leq t \leq T}\{\alpha(t)\}] + 1, m \in N$ provided the right side is point wise defined on $t > 0$.
Let $\alpha(t) > 0$, be a continuous and bounded function, $f(\tau) \in C^m[0,t]$, and $0 \leq \tau \leq t$. then

$$D_t^{\alpha(t)} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha(t))} \int\limits_0^t \frac{f(\tau)}{(t-\tau)^{\alpha(\tau)+1-m}} d\tau, & m-1 \leq \alpha(t) \leq m \\ \frac{d^m}{dt^m} f(t), & \alpha(t) = m. \end{cases} \tag{3}$$

Is called the Caputo variable-order fractional derivative of $f(t)$ where $m = [\max_{0 \leq t \leq T}\{\alpha(t)\}] + 1, m \in N$ and $\Gamma()$ is the Gamma function.

3 The Adams-Bash forth-Moulton Method [16]

In the following we apply the Adams-Bash forth-Moulton predictor-corrector method to implement the numerical solution of variable-order nonlinear FDDEs. Let us consider the following variable-order fractional system:

$$D_t^{\alpha(t)} y(t) = f(y(t), y(t-\tau)), t \in [0,T], 0 < \alpha(t) \le 1, \tag{4}$$

$$y(t) = g(t), t \in [-\tau, 0] \tag{5}$$

where f is in general a nonlinear function.

Also, consider a uniform grid $\{t_n = nh : n = -k, -k+1, .., -1, 0, 1, ..., N\}$ where k and N are integers such that $h = \tau/k$. Let

$$y_h(t_j) = g(t_j), j = -k, -k+1, ..., -1, 0. \tag{6}$$

And note that

$$y_h(t_j - \tau) = y_h(j_h - k_h) = y_h(t_{j-k}), j = 0, 1, 2, ..., N. \tag{7}$$

Applying $I_{t_{n+1}}^{\alpha(t_{n+1})}$ on both sides of (4) and using (5), we claim to:

$$y_h(t_{n+1}) = g(0) + \frac{1}{\Gamma(\alpha(t_{n+1}))} \int_0^{t_{n+1}} (t_{n+1} - \xi)^{\alpha(t_{n+1})-1} f(y(\xi), y(\xi - \tau)) d\xi \tag{8}$$

Further the integral in equation (8) is evaluated using product trapezoidal quadrature formula. Then we have the following corrector formula:

$$y_h(t_{n+1}) = g(0) + \frac{h^{\alpha(t_{n+1})}}{\Gamma(\alpha(t_{n+1})+2)} f(y(t_{n+1}), y((t_{n+1}) - \tau)) + \frac{h^{\alpha(t_{n+1})}}{\Gamma(\alpha(t_{n+1})+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j), y_h(t_j - \tau)),$$

or

$$y_h(t_{n+1}) = g(0) + \frac{h^{\alpha(t_{n+1})}}{\Gamma(\alpha(t_{n+1})+2)} f(y(t_{n+1}), y(t_{n+1-k})) + \frac{h^{\alpha(t_{n+1})}}{\Gamma(\alpha(t_{n+1})+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j), y_h(t_{j-k})), \tag{9}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha(t_{n+1})+1} - (n - \alpha(t_{n+1}))(n+1)^{\alpha(t_{n+1})} & j = 0, \\ (n-j+2)^{\alpha(t_{n+1})+1} + (n-j)^{\alpha(t_{n+1})} - 2(n-j+1)^{\alpha(t_{n+1})+1} & 1 \le j \ge n, \\ 1 & j = n+1, \end{cases} \tag{10}$$

$$y_h(t_{j-k}) \approx v_{n+1} = \begin{cases} \delta y_{n-m+2} + (1-\delta) y_{n-m+1}, & \text{if } m > 1 \\ \delta y_{n+1}^p + (1-\delta) y_n, & \text{if } m = 1 \end{cases} \tag{11}$$

