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On some generalised I-convergent sequence spaces of double interval numbers

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Abstract: In this article we introduce and study some spaces of I-convergent sequences of double interval numbers with the help of a double sequence $\mathscr{F} = (f_{i,j})$ of modulii and double bounded sequence $p = (p_{i,j})$ of positive real numbers. We study some topological and algebraic properties, prove the decomposition theorem and study some inclusion relations on these spaces.

Keywords: Double interval numbers, ideal, filter, double I-convergent sequence spaces, solid and monotone space, Banach space, modulus function.

1 Introduction

Recently, Chiao[4] introduced the sequences of interval numbers and defined usual convergence of sequences of interval numbers. Seng*ö*n*ü*l and Eryimaz[28] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete.

A set(closed interval) of real numbers x such that $a \leq x \leq b$ is called an interval number.[4] A real interval can also be considered as a set. Thus, we can investigate some properties of interval numbers for instance, arithmetic properties or analysis properties. Let us denote the set of all real valued closed intervals by $I\mathbb{R}$. Any element of $I\mathbb{R}$ is called a closed interval and it is denoted by $\bar{A} = [x_l, x_r]$. $I\mathbb{R}$ is a quasilinear space under the algebraic operations and partial order relation for $I\mathbb{R}$ found in [28,31]. and any subspace of $I\mathbb{R}$ is called quasilinear subspace.

The set of all interval numbers $I\mathbb{R}$ is a complete metric space defined by

$$d(\bar{A_1}, \bar{A_2}) = max| x_{1_l} - x_{2_r} |, | x_{2_l} - x_{2_r} |. \text{ See([17,28])} \tag{1}$$

where x_l and x_r are the first and last point of $\bar{A}$ respectively.

Vakeel A. Khan and Mohd. Shafiq defined the transformation f from $\mathbb{N}$ to $I\mathbb{R}$ by $k \to f(k) = \bar{\mathscr{A}}, \bar{\mathscr{A}} = (\bar{A_k})$. The function f is called sequence of interval numbers, where $\bar{A_k}$ is the k^{th} term of the sequence $(\bar{A_k})$. Let us denote the set of sequences of interval numbers with real terms by

$$\omega(\bar{\mathscr{A}}) = \{\bar{\mathscr{A}} = (\bar{A_k}) : \bar{A_k} \in I\mathbb{R}\}. \tag{2}$$

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The following definitions were given by Sengönül and Eryimaz[28]. A sequence $\mathscr{A} = (\bar{A}_k) = ([x_{k_l}, x_{k_r}])$ of interval numbers is said to be convergent to an interval number $\bar{A}_0 = [x_{0_l}, x_{0_r}]$ if for each $\varepsilon > 0$, there exists a positive integer n_0 such that $d(\bar{A}_k, \bar{A}_0) < \varepsilon$, for all $k \geq n_0$ and we denote it as $\lim_k \bar{A}_k = \bar{A}_0$.

Thus, $\lim_k \bar{A}_k = \bar{A}_0 \Leftrightarrow \lim_k x_{k_l} = x_{0_l}$ and $\lim_k x_{k_r} = x_{0_r}$ and it is said to be Cauchy sequence of interval numbers if for each $\varepsilon > 0$, there exists a positive integer k_0 such that $d(\bar{A}_k, \bar{A}_m) < \varepsilon$, whenever $k, m \geq k_0$. Ayhan Esi and B. Hazarika[1] dfined a transformation f from $\mathbb{N} \times \mathbb{N}$ to $I\mathbb{R}$ by $i, j \to f(i, j) = \bar{\mathscr{A}}, \bar{\mathscr{A}} = (\bar{A}_{i,j})$. Then $\bar{\mathscr{A}} = (\bar{A}_{i,j})$ is called double sequence of interval numbers. The $\bar{A}_{i,j}$ is called the $(i, j)^{th}$ term of double sequence of interval numbers $\bar{\mathscr{A}} = (\bar{A}_{i,j})$.

Let us denote the set of double sequence of interval numbers by

$$_2\omega(\bar{\mathscr{A}}) = \{\bar{\mathscr{A}} = (\bar{A}_{i,j}) : \bar{A}_{i,j} \in I\mathbb{R}\}. \tag{3}$$

Definition 1. *An interval valued double sequence $\bar{\mathscr{A}} = (\bar{A}_{i,j})$ is said to be convergent in the Pringsheim's sense or P-convergent to an interval number $\bar{A}_0$, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(\bar{A}_{i,j}, \bar{A}_0) < \varepsilon$, for $i, j > N$ and we denote it by $P - \lim_{i,j} \bar{A}_{i,j} = \bar{A}_0$. The interval number $\bar{A}_0$ is called the Pringsheim limit of $\bar{\mathscr{A}} = (\bar{A}_{i,j})$.*

More exactly, we say that a double sequence of interval numbers $\bar{\mathscr{A}} = (\bar{A}_{i,j})$ converges to a finite interval number $\bar{A}_0$ if $\bar{A}_{i,j}$ tends to $\bar{A}_0$ as both i and j tend to infinity independently of each another. $\bar{\mathscr{A}} = (\bar{A}_{i,j})$ is said to be null if $\bar{A}_0 = \bar{0}$.

Definition 2. *An interval valued double sequence $\bar{\mathscr{A}} = (\bar{A}_{i,j})$ is bounded if there exists a positive number M such that $d(\bar{A}_{i,j}, \bar{A}_0) \leq M$ for all $i, j \in \mathbb{N}$.*

Definition 3. *An interval valued double sequence $\bar{\mathscr{A}} = (\bar{A}_{i,j})$ is said be Cauchy sequence if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(\bar{A}_{i,j}, \bar{A}_{m,n}) < \varepsilon$ whenever $i \geq m \geq N$ and $j \geq n \geq N$.*

Let $p = (p_{i,j})$ be a double sequence positive real numbers. If $0 < p_{i,j} \leq \sup_{i,j} p_{i,j} = H < \infty$ and $D = max(1, 2^{H-1})$, then for $a_{i,j}, b_{i,j} \in \mathbb{R}$ and for all $i, j \in \mathbb{N}$ we have $| a_{i,j} + b_{i,j} |^{p_{i,j}} \leq D(| a_{i,j} |^{p_{i,j}} + | b_{i,j} |^{p_{i,j}})$.

Let us denote the space of all double convergent, double null and double bounded sequences of double interval numbers by $_2c(\bar{\mathscr{A}})$, $_2c_0(\bar{\mathscr{A}})$ and $_2\ell_\infty(\bar{\mathscr{A}})$ respectively.

