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Control of an equation by maximum principle

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Abstract: In this paper, some results, which are related to well posedness, controllability and optimal control of a beam equation, are presented. In order to obtain the optimal control function, maximum principle is employed. Performance index function is defined as quadratic functional of displacement and velocity and also includes a penalty in terms of control function. The solution of the control problem is formulated by using Galerkin expansion. Obtained results are given in the table and graphical forms.

Keywords: Beam, optimal control, maximum principle.

1 Introduction

Control of the undesirable vibrations in the mechanical systems is very active research area due to its wide applications in science and engineering. Some studies in the literature can be summarized as follows, but not limited to: [1]-[9]. The difference of the present study than studies existing in literature that in this study control function depends to time and space variable and it allows to use the smart materials. Also, in the present study is subjected to displacement boundary conditions. Particulary, in this paper, existence and uniqueness of the solution to a beam equation is presented by using energy integral method. Later, the controllability of the system is discussed. Also, optimal control function for the beam equation system in one space dimension is obtained by employing Maximum principle(for more details about maximum principle[5,6,7,8,9]).Performance index functional of the control problem consists of a weighted quadratic functional of the dynamic responses of the system to be minimized and a penalty term defined as the control spent in the control process. By means of the maximum principle, the optimal control problem is transformed to the a coupled system of partial differential equations in terms of state, adjoint and control variables subject to the boundary, initial and terminal conditions. The explicit solution of the problem is sought by Galerkin expansion method. By using MATLAB, numerical results are given demonstrate the robustness and efficiency of the proposed control algorithm.

This paper is organized as follows: in the next section, mathematical formulation and wellposedness of the control problem is given. In section 3, an adjoint equation is introduced and a maximum principle is given as a theorem. In section 4, the solution of the optimal control problem is formulated by means of Galerkin expansion method. In section 5, some numerical examples are presented. Finally, in the last section the main results of the paper is given.

2 Mathematical formulation of the control problem

The second order linear hyperbolic beam equation in one space dimension can be presented as follows[10]:

$$u_{tt} + \alpha u_t + \beta u = u_{xx} + C(x,t) \tag{1}$$

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where $x \in [0,\ell]$ is the space variable, $t \in [0,t_f]$ is the time variable, t_f is the pre-determined final time, u is the displacement function, $\alpha > \beta > 0$ are known constants, $C(x,t)$ is the control function to be determined in optimal way. Eq.(1) is subject to the following boundary conditions, which are named as displacement conditions

$$u(0,t) = T_1(t) \quad (and) \quad u(1,t) = T_2(t) \tag{2}$$

and initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{3}$$

The following assumptions are made on the solution:

$$C(x,t), T_1(t), T_2(t) \quad \text{are continious functions in } S = (0,\ell)\times(0,t_f), \tag{4a}$$

$$u_1(x) \in L^2(0,\ell), \quad u_0(x) \in H^1(0,\ell) = \{u_0(x) \in L^2(0,\ell) : \frac{u_0(x)}{\partial x} \in L^2(0,\ell)\} \tag{4b}$$

$$u, \frac{\partial^j u}{\partial x^j}, \frac{\partial^j u}{\partial t^j} \in \mathbb{S}, \quad j = 0,1,2, \tag{4c}$$

in which $\mathbb{S} \subset L^2(\mathscr{S})$ and $L^2(\mathscr{S})$ denote the Hilbert space of real-valued square-integrable functions defined in the domain $\mathscr{S}$ in the Lebesgue sense with usual inner product and norm defined by

$$\| \zeta \|^2 = <\zeta,\zeta>, \quad <\zeta,\eta>_{\mathscr{S}} = \int_{\mathscr{S}} \zeta\eta \, d\mathscr{S}.$$

Under these assumptions, the system Eqs.(1)-(3) has a solution in the class of analytic functions [12]. For the uniqueness of solution to Eqs.(1)-(3), let us introduce the following lemma.

Lemma 1. *Let u_ε satisfy the system given by Eqs.(1)-(3) corresponding to the control $C_\varepsilon(x,t)$ and $C^\circ(x,t)$ is the optimal control function corresponding to optimal displacement u°. Consider the following difference functions,*

$$\Delta C(x,t) = C_\varepsilon(x,t) - C^\circ(x,t), \quad \Delta u(x,t) = u_\varepsilon(x,t) - u^\circ(x,t). \tag{5}$$

Note that $\Delta u(\overline{x},\overline{t})$ satisfies following equation

$$\Delta u_{tt} + \alpha\Delta u_t + \beta\Delta u - \Delta u_{xx} = \Delta C(x,t) \tag{6}$$

and following homogeneous boundary conditions

$$\Delta u(x,t) = 0 \quad at \quad x = 0,\ell \tag{7}$$

also, zero initial conditions

$$\Delta u(x,t) = 0, \quad \Delta u_t(x,t) = 0 \quad at \quad t = 0. \tag{8}$$

Then

$$\int_0^\ell \Delta u^2(x,t_f)dx = o(\varepsilon), \quad \int_0^\ell \Delta u_t^2(x,t_f)dx = o(\varepsilon)$$