$0 \le \delta < 1$ and the unknown term $y_h(t_{n+1})$ appears on both sides of (9). Due to nonlinearity of f equation (9) can't be solved explicitly for $y_h(t_{n+1})$, so we replace the term $y_h(t_{n+1})$ on the right hand side by an approximation $y_h^p(t_{n+1})$ which called predictor [15]. The product rectangle rule is used in (8) to evaluate predictor term

$$y_h^p(t_{n+1}) = g(0) + \frac{1}{\Gamma(\alpha(t_{n+1}))} \sum_{j=0}^{n} b_{j,n+1} f(y_h(t_j), y_h(t_j - \tau)),$$

or

$$y_h^p(t_{n+1}) = g(0) + \frac{1}{\Gamma(\alpha(t_{n+1}))} \sum_{j=0}^{n} b_{j,n+1} f\left(y_h(t_j), y_h(t_{j-\tau})\right),$$
(12)

where

$$b_{j,n+1} = \frac{h^{\alpha(t_{n+1})}}{\alpha(t_{n+1})} \left((n-j+1)^{\alpha(t_{n+1})} - (n-j)^{\alpha(t_{n+1})}\right).$$
(13)

Now we apply the mentioned method to determine the numerical solution of variable-order fractional system. Consider the following variable-order fractional system:

$$D^{\alpha_1(t)}x = f_1(x,y,z),$$
$$D^{\alpha_2(t)}x = f_2(x,y,z),$$
$$D^{\alpha_3(t)}x = f_3(x,y,z),$$
(14)

With $0 < \alpha_i(t) \leq 1$ and initial condition (x_0, y_0, z_0) According to the previous method, the system (14) can be discredited as follows:

$$x_{n+1} = x_0 + \frac{h^{\alpha_1(t_{n+1})}}{\Gamma(\alpha_1(t_{n+1})+2)} f_1\left(x_{n+1}^p, y_{n+1}^p, z_{n+1}^p\right) + \sum \frac{h^{\alpha_1(t_{n+1})} y_{j,n+1}}{\Gamma(\alpha_1(t_{n+1})+2)} f_1(x_j, y_j, z_j),$$
$$y_{n+1} = y_0 + \frac{h^{\alpha_2(t_{n+1})}}{\Gamma(\alpha_2(t_{n+1})+2)} f_2\left(x_{n+1}^p, y_{n+1}^p, z_{n+1}^p\right) + \sum \frac{h^{\alpha_2(t_{n+1})} y_{2j,n+1}}{\Gamma(\alpha_2(t_{n+1})+2)} f_2(x_j, y_j, z_j),$$
$$z_{n+1} = x_0 + \frac{h^{\alpha_3(t_{n+1})}}{\Gamma(\alpha_3(t_{n+1})+2)} f_3\left(x_{n+1}^p, y_{n+1}^p, z_{n+1}^p\right) + \sum \frac{h^{\alpha_3(t_{n+1})} y_{3j,n+1}}{\Gamma(\alpha_3(t_{n+1})+2)} f_3(x_j, y_j, z_j),$$
(15)

where

$$x_{n+1}^p = x_0 + \sum \frac{\beta_{1j,n+1}}{\Gamma(\alpha_1(t_{n+1}))} f_1(x_j, y_j, z_j)$$
$$y_{n+1}^p = y_0 + \sum \frac{\beta_{2j,n+1}}{\Gamma(\alpha_2(t_{n+1}))} f_2(x_j, y_j, z_j)$$
$$z_{n+1}^p = z_0 + \sum \frac{\beta_{3j,n+1}}{\Gamma(\alpha_3(t_{n+1}))} f_3(x_j, y_j, z_j).$$
(16)

4 4- SI model

The *SI* Model is the simplest one among the epidemic models. That is why it is also called the Simple Model. We divide the population just in the susceptible compartment $S(t)$ and the infectious compartment $I(t)$. We do assume the disease to be highly infectious but not serious, which means that the ineffective remain in contact with susceptible for all time $t \geq 0$. We also assume that the ineffective continue to spread the disease till the end of the epidemic, the population size to be constant $(S(t) + I(t) = N)$ and homogeneous mixing of population. Infection rate is proportional to the number of infective, i.e. $\beta = r\lambda I$ We have a pair of ordinary differential equations for this model:

$$\frac{dS(t)}{dt} = -r\lambda S(t) I(t)$$
$$\frac{dI(t)}{dt} = r\lambda S(t) I(t)$$
(17)

where

$$N = S(t) + I(t)$$
$$S(t) = N - I(t) \tag{18}$$

and therefore we get

$$\frac{dI}{dt} = r\lambda(t)[N - I(t)]. \tag{19}$$

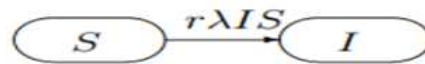What is known as the logistic growth equation.