The spaces $_2c(\bar{\mathscr{A}})$, $_2c_0(\bar{\mathscr{A}})$ and $_2\ell_\infty(\bar{\mathscr{A}})$ are complete metric spaces with the metric

$$\hat{d}(\bar{A}_{i,j}, \bar{B}_{i,j}) = \sup_{i,j} max\{| x_{(i,j)_l} - y_{(i,j)_l} |, | x_{(i,j)_r} - y_{(i,j)_r} |\} \tag{4}$$

If we take $\bar{B}_{i,j} = \bar{0}$ in (4), then the metric $\hat{d}$ reduces to

$$\hat{d}(\bar{A}_{i,j}, \bar{0}) = \sup_{i,j} max\{| x_{(i,j)_l} |, | x_{(i,j)_r} |\} \tag{5}$$

In this paper we assume that a norm $\|\bar{A}_{i,j}\|$ of the double sequence of interval numbers $(\bar{A}_{i,j})$ is the distance from $(\bar{A}_{i,j})$ to $\bar{0}$ and satisfies the following properties: For all $\bar{A}_{i,j}, \bar{B}_{i,j} \in {}_2\lambda(\bar{\mathscr{A}})$ and for all $\alpha \in \mathbb{R}$,

(N1) $\|\bar{A}_{i,j}\|_{_2\lambda(\bar{\mathscr{A}})} > 0, \quad \forall \quad \bar{A}_{i,j} \in {}_2\lambda(\bar{\mathscr{A}}) - \{\bar{0}\}$,
(N2) $\|\bar{A}_{i,j}\|_{_2\lambda(\bar{\mathscr{A}})} = 0 \Leftrightarrow \bar{A}_{i,j} = \bar{0}$,
(N3) $\|\bar{A}_{i,j} + \bar{B}_{i,j}\|_{_2\lambda(\bar{\mathscr{A}})} \leq \|\bar{A}_{i,j}\|_{_2\lambda(\bar{\mathscr{A}})} + \|\bar{B}_{i,j}\|_{_2\lambda(\bar{\mathscr{A}})}$

(N4) $\|\alpha\overline{A}_{i,j}\|_{2\lambda(\overline{\mathscr{A}})} = |\alpha| \, \|\overline{A}_{i,j}\|_{2\lambda(\overline{\mathscr{A}})}$, where ${}_2\lambda(\overline{\mathscr{A}})$ is a subset of ${}_2\omega(\overline{\mathscr{A}})$.
Let $\overline{\mathscr{A}} = (\overline{A}_{i,j}) = ([x_{(i,j)_l}, x_{(i,j)_r}])$ be the element of ${}_2c(\overline{\mathscr{A}})$, ${}_2c_0(\overline{\mathscr{A}})$ and ${}_2\ell_\infty(\overline{\mathscr{A}})$. Then the classes of sequences ${}_2c(\overline{\mathscr{A}})$, ${}_2c_0(\overline{\mathscr{A}})$ and ${}_2\ell_\infty(\overline{\mathscr{A}})$ are double normed interval spaces normed by

$$\|\overline{\mathscr{A}}\| = \sup_{i,j} max\{| x_{(i,j)_l}, x_{(i,j)_r} |\}. \tag{6}$$

The notion of I-convergence was initially introduced by Kostyrko, et. al[15] as generalization of statistical convergence(See [6],[27]) which is based on the structure of the ideal I of subsets of natural numbers $\mathbb{N}$. Kostyrko, et. al. gave some of basic properties of I-convergence and dealt with extremal I-limit points. Although an ideal is defined as a heredity and additive family of subsets of a non-empty arbitrary set X, here in our study it suffices to take I as a family of subsets of $\mathbb{N}$, positive integers, i.e.$I \subset 2^{\mathbb{N}}$, such that $A \cup B \in I$ for each $A, B \in I$, and each subset of an element of I is an element of I.

A non-empty family of sets $\mathscr{F} \subset 2^{\mathbb{N}}$ is a filter on $\mathbb{N}$ if and only if $\phi \notin \mathscr{F}, A \cap B \in \mathscr{F}$, for each $A, B \in \mathscr{F}$, and any superset of an element of $\mathscr{F}$ is an element of $\mathscr{F}$. An ideal I is called non-trivial if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly I is non-trivial ideal if and only if $\mathscr{F} = \mathscr{F}(I) = \{\mathbb{N} - A : A \in I\}$ is a filter in $\mathbb{N}$, called the filter associated with the ideal I. A non-trivial ideal I is called admissible if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$. A non-trivial ideal I is maximal if there can not exist any non-trivial ideal $J \neq I$ containing I as a subset. Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to a real number ℓ if $\{k \in \mathbb{N} : | x_k - \ell | \geq \varepsilon\} \in I$ for every $\varepsilon > 0$([15]). In this case we write $I - lim x_k = \ell$. The notion of I-convergence double sequence was initially introduced by Tripathy and Tripathy(See[31]).

Let I be an ideal of $\mathbb{N} \times \mathbb{N}$. Then a double sequence of interval numbers $\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}}) \subset {}_2\omega(\overline{\mathscr{A}})$,

(i) is said to be I-convergent to an interval number $\overline{A}_0$ if for every $\varepsilon > 0$,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : \|\overline{A}_{i,j} - \overline{A}_0\| \geq \varepsilon\} \in I.$$

In this case we write $I - lim\overline{A}_{i,j} = \overline{A}_0$. If $\overline{A}_0 = \overline{0}$. Then the sequence $\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}})$ is said to be I-null. In this case we write $I - lim\overline{A}_{i,j} = \overline{0}$.

(ii) is said to be I-Cauchy, if for every $\varepsilon > 0$, there exist numbers $m = m(\varepsilon), n = n(\varepsilon)$ such that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : \|\overline{A}_{i,j} - \overline{A}_{m,n}\| \geq \varepsilon\} \in I,$$

(iii) is said to be I-bounded, if there exists some $M > 0$ such that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : \|\overline{A}_{i,j}\| \geq M\} \in I.$$

Definition 4. *A sequence space ${}_2\lambda(\overline{\mathscr{A}})$ of double sequence of interval numbers,*

(i) *is said be solid(normal), if $(\alpha_{i,j}\overline{A}_{i,j}) \in {}_2\lambda(\overline{\mathscr{A}})$, whenever $(\overline{A}_{i,j}) \in {}_2\lambda(\overline{\mathscr{A}})$ and for any double sequence $(\alpha_{i,j})$ of scalars with $| \alpha_{i,j} | \leq 1$, for all $(i,j) \in \mathbb{N} \times \mathbb{N}$,*
(ii) *is said be symmetric, if $(\overline{A}_{\pi(i,j)}) \in {}_2\lambda(\overline{\mathscr{A}})$, whenever $(\overline{A}_{i,j}) \in {}_2\lambda(\overline{\mathscr{A}})$ where π is permutation on $\mathbb{N} \times \mathbb{N}$,*
(iii) *is said be sequence algebra, if $(\overline{A}_{i,j}) * (\overline{B}_{i,j}) = (\overline{A}_{i,j}.\overline{B}_{i,j}) \in {}_2\lambda(\overline{\mathscr{A}})$, whenever $(\overline{A}_{i,j}), (\overline{B}_{i,j}) \in {}_2\lambda(\overline{\mathscr{A}})$,*
(iv) *is said be convergence free, if $(\overline{B}_{i,j}) \in {}_2\lambda(\overline{\mathscr{A}})$ whenever $(\overline{A}_{i,j}) \in {}_2\lambda(\overline{\mathscr{A}})$ and $\overline{A}_{i,j} = \overline{0}$ implies $\overline{B}_{i,j} = \overline{0}$, for all i,j.*

Definition 5. *Let $K = \{(i_1, j_1) < (i_2, j_2) < (i_3, j_3) < \cdots\} \subset \mathbb{N} \times \mathbb{N}$. The K-step space of ${}_2\lambda(\overline{\mathscr{A}})$, is a sequence space*

$${}_2\mu_k^{2\lambda(\overline{\mathscr{A}})} = \{(\overline{A}_{(i_n,j_n)} \in {}_2\omega(\overline{\mathscr{A}}) : (\overline{A}_{i,j}) \in {}_2\lambda(\overline{\mathscr{A}})\}.$$

Definition 6. *A canonical preimage of a double sequence of interval numbers* $(\bar{A}_{i_n,j_n}) \in {}_2\mu_k^{2\lambda(\bar{\mathscr{A}})}$ *is double sequence* $(\bar{B}_{i,j}) \in {}_2\omega(\bar{\mathscr{A}})$ *defined by*

$$\bar{B}_{i,j} = \begin{cases} \bar{A}_{i,j}, & \text{if } (i,j) \in K, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space ${}_2\mu_k^{2\lambda(\bar{\mathscr{A}})}$ is a set of canonical preimages of all elements in ${}_2\mu_k^{2\lambda(\bar{\mathscr{A}})}$. That is $\bar{\mathscr{B}}$ is the canonical preimage of ${}_2\mu_k^{2\lambda(\bar{\mathscr{A}})}$ if and only if $\bar{\mathscr{B}}$ is the canonical preimage of some $\bar{\mathscr{A}} \in {}_2\mu_k^{2\lambda(\bar{\mathscr{A}})}$.

Definition 7. *A sequence space* ${}_2\lambda(\bar{\mathscr{A}})$ *is said to be monotone if it contains the canonical preimage of its step space.*

Definition 8. *A function* $f : [0,\infty) \to [0,\infty)$ *is called a modulus function if*

(i) $f(t) = 0$ *if and only if* $t = 0$,
(ii) $f(t+u) \le f(t) + f(u)$ *for all* $t,u \ge 0$,
(iii) f *is increasing,*
(iv) f *is continuous from the right at zero.*

A modulus function f is said to satisfy $\triangle_2$-condition for all values of u if there exists a constant $K > 0$ such that $f(Lu) \le KLf(u)$ for all values of $L > 1$. The idea of modulus function was introduced by Nakano in 1953,(see[20], Nakano, 1953).

For any modulus function f, we have the inequalities $| f(x) - f(y) | \le f(x-y)$ and $f(nx) \le nf(x)$, for all $x,y \in [0,\infty)$.

Ruckle[21-23] used the idea of modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(| x_k |) < \infty\} = \{x = x_k : (f(| x_k |) \in X\}. \tag{7}$$

After then, E. Kolk[12,13] gave an extension of $X(f)$ by considering a sequence of moduli $\mathscr{F} = (f_k)$ and defined the sequence space

$$X(f) = \{x = (x_k) : (f_k(| x_k |)) \in X\}. \tag{8}$$

Now we give an extension of $X(f)$ by considering a double sequence of modulii $\mathscr{F} = (f_{i,j})$ and define the sequence space

$${}_2X(f) = \{x = (x_{i,j}) : (f_{i,j}(| x_{i,j} |)) \in X\}. \tag{9}$$

Mursaleen and Naman[18] introduced the notion of λ-convergent and λ-bounded sequences.

Vakeel A. Khan and Mohd. shafiq extended this concept to the sequence of interval numbers as follows: Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity. That is

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \lambda_k \to \infty \qquad \text{as} \qquad k \to \infty. \tag{10}$$

The sequence $\bar{\mathscr{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathscr{A}})$ is λ-convergent to an interval number $\bar{A}_0$, called the λ-limit of $\bar{\mathscr{A}}$, if $\wedge_m(\bar{\mathscr{A}}) \to \bar{A}_0$ as $m \to \infty$, where

$$\wedge_m(\bar{\mathscr{A}}) = \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})\bar{A}_k, \qquad k \in \mathbb{N}.$$

Any term with a negative subscript is equal to naught. For example $\lambda_{-1} = 0$.

In particular, $\overline{\mathscr{A}} = (\overline{A}_k) \in \ell_\infty(\overline{\mathscr{A}})$ is said to be λ-null, if $\wedge_m(\overline{\mathscr{A}}) \to 0$ as $m \to \infty$.

The sequence $\overline{\mathscr{A}} = (\overline{A}_k) \in \ell_\infty(\overline{\mathscr{A}})$ is λ-bounded if $\sup_m \| \wedge_m (\overline{\mathscr{A}})\| < \infty$. It can be seen that if $\lim_m \overline{A}_m = \overline{A}$ in the ordinary sense of convergence of interval numbers, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) \|\overline{A}_k - \overline{A}\| \right) \right) = 0. \tag{11}$$

This implies that

$$\lim_m \| \wedge_m (\overline{\mathscr{A}}) - \overline{A}\| = \lim_m \| \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(\overline{A}_k - \overline{A})\| = 0, \tag{12}$$

which yields that

$\lim_m \wedge_m(\overline{\mathscr{A}}) = \overline{A}$ and hence $\overline{\mathscr{A}} = (\overline{A}_k) \in \ell_\infty(\overline{\mathscr{A}})$ is λ-convergent to $\overline{A}$.

On generalizing the above notation we introduce the concept of λ- convergence and λ-boundedness for double sequence of interval numbers.

Let $\lambda = (\lambda_{i,j})$ be a strictly increasing double sequence of positive real numbers tending to infinity. That is, $0 < \lambda_{i_0,j_0} < \lambda_{i_1,j_1} < \cdots < \lambda_{i_k,j_k} < \cdots \quad \lambda_{i_k,j_k} \to \infty$ as $i_k, j_k \to \infty$.

The double sequence $\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}})$ is said to be λ-convergent to an interval number $\overline{A}_0$, called the λ-limit of $\overline{\mathscr{A}}$, if $\wedge_{i,j}(\overline{\mathscr{A}}) \to \overline{A}_0$, as $i, j \to \infty$, where

$$\wedge_{i,j}(\overline{\mathscr{A}}) = \frac{1}{\lambda_{m,n}} \sum_{i=1}^{m} \sum_{j=1}^{n} (\lambda_{i,j} - \lambda_{i-1,j-1}) \overline{A}_{i,j}, \qquad (i,j) \in \mathbb{N} \times \mathbb{N}.$$

Here and in the sequal, we shall use $\lambda_{-1,-1} = 0$.

In particular, $\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}})$ is said to be λ-null, if $\wedge_{i,j}(\overline{\mathscr{A}}) \to 0$, as $i, j \to \infty$.

The double sequence $\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}})$ is λ-bounded, if $\sup_{i,j} \| \wedge_{i,j} (\overline{\mathscr{A}})\| < \infty$. It can be seen that if $\lim_{i,j} \overline{A}_{i,j} = \overline{A}$ in the Pringsheim's sense of convergence of double interval numbers, then

$$\lim_{i,j} \left(\frac{1}{\lambda_{m,n}} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} (\lambda_{i,j} - \lambda_{i-1,j-1}) \|\overline{A}_{i,j} - \overline{A}\| \right) \right) = 0 \tag{13}$$

This implies that

$$\lim_{i,j} \| \wedge (\overline{\mathscr{A}}) - \overline{A}\| = \lim_{i,j} \| \frac{1}{\lambda_{m,n}} \sum_{i=1}^{m} \sum_{j=1}^{n} (\lambda_{i,j} - \lambda_{i-1,j-1})(\overline{A}_{i,j} - \overline{A})\| = 0 \tag{14}$$

which yields that $\lim_{i,j} \wedge_{i,j}(\overline{\mathscr{A}}) = \overline{A}$ and hence $\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}})$ is λ-convergent to $\overline{A}$.