Fig. 1

4.1 Procedure solution of SI model by using Adams-Bash forth-Moulton method

At Now, we define the equations (17) at differential order $\alpha(t), \alpha(t)$ then this model can be taking the following form:

$$D^{\alpha_1(t)}S = -r\lambda S(t)I(t)$$
$$D^{\alpha_2(t)}S = r\lambda S(t)I(t). \tag{20}$$

With initial conditions $S(0) = 1, I(0) = 1$ where

$$f_1(S,I) = -r\lambda S(t)I(t)$$
$$f_2(S,I) = r\lambda S(t)I(t). \tag{21}$$

By applying the Adams-Bash forth-Moulton Method on SI Model we see the following result

$$S_{n+1} = S_0 + \frac{h^{\alpha_1(t_{n+1})}y_{j,n+1}}{\Gamma(\alpha_1(t_{n+1})+2)}f_1\left(S_{n+1}^p,I_{n+1}^p\right) + \sum \frac{h^{\alpha_2(t_{n+1})}y_{2j,n+1}}{\Gamma(\alpha_2(t_{n+1})+2)}f_2(S_j,I_j),$$

$$I_{n+1} = I_0 + \frac{h^{\alpha_1(t_{n+1})}y_{j,n+1}}{\Gamma(\alpha_1(t_{n+1})+2)}f_1\left(S_{n+1}^p,I_{n+1}^p\right) + \sum \frac{h^{\alpha_2(t_{n+1})}y_{2j,n+1}}{\Gamma(\alpha_2(t_{n+1})+2)}f_2(S_j,I_j), \tag{22}$$

So

$$S_{n+1} = 1 + \frac{h^{\alpha_1(t_{n+1})}y_{j,n+1}}{\Gamma(\alpha_1(t_{n+1})+2)}\left(-r\lambda S_{n+1}^p I_{n+1}^p\right) + \sum \frac{h^{\alpha_2(t_{n+1})}y_{2j,n+1}}{\Gamma(\alpha_2(t_{n+1})+2)}\left(-r\lambda S_j I_j\right)$$

$$I_{n+1} = 1 + \frac{h^{\alpha_1(t_{n+1})}y_{j,n+1}}{\Gamma(\alpha_1(t_{n+1})+2)}\left(-r\lambda S_{n+1}^p I_{n+1}^p\right) + \sum \frac{h^{\alpha_2(t_{n+1})}y_{2j,n+1}}{\Gamma(\alpha_2(t_{n+1})+2)}\left(r\lambda S_j I_j\right). \tag{23}$$

Figs. 1 and 2 show the solutions $S(t)$ and $I(t)$ of system (17) for $\alpha = 1, h = 0.1$ and $\alpha = 0.5, h = 0.1$.
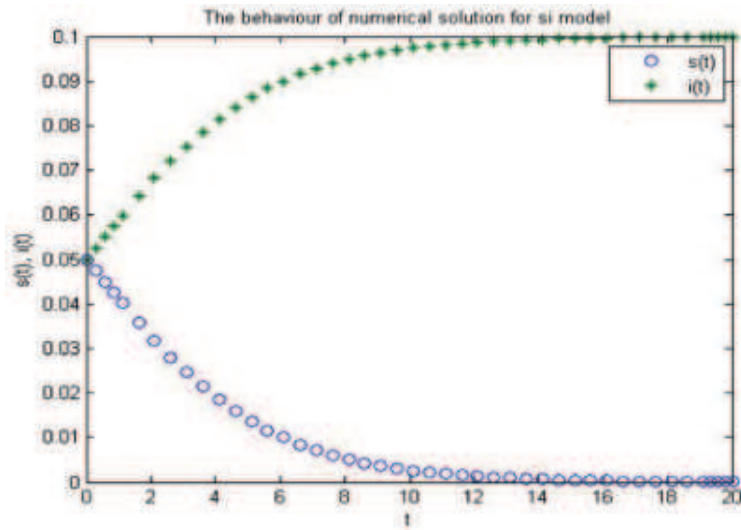
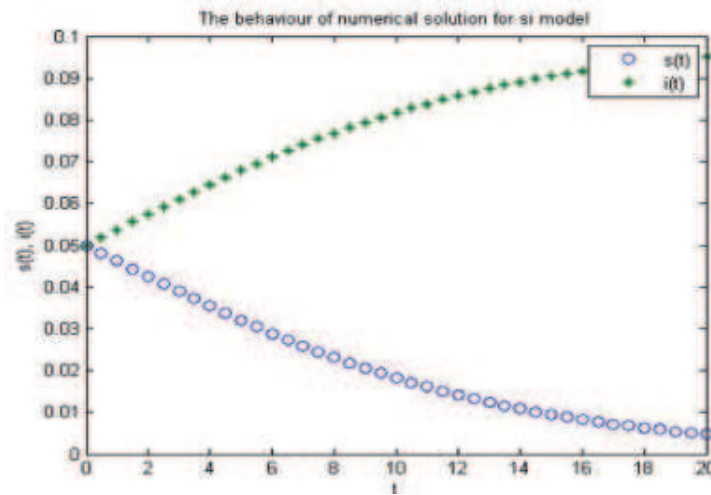**Fig. 2:** The numerical behavior os system (17) at $\alpha = 1$ and $h = 0.1$.