Let us denote the classes of double I-convergent, double I-null, double bounded I-convergent and double bounded I-null sequences of double interval numbers by ${}_2c^I(\overline{\mathscr{A}})$, ${}_2c_0^I(\overline{\mathscr{A}})$, ${}_2\mathscr{M}_c^I(\overline{\mathscr{A}})$ and ${}_2\mathscr{M}_{c_0}^I(\overline{\mathscr{A}})$, respectively.

Now we give some important lemmas.

Lemma 1. *Every solid space is monotone.*

Lemma 2. *Let $K \in \mathscr{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$ where $\mathscr{F}(I) \subseteq 2^{\mathbb{N}}$ filter on $\mathbb{N}$.*

Lemma 3. *If $I \subseteq 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap \mathbb{N} \notin I$.*

Definition 9. [30] *Let $\overline{X}$ be the space of interval numbers. A function $g : \overline{X} \to \mathbb{R}$ is called a paranorm on $\overline{X}$, if for all $A, B \in \overline{X}$, (P_1) $g(A) = 0$, if $A = \overline{0}$, (P_2) $g(A) \geq 0$, (P_3) $g(-A) = g(A)$, (P_4) $g(A+B) \leq g(A) + g(B)$, (P_5) if λ_n is a sequence of scalars with $\lambda_n \to \lambda \quad (n \to \infty)$ and $(A_n), A_0 \in \overline{X}$ with $g(A_n) \to g(A_0) (n \to \infty)$ then $g(\lambda_n A_n - \lambda A_0) \to 0 \quad (n \to \infty)$, (P_6) If $A \leq B$, then $g(A) \leq g(B)$.*

In this article, we introduce and study the following classes of double sequences:

Let I be an ideal of $\mathbb{N} \times \mathbb{N}$ and $(p_{i,j})$ be a double bounded sequence positive real numbers.

$$ {}_2C^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) = \{\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}}) : \{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}}) - \overline{A}\|)^{p_{i,j}} \geq \varepsilon\} \in I, \text{for some} \overline{A}\}, \tag{15} $$
$$ {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) = \{\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}}) : \{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}})\|)^{p_{i,j}} \geq \varepsilon\} \in I\} \tag{16} $$
$$ {}_2\ell_\infty^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) = \{\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}}) : \exists K > 0 s.t. \{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}})\|)^{p_{i,j}} \geq K\} \in I\} \tag{17} $$
$$ {}_2\ell_\infty(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) = \{\mathscr{A} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\mathscr{A}) : \sup_{i,j} f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}})\|)^{p_{i,j}} < \infty\}. \tag{18} $$

We also denote

$$ {}_2\mathscr{M}_c^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) = {}_2\ell_\infty(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) \cap {}_2C^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p), $$

and

$$ {}_2\mathscr{M}_{c_0}^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) = {}_2\ell_\infty(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) \cap {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p), $$

where $\mathscr{F} = (f_{i,j})$ is a double sequence of moduli and $\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}}) \subset {}_2\omega(\overline{\mathscr{A}})$ is a double bounded sequence of interval numbers. If we take $p = (p_{i,j}) = 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$, we have

$$ {}_2C^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}) = \{\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}}) : \{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}}) - \overline{A}\|) \geq \varepsilon\} \in I, \text{for some} \overline{A}\}, \tag{19} $$
$$ {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}) = \{\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}}) : \{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}})\|) \geq \varepsilon\} \in I\} \tag{20} $$
$$ {}_2\ell_\infty^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}) = \{\overline{\mathscr{A}} = (\overline{A}_{i,j}) \in {}_2\ell_\infty(\overline{\mathscr{A}}) : \exists K > 0 s.t. \{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}})\|) \geq K\} \in I\} \tag{21} $$
$$ {}_2\ell_\infty(\overline{\mathscr{A}}, \wedge, \mathscr{F}) = \{\mathscr{A} = (\overline{A}_{i,j} \in {}_2\ell_\infty(\mathscr{A}) : \sup_{i,j} f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}})\|) < \infty\} \tag{22} $$

2 Main results

Theorem 1. *Let $\mathscr{F} = (f_{i,j})$ be a double sequence of modulus functions and $p = (p_{i,j})$ be the double bounded sequence of positive real numbers. Then the classes of sequences ${}_2\mathscr{M}_c^I(\mathscr{A}, \wedge, \mathscr{F}, p)$ and ${}_2\mathscr{M}_{c_0}^I(\mathscr{A}, \wedge, \mathscr{F}, p)$ are paranormed spaces, paranormed by*

$$ g(\overline{\mathscr{A}}) = g((\overline{A}_{i,j})) = \sup_{i,j} f_{i,j}(\| \wedge_{i,j} (\overline{A}_{i,j})\|)^{\frac{p_{i,j}}{M}}, $$

where $M = max\{1, \sup_{i,j} p_{i,j}\}$.

Proof. Let $\overline{\mathscr{A}} = (\overline{A}_{i,j}), \overline{\mathscr{B}} = (\overline{B}_{i,j}) \in {}_2\mathscr{M}_c^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$.

(P1) It is clear that $g(\overline{\mathscr{A}}) = \overline{0}$, if $\overline{A} = \overline{0}$.

(P2) It is also obvious that $g(\overline{\mathscr{A}}) \geq 0$.
(P3) $g(\overline{\mathscr{A}}) = g(-\overline{\mathscr{A}})$ is obvious.
(P4) Since $\frac{p_{i,j}}{M} \leq 1$ and $M > 1$, using Minkowski's inequality, we have

$$\begin{aligned} g(\overline{\mathscr{A}}+\overline{\mathscr{B}}) = g(\overline{A}_{i,j}+\overline{B}_{i,j}) &= \sup_{i,j} f_{i,j}(\| \wedge_{i,j} (\overline{A}_{i,j}+\overline{B}_{i,j})\|)^{\frac{p_{i,j}}{M}} \\ &= \sup_{i,j} f_{i,j}(\| \wedge_{i,j} (\overline{A}_{i,j}) + \wedge_{i,j}(\overline{B}_{i,j})\|)^{\frac{p_{i,j}}{M}} \\ &\leq \sup_{i,j} f_{i,j}(\| \wedge_{i,j} (\overline{A}_{i,j})\|^{\frac{p_{i,j}}{M}} + \|(\wedge_{i,j}(\overline{B}_{i,j})\|^{\frac{p_{i,j}}{M}}) \\ &\leq \sup_{i,j} f_{i,j}(\| \wedge_{i,j} (\overline{A}_{i,j}\|^{\frac{p_{i,j}}{M}}) + \sup_{i,j} f_{i,j}(\| \wedge_{i,j} (\overline{B}_{i,j})\|^{\frac{p_{i,j}}{M}}) \\ &= g(\overline{\mathscr{A}}) + g(\overline{\mathscr{B}}). \end{aligned}$$