Fig. 3: The numerical behavior os system (17) at $\alpha = 0.5$ and $h = 0.1$.

4.2 Convergent of SI model

Using equation (17), put $\alpha = \alpha = \alpha$ and use

$$D^{\alpha_1} S(t) = \sigma_{\alpha,k} \sum_{j=0}^{n} \omega_j^{(\alpha)} \left[S_{n-j+1} - S_{n-j} \right]$$

and

$$D^{\alpha_2} S(t) = \sigma_{\alpha,k} \sum_{j=0}^{n} \omega_j^{(\alpha)} \left[I_{n-j+1} - I_{n-j} \right],$$

in [14]. Then by using the latest equations. So we have got

$$\sigma_{\alpha,k} \sum_{j=0}^{n} \omega_j^{(\alpha)} \left[S_{n-j+1} - S_{n-j} \right] + r\lambda S_n I_n = 0 \qquad F_1$$

$$\sigma_{\alpha,k} \sum_{j=0}^{n} \omega_j^{(\alpha)} \left[I_{n-j+1} - I_{n-j} \right] - r\lambda S_n I_n = 0 \qquad F_2. \tag{24}$$

We substituting for value of $n = 1, 2, 3, ...,$ at $n = 1$ equation (24) become

$$\sigma_{\alpha,k} \left(\omega_1^{(\alpha)} \left[S_1 - S_0 \right] \right) + r\lambda S_1 I_1 = 0$$

$$\sigma_{\alpha,k} \left(\omega_1^{(\alpha)} \left[I_1 - I_0 \right] \right) - r\lambda S_1 I_1 = 0. \tag{25}$$

By using Jacobin matrix

$$(J_n) = \begin{pmatrix} \frac{\partial F_1}{\partial S_n} & \frac{\partial F_2}{\partial S_n} \\ \frac{\partial F_1}{\partial I_n} & \frac{\partial F_1}{\partial I_n} \end{pmatrix}. \tag{26}$$

Use (25) and (26) to evaluate

$$|J_1| = \begin{pmatrix} \frac{\partial F_1}{\partial S_1} & \frac{\partial F_2}{\partial S_1} \\ \frac{\partial F_1}{\partial I_1} & \frac{\partial F_2}{\partial I_1} \end{pmatrix} = \begin{pmatrix} \sigma_{\alpha,k} \omega_1^{(\alpha)} + r\lambda I_1 & -r\lambda I_1 \\ r\lambda S_1 & \sigma_{\alpha,k} \omega_1^{(\alpha)} + r\lambda S_1 \end{pmatrix}$$

So the determinant of Jacobin matrix equal to

$$|J_1| = \begin{vmatrix} \sigma_{\alpha,k} \omega_1^{(\alpha)} + r\lambda I_1 & -r\lambda I_1 \\ r\lambda S_1 & \sigma_{\alpha,k} \omega_1^{(\alpha)} + r\lambda S_1 \end{vmatrix} = \left(\sigma_{\alpha,k} \omega_1^{(\alpha)} + r\lambda I_1 \right) \left(\sigma_{\alpha,k} \omega_1^{(\alpha)} - r\lambda S_1 \right) + r^2 \lambda^2 S_1 I_1$$

$$= \sigma_{\alpha,k}^2 \omega_1^{2\,(\alpha)} + \sigma_{\alpha,k} \omega_1^{(\alpha)} \left(r\lambda I_1 - r\lambda S_1 \right) - r^2 \lambda^2 S_1 I_1 + r^2 \lambda^2 S_1 I_1 = \sigma_{\alpha,k}^2 \omega_1^{2\,(\alpha)} + \sigma_{\alpha,k} \omega_1^{(\alpha)} \left(r\lambda I_1 - r\lambda S_1 \right) \ (i).$$