Thus $g(\overline{\mathscr{A}}+\overline{\mathscr{B}}) \leq g(\overline{\mathscr{A}}) + g(\overline{\mathscr{B}})$, for all $\overline{\mathscr{A}}, \overline{\mathscr{B}} \in {}_2\mathscr{M}_c^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$.
(P5) Let $(\lambda_{i,j})$ be a double sequence of scalars with $(\lambda_{i,j}) \to \lambda \quad (i,j \to \infty)$ and $(\overline{A}_{i,j}), \overline{A}_0 \in {}_2\mathscr{M}_c^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$ with $g(\overline{A}_{i,j}) \to g(\overline{A}_0), (i,j \to \infty)$. Note that $g(\lambda\overline{\mathscr{A}}) \leq max\{1, |\lambda|\} g(\overline{\mathscr{A}})$. Then, since the inequality $g(\overline{A}_{i,j}) \leq g(\overline{A}_{i,j} - \overline{A}_0) + g(\overline{A}_0)$ holds by subadditivity of g, the sequence $\{g(\overline{A}_{i,j})\}$ is bounded.

Therefore

$$| g(\lambda_{i,j}\overline{A}_{i,j}) - g(\lambda\overline{A}_0) | = | g(\lambda_{i,j}\overline{A}_{i,j}) - g(\lambda\overline{A}_{i,j}) + g(\lambda\overline{A}_{i,j}) - g(\lambda\overline{A}_0) |$$

$\leq | \lambda_{i,j} - \lambda |^{\frac{p_{i,j}}{M}} | g(\lambda_{i,j}\overline{A}_{i,j}) | + | \lambda |^{\frac{p_{i,j}}{M}} | g(\overline{A}_{i,j}) - g(\overline{A}_0) | \to 0$, as $(i,j \to \infty)$. That is to say that scalar multiplication is continuous.
(P6) Since each $f_{i,j}, (i,j) \in \mathbb{N}\times\mathbb{N}$ is an increasing function, it is clear that $g(\overline{\mathscr{A}}) \leq g(\overline{\mathscr{B}})$, if $\overline{\mathscr{A}} \subseteq \overline{\mathscr{B}}$.

Hence ${}_2\mathscr{M}_c^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$ is a paranormed space. For ${}_2\mathscr{M}_{c_0}^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$ the result is similar.

Theorem 2. *The set ${}_2\mathscr{M}_c^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$ is closed subspace of ${}_2\ell_\infty(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$.*

Proof. Let $\overline{\mathscr{A}}^{(n)} = (\overline{A}_{i,j}^{(n)})$ be a Cauchy sequence in ${}_2\mathscr{M}_c^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$ such that $\overline{A}_{i,j}^{(n)} \to \overline{A}_0$. We show that $\overline{A} \in {}_2\mathscr{M}_c^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$. Since $\overline{\mathscr{A}}^{(n)} = (\overline{A}_{i,j}^{(n)}) \in {}_2\mathscr{M}_c^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$. Then, there exists $\overline{A}_n$ such that $\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}}^{(n)}) - \overline{A}_n\|)^{p_{i,j}} \geq \varepsilon\} \in I$. We need to show that

(1) $(\overline{A}_n)$ converges to $\overline{A}_0$.
(2) If $U = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}}) - \overline{A}_0\|)^{p_{i,j}} < \varepsilon\}$, then $U^c \in I$.

(1) Since $\overline{\mathscr{A}}^{(n)} = (\overline{A}_{i,j}^{(n)})$ is Cauchy sequence in ${}_2\mathscr{M}_c^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) \Rightarrow$ for a given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sup_{i,j} f(\| \wedge_{i,j} (\overline{\mathscr{A}}^{(n)}) - \wedge_{i,j}(\overline{\mathscr{A}}^{(q)})\|)^{\frac{p_{i,j}}{M}} < \frac{\varepsilon}{3}$, for all $n,q \geq k_0$, where $M = max\{1, \sup_{i,j} p_{i,j}\}$.

For $\varepsilon > 0$, we have

$$\begin{aligned} B_{nq} &= \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}}^{(n)}) - \wedge_{i,j}\overline{\mathscr{A}}^{(q)})\|)^{p_{i,j}} < (\frac{\varepsilon}{3})^M\}, \\ B_q &= \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}}^{(q)}) - \overline{A}_q\|)^{p_{i,j}} < (\frac{\varepsilon}{3})^M\}, \\ B_n &= \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{\mathscr{A}}^{(n)}) - \overline{A}_n\|)^{p_{i,j}} < (\frac{\varepsilon}{3})^M\}. \end{aligned}$$

Then $B^c_{nq}, B^c_q, B^c_n \in I$. Let $B^c = B^c_{nq} \cup B^c_q \cup B^c_n$, where $B = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\|\bar{A}_q - \bar{A}_n\|)^{p_{i,j}} < \varepsilon\}$. Then $B^c \in I$. We choose $(i_0, j_0) \in B^c$. Then for each $n \geq i_0, q \geq j_0$, we have

$$\begin{aligned}
&\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\|\bar{A}_q - \bar{A}_n\|)^{p_{i,j}} < \varepsilon\} \\
&\supseteq \Big[\{(i,j) \in \mathbb{N}\times\mathbb{N} f_{i,j}(\|\bar{A}_q - \wedge_{i,j}(\overline{\mathscr{A}}^{(q)})\|)^{p_{i,j}} < (\frac{\varepsilon}{3})^M\} \\
&\cap \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j}(\overline{\mathscr{A}}^{(n)}) - \wedge_{i,j}(\overline{\mathscr{A}}^{(q)})\|)^{p_{i,j}} < (\frac{\varepsilon}{3}^M\} \\
&\cap \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j}(\overline{\mathscr{A}}^{(n)} - \bar{A}_n\|)^{p_{i,j}} < (\frac{\varepsilon}{3})^M\}\Big].
\end{aligned}$$

Then, $(\bar{A}_n)$ is a Cauchy sequence of interval numbers, so there exists some interval number $\bar{A}_0$ such that $\bar{A}_n \to \bar{A}_0$. as $n \to \infty$.