At $n = 2$ equation (24) become

$$\sigma_{\alpha,k} \left(\omega_1^{(\alpha)} \left[S_1 - S_0 \right] + \omega_2^{(\alpha)} \left[S_2 - S_1 \right] \right) + r\lambda S_2 I_2 = 0$$

$$\sigma_{\alpha,k} \left(\omega_1^{(\alpha)} \left[I_1 - I_0 \right] + \omega_2^{(\alpha)} \left[I_2 - I_1 \right] \right) - r\lambda S_2 I_2 = 0. \tag{27}$$

By using Jacobin matrix

$$(J_2) = \begin{pmatrix} \frac{\partial F_1}{\partial S_2} & \frac{\partial F_2}{\partial S_2} \\ \frac{\partial F_1}{\partial I_2} & \frac{\partial F_1}{\partial I_2} \end{pmatrix} = \begin{pmatrix} \sigma_{\alpha,k} \omega_2^{(\alpha)} + r\lambda I_2 & -r\lambda I_2 \\ r\lambda S_2 & \sigma_{\alpha,k} \omega_2^{(\alpha)} + r\lambda S_2 \end{pmatrix}. \tag{28}$$

So the determinant of Jacobin matrix equal to

$$|J_2| = \begin{vmatrix} \sigma_{\alpha,k} \omega_2^{(\alpha)} + r\lambda I_2 & -r\lambda I_2 \\ r\lambda S_2 & \sigma_{\alpha,k} \omega_2^{(\alpha)} + r\lambda I_2 \end{vmatrix} = \left(\sigma_{\alpha,k} \omega_2^{(\alpha)} + r\lambda I_2 \right) \left(\sigma_{\alpha,k} \omega_2^{(\alpha)} - r\lambda S_2 \right) + r^2 \lambda^2 S_2 I_2$$

$$= \sigma_{\alpha,k}^2 \omega_2^{2\,(\alpha)} + \sigma_{\alpha,k} \omega_2^{(\alpha)} \left(r\lambda I_2 - r\lambda S_2 \right) - r^2 \lambda^2 S_2 I_2 + r^2 \lambda^2 S_2 I_2 = \sigma_{\alpha,k}^2 \omega_2^{2\,(\alpha)} + \sigma_{\alpha,k} \omega_2^{(\alpha)} \left(r\lambda I_2 - r\lambda S_2 \right) \ (ii),$$

and so on

$$\vdots$$

At n equation (24) become

$$\sigma_{\alpha,k}\left(\omega_1^{(\alpha)}\left[S_1 - S_0\right] + ... + \omega_n^{(\alpha)}\left[S_n - S_{n-1}\right]\right) + r\lambda S_n I_n = 0$$

$$\sigma_{\alpha,k}\left(\omega_1^{(\alpha)}\left[I_1 - I_0\right] + ... + \omega_n^{(\alpha)}\left[I_n - I_{n-1}\right]\right) + r\lambda S_n I_n = 0. \tag{29}$$

By using Jacobin matrix

$$(J_n) = \begin{pmatrix} \frac{\partial F_1}{\partial S_n} & \frac{\partial F_2}{\partial S_n} \\ \frac{\partial F_1}{\partial I_n} & \frac{\partial F_1}{\partial I_n} \end{pmatrix} = \begin{pmatrix} \sigma_{\alpha,k}\omega_n^{(\alpha)} + r\lambda I_n & -r\lambda I_n \\ r\lambda S_n & \sigma_{\alpha,k}\omega_n^{(\alpha)} + r\lambda S_n \end{pmatrix}. \tag{30}$$