(2) Let $0 < \delta < 1$ be given. Then, we show that, if $U = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j}(\overline{\mathscr{A}}) - \bar{A}_0\|)^{p_{i,j}} < \delta\}$, then $U^c \in I$. Since $\overline{\mathscr{A}}^{(n)} = (\bar{A}^{(n)}_{i,j}) \to \bar{A}$, then there exists $q_0 \in \mathbb{N}$ such that

$$P = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j}(\overline{\mathscr{A}}^{(q_0)}) - \wedge_{i,j}(\overline{\mathscr{A}})\|)^{p_{i,j}} < (\frac{\delta}{3D})^M\} \tag{23}$$

implies $P^c \in I$, where $D = max\{1, 2^{H-1}\}, H = \sup_{i,j} p_{i,j} \geq 0$. The number q_0 can be chosen that together with (23), we have

$$Q = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j}(\bar{A}_{q_0}) - \bar{A}_0\|)^{p_{i,j}} < (\frac{\delta}{3D})^M\} \text{ such that } Q^c \in I.$$

Since $\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j}(\overline{\mathscr{A}}^{(q_0)}) - \wedge_{i,j}(\bar{A}_{q_0})\|)^{p_{i,j}} \geq \delta\} \in I$. Then, we a subset S of $\mathbb{N}$ such that $S^c \in I$, where

$$S = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j}(\overline{\mathscr{A}}^{(q_0)}) - \wedge_{i,j}(\bar{A}_{q_0})\|)^{p_{i,j}} < (\frac{\delta}{3D})^M\}.$$

Let $U^c = P^c \cup Q^c \cup S^c$, where $U = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j}(\overline{\mathscr{A}}) - \bar{A}_0\|)^{p_{i,j}} < \delta\}$. Therefore, for each $(i,j) \in U^c$, we have

$$\begin{aligned}
&\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j}(\overline{\mathscr{A}}) - \bar{A}_0\|)^{p_{i,j}} < \delta\} \\
&\supseteq \Big[\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j}(\overline{\mathscr{A}}^{(q_0)}) - \wedge_{i,j}(\overline{\mathscr{A}})\|)^{p_{i,j}} < (\frac{\delta}{3D})^M\} \\
&\cap \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j}(\overline{\mathscr{A}}^{(q_0)}) - \wedge_{i,j}(\bar{A}_{q_0})\|)^{p_{i,j}} < (\frac{\delta}{3D})^M\} \\
&\cap \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\|\bar{A}_{q_0} - \bar{A}_0\|)^{p_{i,j}} < (\frac{\delta}{3D})^M\}\Big].
\end{aligned} \tag{24}$$

Then, the result follows from (24). Since the inclusions ${}_2\mathscr{M}^I_c(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) \subset {}_2\ell_\infty(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$ and ${}_2\mathscr{M}^I_{c_0}(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) \subset {}_2\ell_\infty(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$ are strict so in view of Theorem 2.2 we have the following result.

Theorem 3. *The spaces ${}_2\mathscr{M}^I_c(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$ and ${}_2\mathscr{M}^I_{c_0}(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$ are nowhere dense subsets of ${}_2\ell_\infty(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$.*

Theorem 4. *The spaces ${}_2C^I_0(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$ and ${}_2\mathscr{M}^I_{C_0}(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$ are both solid and monotone.*

Proof. We shall prove the result for ${}_2C^I_0(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$. For ${}_2\mathscr{M}^I_{c_0}(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$, the result follows similarly. For, let $\overline{\mathscr{A}} = (\bar{A}_{i,j}) \in {}_2C^I_0(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$ and $(\alpha_{i,j})$ be sequence of scalars with $|\alpha_{i,j}| \leq 1$, for all $(i,j) \in \mathbb{N}\times\mathbb{N}$. Since $|\alpha_{i,j}|^{p_{i,j}} \leq max\{1, |\alpha_{i,j}|^G\} \leq 1$, for all $(i,j) \in \mathbb{N}\times\mathbb{N}$, we have

$$f_{i,j}(\|\alpha_{i,j}\wedge_{i,j}(\bar{A}_{i,j})\|)^{p_{i,j}} \leq f_{i,j}(\|\wedge_{i,j}(\bar{A}_{i,j})\|)^{p_{i,j}}, \text{ for all } (i,j)\in\mathbb{N}\times\mathbb{N},$$

which further implies that

$$\{(i,j)\in\mathbb{N}\times\mathbb{N}: f_{i,j}(\|\wedge_{i,j}(\bar{A}_{i,j})\|)^{p_{i,j}}\geq\varepsilon\}\supseteq\{(i,j)\in\mathbb{N}\times\mathbb{N}: f_{i,j}(\|\alpha_{i,j}\wedge_{i,j}(\bar{A}_{i,j})\|)^{p_{i,j}}\geq\varepsilon\}.$$

Thus, $\alpha_{i,j}(\bar{A}_{i,j})\in {}_2C_0^I(\bar{\mathscr{A}},\wedge,\mathscr{F},p)$. Therefore, the space ${}_2C_0^I(\bar{\mathscr{A}},\wedge,\mathscr{F},p)$ is solid and hence by Lemma 1.1 it is monotone.

Theorem 5. *Let $G=\sup_{i,j} p_{i,j}<\infty$ and I be an admissible ideal. Then the following are equivalent.*

(a) $\bar{\mathscr{A}}=(\bar{A}_{i,j})\in {}_2C^I(\bar{\mathscr{A}},\wedge,\mathscr{F},p)$;

(b) *there exists $\bar{\mathscr{B}}=(\bar{B}_{i,j})\in {}_2C(\bar{\mathscr{A}},\wedge,\mathscr{F},p)$ such that $\bar{A}_{i,j}=\bar{B}_{i,j}$ for a.a.(i,j) r.I;*

(c) *there exists $\bar{\mathscr{B}}=(\bar{B}_{i,j})\in {}_2C(\bar{\mathscr{A}},\wedge,\mathscr{F},p)$ and $\bar{\mathscr{C}}=\bar{C}_{i,j}\in {}_2C_0(\bar{\mathscr{A}},\wedge,\mathscr{F},p)$ such that*

$$\bar{A}_{i,j}=\bar{B}_{i,j}+\bar{C}_{i,j} \text{ for all } (i,j)\in\mathbb{N}\times\mathbb{N}$$

and

$$\{(i,j)\in\mathbb{N}\times\mathbb{N}: f_{i,j}(\|\wedge_{i,j}(\bar{B}_{i,j})-\bar{A}\|)^{p_{i,j}}\geq\varepsilon\}\in I;$$

(d) *there exists a subset $K=\{(i_1,j_1)<(i_2,j_2)<\cdots\}$ of $\mathbb{N}\times\mathbb{N}$ such that $K\in\mathscr{F}(I)$ and $\lim_{n\to\infty} f_{i,j}(\|\wedge_{i,j}(\bar{A})_{i_n,j_n}\|)^{p_{i_n,j_n}}=0$.*

Proof. (a) implies (b). Let $\bar{\mathscr{A}}=(\bar{A}_{i,j})\in {}_2C^I(\bar{\mathscr{A}},\wedge,\mathscr{F},p)$. Then, there exists interval number $\bar{A}$ such that the set

$$\{(i,j)\in\mathbb{N}\times\mathbb{N}: f_{i,j}(\|\wedge_{i,j}(\bar{A}_{i,j})-\bar{A}\|)^{p_{i,j}}\geq\varepsilon\}\in I.$$

Let (m_t,n_t) be an increasing double sequence with

$$(m_t,n_t)\in\mathbb{N}\times\mathbb{N} \text{ such that } \{(i,j)\leq(m_t,n_t): f_{i,j}(\|\wedge_{i,j}(\bar{A}_{i,j})-\bar{A}\|)^{p_{i,j}}\geq t^{-1}\}\in I.$$

Define a sequence $\bar{\mathscr{B}}=(\bar{B}_{i,j})$ as $\bar{B}_{i,j}=\bar{A}_{i,j}$ for all $(i,j)\leq(m_1,n_1)$. For $(m_t,n_t)<(i,j)\leq(m_{t+1},n_{t+1})$, $t\in\mathbb{N}$.

$$\bar{B}_{i,j}=\begin{cases}\bar{A}_{i,j}, & \text{if } f_{i,j}(\|\wedge_{i,j}(\bar{A}_{i,j})-\bar{A}\|)^{p_{i,j}}<t^{-1},\\ \bar{A}, & \text{otherwise}\end{cases}$$

Then, $\bar{\mathscr{B}}=(\bar{B}_{i,j})\in {}_2C(\bar{\mathscr{A}},\wedge,\mathscr{F},p)$ and from the inclusion

$$\{(i,j)\leq(m_t,n_t):\bar{A}_{i,j}\neq\bar{B}_{i,j}\}\subseteq\{(i,j)\leq(m_t,n_t): f_{i,j}(\|\wedge_{i,j}(\bar{A}_{i,j})-\bar{A}\|)^{p_{i,j}}\geq\varepsilon\}\in I.$$

We get $\bar{A}_{i,j}=\bar{B}_{i,j}$ for a.a.(i,j).r I.

(b) implies (c). For $\bar{\mathscr{A}}=(\bar{A}_{i,j})\in {}_2C^I(\bar{\mathscr{A}},\wedge,\mathscr{F},p)$, then, there exists $\bar{\mathscr{B}}=(\bar{B}_{i,j})\in {}_2C(\bar{\mathscr{A}},\wedge,\mathscr{F},p)$ such that $\bar{A}_{i,j}=\bar{B}_{i,j}$, for a.a$(i,j)$.r.$I$. Let $K=\{(i,j)\in\mathbb{N}\times\mathbb{N}:\bar{A}_{i,j}\neq\bar{B}_{i,j}\}$ then $K\in I$. Define $\bar{\mathscr{C}}=\bar{C}_{i,j}$ as follows:

$$\bar{C}_{i,j}=\begin{cases}\bar{A}_{i,j}-\bar{B}_{i,j}, & \text{if } (i,j)\in K,\\ \bar{0}, & \text{if } (i,j)\notin K\end{cases}$$

Then, $\bar{\mathscr{C}}=(\bar{C}_{i,j})\in {}_2C_0^I(\bar{\mathscr{A}},\wedge,\mathscr{F},p)$ and $\bar{\mathscr{B}}=(\bar{B}_{i,j})\in {}_2C(\bar{\mathscr{A}},\wedge,\mathscr{F},p)$.

(c) implies (d). Suppose (c) holds. Let $\varepsilon>0$ be given. Let

$$P_1=\{(i,j)\in\mathbb{N}\times\mathbb{N}: f_{i,j}(\|\wedge_{i,j}(\bar{C}_{i,j})\|)^{p_{i,j}}\geq\varepsilon\}\in I$$

and

$$K = P_1^c = \{(i_1, j_1) < (i_2, j_2) < (i_3, j_3) < \cdots\} \in \mathscr{F}(I).$$

Then, we have $\lim_{n\to\infty} f_{i,j}(\| \wedge_{i,j} (\bar{A}_{i_n,j_n}) - \bar{A}\|)^{p_{i_n,j_n}} = 0$.

(d) implies (a). Let $K = \{(i_1, j_1) < (i_2, j_2) < (i_3, j_3) < \cdots\} \in \mathscr{F}(I)$ and

$$\lim_{n\to\infty} f_{i,j}(\| \wedge_{i,j} (\bar{A}_{i_n,j_n}) - \bar{A}\|)^{p_{i_n,j_n}} = 0.$$

Then for any $\varepsilon > 0$, and Lemma 1.2, we have

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\bar{A}_{i,j}) - \bar{A}\|)^{p_{i,j}} \geq \varepsilon\} \subseteq K^c \cup \{(i,j) \in K : f_{i,j}(\| \wedge_{i,j} (\bar{A}_{i,j}) - \bar{A}\|)^{p_{i,j}} \geq \varepsilon\}.$$

Thus, $\bar{\mathscr{A}} = (\bar{A}_{i,j}) \in {}_2C^I(\bar{\mathscr{A}}, \wedge, \mathscr{F}, p)$.

Theorem 6. *Let $\mathscr{F} = (f_{i,j})$ and $\mathscr{G} = (g_{i,j})$ be two sequences of modulus functions and for each $(i,j) \in \mathbb{N}\times\mathbb{N}, (f_{i,j})$ and $(g_{i,j})$ satisfying $\triangle_2$-condition and $p = (p_{i,j}) \in {}_2\ell_\infty$ be a bounded sequence of positive real numbers. Then*

(a) ${}_2\chi(\bar{\mathscr{A}}, \wedge, \mathscr{G}, p) \subseteq {}_2\chi(\bar{\mathscr{A}}, \wedge, \mathscr{F} \circ \mathscr{G}, p)$,
(b) ${}_2\chi(\bar{\mathscr{A}}, \wedge, \mathscr{F}, p) \cap {}_2\chi(\bar{\mathscr{A}}, \wedge, \mathscr{G}, p) \subseteq {}_2\chi(\bar{\mathscr{A}}, \wedge, \mathscr{F} + \mathscr{G}, p)$ *for* ${}_2\chi = {}_2C^I$, ${}_2C_0^I$, ${}_2\mathscr{M}_c^I$ *and* ${}_2\mathscr{M}_{c_0}^I$.

Proof. (a) Let $\bar{\mathscr{A}} = (\bar{A}_{i,j}) \in {}_2C_0^I(\bar{\mathscr{A}}), \wedge, \mathscr{G}, p)$ be any arbitrary element. Then, the set

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : g_{i,j}(\| \wedge_{i,j} (\bar{A}_{i,j})\|)^{p_{i,j}} \geq \varepsilon\} \in I. \tag{25}$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_{i,j}(t) < \varepsilon, 0 \leq t \leq \delta$. Let us denote

$$\bar{B}_{i,j} = g_{i,j}(\| \wedge_{i,j} (\bar{A}_{i,j})\|)^{p_{i,j}} \tag{26}$$

and consider

$$\lim_{i,j} f_{i,j}(\bar{B}_{i,j}) = \lim_{\bar{B}_{i,j}\leq\delta,(i,j)\in\mathbb{N}\times\mathbb{N}} f_{i,j}(\bar{B}_{i,j}) + \lim_{\bar{B}_{i,j}>\delta,(i,j)\in\mathbb{N}\times\mathbb{N}} f_{i,j}(\bar{B}_{i,j}).$$

Now, since $f_{i,j}$ for each $(i,j) \in \mathbb{N}\times\mathbb{N}$ is modulus function, we have

$$\lim_{\bar{B}_{i,j}\leq\delta,(i,j)\in\mathbb{N}\times\mathbb{N}} f_{i,j}(\bar{B}_{i,j}) \leq f_{i,j}(2) \lim_{\bar{B}_{i,j}\leq\delta,(i,j)\in\mathbb{N}\times\mathbb{N}} (\bar{B}_{i,j}). \tag{27}$$

For $\bar{B}_{i,j} > \delta$, we have $\bar{B}_{i,j} < \frac{\bar{B}_{i,j}}{\delta} < 1 + \frac{\bar{B}_{i,j}}{\delta}$. Now, since each $f_{i,j}$ is non-decreasing and modulus, it follows that

$$f_{i,j}(\bar{B}_{i,j}) < f_{i,j}\big(1 + \frac{\bar{B}_{i,j}}{\delta}\big) < \frac{1}{2} f_{i,j}(2) + \frac{1}{2} f_{i,j}\big(\frac{2\bar{B}_{i,j}}{\delta}\big).$$