So the determinant of Jacobin matrix equal to

$$|J_n| = \begin{vmatrix} \sigma_{\alpha,k}\omega_n^{(\alpha)} + r\lambda I_n & -r\lambda I_n \\ r\lambda S_n & \sigma_{\alpha,k}\omega_n^{(\alpha)} + r\lambda I_n \end{vmatrix} = \left(\sigma_{\alpha,k}\omega_n^{(\alpha)} + r\lambda I_n\right)\left(\sigma_{\alpha,k}\omega_n^{(\alpha)} - r\lambda S_n\right) + r^2\lambda^2 S_n I_n$$

$$= \sigma_{\alpha,k}^2\omega_n^{2\,(\alpha)} + \sigma_{\alpha,k}\omega_n^{(\alpha)}\left(r\lambda I_n - r\lambda S_n\right) - r^2\lambda^2 S_n I_n + r^2\lambda^2 S_n I_n = \sigma_{\alpha,k}^2\omega_2^{2\,(\alpha)} + \sigma_{\alpha,k}\omega_2^{(\alpha)}\left(r\lambda I_2 - r\lambda S_2\right) \ (iii).$$

By $((i),(ii),(iii))$ *SI* model convergent uniformly If $|J_n| = 0$ then weather or $\sigma_{\alpha,k}\omega_n^{(\alpha)} = 0$.

$$\sigma_{\alpha,k}\omega_n^{(\alpha)}\left(r\lambda I_n - r\lambda S_n\right) = 0$$

$$\sigma_{\alpha,k}\omega_n^{(\alpha)} = r\lambda\left(I_n - S_n\right).$$

In this section, we study the numerical solution of the non-linear fractional SI model of the form:

$$\begin{aligned} D^{\alpha_1(t)}S &= -r\lambda S(t)I(t), \\ D^{\alpha_2(t)}S &= r\lambda S(t)I(t), \end{aligned} \quad t > 0, \ 0 < \alpha_1, \alpha_2 \le 1. \tag{31}$$

The parameter $\alpha(t), \alpha(t)$ refers to the fractional order of the time derivative. With initial condition.

$$S(0) = S_0, I(0) = I_0. \tag{32}$$

4.3 Existence and uniqueness of SI model

Let $J = [0,1], T < \infty$ and C(J) be the class of all continuous functions defined on J, with the norm. $\|S\| = \sup_{t \in J}\left|e^{-Nt}S(t)\right|$, $\|I\| = \sup_{t \in J}\left|e^{-Nt}I(t)\right|$, $N > 0$, which is equivalent to the sup-norm $\|S\| = \sup_{t \in J}|s(t)|, \|I\| = \sup_{t \in J}|I(t)|$. To study the existence and the uniqueness of the initial value problem of the fractional SI model (30), we suppose that the solution S(t),I(t) belongs to the space $B = \{S, I \in \Re : |S, I| \le b$ for any constant b The initial Value problem (32) has a unique solution $S, I \in C(J)$.

$$S^l, I^l \in X = \left\{S, I \in L[0,1], \|S\| = \left\|e^{-Nt}S(t)\right\| L_1, \|I\| = \left\|e^{-Nt}S(t)\right\| L_1\right\}.$$

Proof. Form properties of fractional calculus. The fractional differential equation (30) can be written as.

$$\begin{aligned} \mathrm{I}^{1-\alpha}\frac{dS(t)}{dt} &= -r\lambda S(t)I(t) \\ \mathrm{I}^{1-\alpha}\frac{dI(t)}{dt} &= r\lambda S(t)I(t). \end{aligned} \tag{33}$$

Operating with I^α so we obtain

$$S(t) = I^\alpha [-r\lambda S(t) I(t)]$$
$$I(t) = I^\alpha [r\lambda S(t) I(t)]. \qquad (34)$$

Now let us defined the operator $F : C(J) \to C(J)$ by

$$FS(t) = I^\alpha [-r\lambda S(t) I(t)]$$
$$FI(t) = I^\alpha [r\lambda S(t) I(t)]. \qquad (35)$$

Then

(1)

$$e^{-Nt} (FS_1 - FS_2) = e^{-Nt} I^\alpha [-r\lambda S_1(t) I(t) + r\lambda S_2(t) I(t)]$$
$$= e^{-Nt} I^\alpha r\lambda I(t) [S_2(t) - S_1(t)]$$
$$\leq \int_0^t \frac{(t-S)^{\alpha-1}}{\Gamma(\alpha)} e^{N(t-s)} (-r\lambda) I^\alpha I(t) [S_2(t) - S_1(t)] e^{-Ns} ds$$
$$\leq \|I_2 - I_1\| \int_0^t \frac{-r\lambda S^{\alpha-1} e^{-Ns}}{\Gamma(\alpha)} ds$$

There for we obtain

$$e^{-Nt} (FS_1 - FS_2) \leq \|S_2 - S_1\|.$$

(2)

$$e^{-Nt} (FI_1 - FI_2) = e^{-Nt} I^\alpha [-r\lambda I_1(t) S(t) + r\lambda I_2(t) S(t)]$$
$$= e^{-Nt} I^\alpha r\lambda S(t) [I_2(t) - I_1(t)]$$
$$\leq \int_0^t \frac{(t-S)^{\alpha-1}}{\Gamma(\alpha)} e^{N(t-s)} (-r\lambda) I^\alpha S(t) [I_2(t) - I_1(t)] e^{-Ns} ds$$
$$\leq \|I_2 - I_1\| \int_0^t \frac{-r\lambda S^{\alpha-1} e^{-Ns}}{\Gamma(\alpha)} ds.$$

There for we obtain

$$e^{-Nt} (FI_1 - FI_2) \leq \|I_2 - I_1\|.$$

And operator F given by (35) has a unique fixed point. Consequently the integral equation (32) has a unique solution, $S, I \in C(J)$.

Now from equation (32), we formally have

(1)

$$S(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} \left[(-r\lambda S_0 I) + I^\alpha \left(-r\lambda S^l I \right) \right],$$

and

$$\frac{dS(t)}{dt} = \frac{t^{\alpha-1}}{\Gamma(\alpha+1)} \left[(-r\lambda S_0 I) + I^\alpha \left(-r\lambda S^l I \right) \right]$$
$$e^{-Nt} S^l(t) = e^{-Nt} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha+1)} \left[(-r\lambda S_0 I) + I^\alpha \left(-r\lambda S^l I \right) \right] \right].$$

From which we can deduce that $S^l \in C(J)$ and $S^l \in X$ Now from Eq.(35), we get

$$\frac{dS(t)}{dt} = \frac{d}{dt} I^\alpha [-r\lambda SI],$$

$$I^{\alpha-1} \frac{dS(t)}{dt} = I^{1-\alpha} \frac{d}{dt} [-r\lambda SI] = \frac{d}{dt} I^{1-\alpha} I^\alpha [-r\lambda SI],$$

$$D^\alpha S(t) = \frac{d}{dt} I[-r\lambda SI] = -r\lambda SI,$$

and

$$S(0) = I^\alpha [-r\lambda SI]_{t=0} = 0.$$

(2)

$$I(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} \left[(-r\lambda SI_0) + I^\alpha \left(-r\lambda SI^l \right) \right]$$

and

$$\frac{dI(t)}{dt} = \frac{t^{\alpha-1}}{\Gamma(\alpha+1)} \left[(-r\lambda SI_0) + I^\alpha \left(-r\lambda SI^l \right) \right]$$

$$e^{-Nt} I^l(t) = e^{-Nt} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha+1)} \left[\left(-r\lambda SI^l \right) + I^\alpha \left(-r\lambda SI^l \right) \right] \right].$$

From which we can deduce that $I^l \in C(J)$ and $I^l \in X$ now from Eq. (35), we get

$$\frac{dI(t)}{dt} = \frac{d}{dt} I^\alpha [r\lambda SI],$$

$$I^{\alpha-1} \frac{dI(t)}{dt} = I^{1-\alpha} \frac{d}{dt} [r\lambda SI] = \frac{d}{dt} I^{1-\alpha} I^\alpha [r\lambda SI],$$

$$D^\alpha I(t) = \frac{d}{dt} I[r\lambda SI] = r\lambda SI,$$

and

$$I(0) = I^\alpha [r\lambda SI]_{t=0} = 0.$$

Then the integral equation (32) is equivalent to the initial value problem (31) and theorem is proved.

5 Conclusions

Variable-order fractional calculus has been highly neglected since it was proposed. Nevertheless, the scientific community has found a large variety of applications which can be modeled and more clearly understood by using this branch of mathematics. By using the Adams-Bash forth-Moulton method, we obtain the numerical solution of the variable-order fractional SI Model We present a convergent of SI Model in section 4, also we prove that the Existence and Uniqueness of SI Model. Moreover, we hope our work about variable-order fractional calculus would generate interest from related scholars in the future and also hope that their work may result in significant contributions to this field.

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