Again, since each $f_{i,j}, (i,j) \in \mathbb{N}\times\mathbb{N}$ satisfies Δ_2-condition, we have

$$f_{i,j}(\bar{B}_{i,j}) < \frac{1}{2} K \frac{(\bar{B}_{i,j})}{\delta} f_{i,j}(2) + \frac{1}{2} K \frac{(\bar{B}_{i,j})}{\delta} f_{i,j}(2).$$

Thus, $f_{i,j}(\bar{B}_{i,j}) < K \frac{(\bar{B}_{i,j})}{\delta} f_{i,j}(2)$. Hence

$$\lim_{\bar{B}_{i,j}>\delta,(i,j)\in\mathbb{N}\times\mathbb{N}} f_{i,j}(\bar{B}_{i,j}) \leq max\{1, (K\delta^{-1} f_{i,j}(2))^H\} \lim_{\bar{B}_{i,j}\geq\delta,(i,j)\in\mathbb{N}\times\mathbb{N}} (\bar{B}_{i,j}), \quad H = max\{1, \sup_{i,j} p_{i,j}\}. \tag{28}$$

Therefore, from (26), (27) and (28), we have $\overline{\mathscr{A}} = (\overline{A_{i,j}}) \in {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{F} \circ \mathscr{G}, p)$. Thus,

$$ {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{G}, p) \subseteq {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{F} \circ \mathscr{G}, p). $$

Hence,

$$ {}_2\chi(\overline{\mathscr{A}}, \wedge, \mathscr{G}, p) \subseteq_2 \chi(\overline{\mathscr{A}}, \wedge, \mathscr{F} \circ \mathscr{G}, p), \text{ for } {}_2\chi = {}_2C_0^I. $$

For ${}_2\chi = {}_2C^I, {}_2\mathscr{M}_c^I$ and ${}_2\mathscr{M}_{c_0}^I$ the inclusions can be established similarly.

(b) Let

$$ \overline{\mathscr{A}} = (\overline{A_{i,j}}) \in {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) \cap {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{G}, p). $$

Let $\varepsilon > 0$ be given. Then, the sets

$$ \{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{A_{i,j}})\|)^{p_{i,j}} \geq \varepsilon\} \in I \tag{29} $$

and

$$ \{(i,j) \in \mathbb{N} \times \mathbb{N} : g_{i,j}(\| \wedge_{i,j} (\overline{A_{i,j}})\|)^{p_{i,j}} \geq \varepsilon\} \in I \tag{30} $$

Therefore, from (29) and (30), we have

$$ \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathscr{F} + \mathscr{G}(\| \wedge_{i,j} (\overline{A_{i,j}})\|)^{p_{i,j}} \geq \varepsilon\} \in I. $$

Thus, $\overline{\mathscr{A}} = (\overline{A_{i,j}}) \in {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{F} + \mathscr{G}, p)$. Hence,

$$ {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) \cap {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{G}, p) \subseteq {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{F} + \mathscr{G}, p). $$

For ${}_2\chi = {}_2C^I, {}_2\mathscr{M}_c^I$ and ${}_2\mathscr{M}_{c_0}^I$ the inclusions are similar. For $g_{i,j}(x) = x$ and $f_{i,j}(x) = f(x), \forall x \in [0, \infty)$, we have the following corollary.

Corollary 1. *${}_2\chi(\overline{\mathscr{A}}, \wedge, p) \subseteq {}_2\chi(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p)$, for ${}_2\chi = {}_2C^I, {}_2C_0^I, {}_2\mathscr{M}_c^I$ and ${}_2\mathscr{M}_{c_0}^I$.*

Theorem 7. *Let $\mathscr{F} = (f_{i,j})$ be a double sequence of modulus function. Then the inclusions*

$$ {}_2C_0^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) \subset {}_2C^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) \subset {}_2\ell_\infty(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p) $$

hold.

Proof. Let $\overline{\mathscr{A}} = (\overline{A_{i,j}}) \in {}_2C^I(\overline{\mathscr{A}}, \wedge, \mathscr{G}, p)$ be an arbitrary element. Then there exists some double interval number $\overline{A}$ such that the set

$$ \{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{A_{i,j}}) - \overline{A}\|)^{p_{i,j}} \geq \varepsilon\} \in I. \text{ Since each } f_{i,j} $$

is modulus, we have

$$ f_{i,j}(\| \wedge_{i,j} (\overline{A_{i,j}})\|)^{p_{i,j}} = f_{i,j}(\| \wedge_{i,j} (\overline{A_{i,j}}) - \overline{A} + \overline{A}\|)^{p_{i,j}} \leq f_{i,j}(\| \wedge_{i,j} (\overline{A_{i,j}}) - \overline{A}\|)^{p_{i,j}} + f_{i,j}(\|\overline{A}\|)^{p_{i,j}}. $$

Taking the supremum over (i,j) on both sides, we get

$$ \overline{\mathscr{A}} = (\overline{A_{i,j}}) \in {}_2\ell_\infty^I(\overline{\mathscr{A}}, \wedge, \mathscr{F}, p). $$

The inclusion

$$_2C_0^I(\overline{\mathscr{A}},\wedge,\mathscr{F},p) \subset {}_2C^I(\overline{\mathscr{A}},\wedge,\mathscr{F},p)$$

is obvious. Hence

$$_2C_0^I(\overline{\mathscr{A}},\wedge,\mathscr{F},p) \subset {}_2C^I(\overline{\mathscr{A}},\wedge,\mathscr{F},p) \subset {}_2\ell_\infty^I(\overline{\mathscr{A}},\wedge,\mathscr{F},p).$$

Theorem 8. *The spaces* $_2C_0^I(\overline{\mathscr{A}},\wedge,\mathscr{F},p)$ *and* $_2C^I(\overline{\mathscr{A}},\wedge,\mathscr{F},p)$ *are sequence algebra.*

Proof. Let $\overline{\mathscr{A}} = (\overline{A}_{i,j}), \overline{\mathscr{B}} = (\overline{B}_{i,j}) \in {}_2C_0^I(\overline{\mathscr{A}},\wedge,\mathscr{F},p)$, then the sets

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{A}_{i,j})\|)^{p_{i,j}} \geq \varepsilon\} \in I \tag{31}$$

and

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{B}_{i,j})\|)^{p_{i,j}} \geq \varepsilon\} \in I \tag{32}$$

Therefore, from (31) and (32), we have

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{i,j}(\| \wedge_{i,j} (\overline{A}_{i,j}\overline{B}_{i,j})\|)^{p_{i,j}} \geq \varepsilon\} \in I.$$

Thus, $\overline{\mathscr{A}}.\overline{\mathscr{B}} \in {}_2C_0^I(\overline{\mathscr{A}},\wedge,\mathscr{F},p)$. Hence $_2C_0^I(\overline{\mathscr{A}},\wedge,\mathscr{F},p)$ is a sequence algebra. Similarly, we can prove that $_2C^I(\overline{\mathscr{A}},\wedge,\mathscr{F},p)$ is a sequence algebra.

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