and

$$\int_0^{t_f}\int_0^{\ell} \Delta u^2(x,t)dxdt = o(\varepsilon).$$

$o(\varepsilon)$ *is a quantity such that*

$$\lim_{\varepsilon\to 0}(o(\varepsilon)/|\varepsilon|) = 0.$$

Proof. Let $(x_1,t_1),...,(x_P,t_P)$ be P arbitrary points in the open region $(0,\ell)\times(0,t_f)$ and ε_j are the coefficients that the rectangles $R_j : [x_j, x_j+\sqrt{\varepsilon_j}]\times[t_j,t_j+\sqrt{\varepsilon_j}]$ do not have any intersection for $1\le j\le P$. Let us define the following energy integral like in [7,11],

$$E(t) = \frac{1}{2}\int_0^{\ell}\{(\Delta u_t)^2+\beta(\Delta u)^2-\nabla^2(\Delta u)^2\}dx. \tag{9}$$

Eq.(9) can be written in the following form;

$$E(t) = \int_0^{\bar{t}}\frac{dE(\delta)}{d\delta}d\delta = \frac{1}{2}\int_0^{t}\int_0^{\ell}\{2\Delta u_{tt}\Delta u_t + 2\beta\Delta u\Delta u_t - 2\nabla^2\Delta u\Delta u_t\}dxdt. \tag{10}$$

With integration by parts and using homogeneous boundary conditions given by Eq.(7), Eq.(10) becomes

$$E(\bar{t}) = \int_0^{t}\int_0^{\ell}\{\Delta u_{tt}+\beta\Delta u-\nabla^2(\Delta u)\}\Delta u_t dxdt = \int_0^{t}\int_0^{\ell}\{\Delta C(x,t)-\alpha\Delta u_t\}\Delta u_t dxd\delta$$

$$\le \int_0^{t}\int_0^{\ell}\Delta C(x,t)\Delta u_t(x,\delta)dxd\delta.$$

Applying the Cauchy-Schwartz inequality to the space integral, one obtains

$$E(t)\le\int_0^{t}\left[\int_0^{\ell}(\Delta u_t)^2dx\right]^{1/2}\left[\int_0^{\ell}(\Delta C(x,\delta)^2dx\right]^{1/2}d\delta \le \int_0^{t}E^{1/2}(\delta)\left[\int_0^{\ell}(\Delta C(x,\delta)^2\right]^{1/2}d\delta. \tag{11}$$

Taking the sup of both sides of Eq.(11) leads to

$$\sup E(t)\le \sup E^{1/2}(t)\int_0^{t}\left[\int_0^{\ell}(\Delta C(x,\delta)^2dx\right]^{1/2}d\delta = \sup E^{1/2}(t)\sum_{i=1}^{P}O(\varepsilon_i^{5/4}) \tag{12}$$

where $O(r)$ is a quantity such that

$$\lim_{r\to 0^+}(O(r)/r) = \text{constant}.$$

By means of Eq.(12), the following inequality is observed for each $t\in[0,t_f]$

$$0\le E^{1/2}(t)\le O(\varepsilon^{5/4}).$$

Because $5/4 > 1$ [9], the following equality is obtained

$$E(t) = o(\varepsilon). \tag{13}$$

Because the coefficients of Eq.(1) are bounded away from zero, the conclusion of the **Lemma 1** is obtained from Eq.(13).

Corollary 1. *It is concluded from **Lemma 1** that*

$$\lim_{\Delta C(x,t)\to 0} \Delta u(x,t) = 0$$

Namely, Eqs.(1)-(3) has a unique solution.

3 Maximum principle

The aim of the optimal control problem is to determine an optimum voltage function $C(x,t)$ to minimize the dynamic response of the beam at t_f with the minimum expenditure of the control voltage. Therefore, performance index is defined by the weighted dynamic response of the beam and the expenditure of the control over $(0,t_f)$. The set of admissible control functions is given by

$$C_{ad} = \{C(x,t)|C(x,t) \in L^2(\mathbb{S}), \quad |C(x,t)| \le c_0 < \infty\} \tag{14}$$

and the performance index of the controlled system is defined as follows;

$$\mathscr{J}(C(x,t)) = \int_0^1 [\mu_1 u^2(x,t_f) + \mu_2 u_t^2(x,t_f)]dx + \int_0^{t_f}\int_0^{\ell} \mu_3 C^2(x,t)dt \tag{15}$$

where $\mu_1,\mu_2 \ge 0, \quad \mu_1+\mu_2 \ne 0$ and $\mu_3 > 0$ are weighting constants. The first integral in Eq.(15) is the modified dynamic response of the beam and last integral represents the measure of the total control energy that accumulates over $(0,t_f)$. The optimal control of a beam is expressed as

$$\mathscr{J}(C^\circ(x,t)) = \min_{C(x,t)\in C_{ad}} \mathscr{J}(C(x,t)) \tag{16}$$

subject to the Eqs.(1)-(3). In order to achieve the maximum principle, let us introduce an adjoint variable $v(x,t)$ satisfying the following equation

$$v_{tt} - \alpha v_t + \beta v = v_{xx} \tag{17}$$

and subjects to the following homogeneous boundary conditions

$$v(x,t) = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad v(x,t) = 0 \quad \text{at} \quad x = \ell = 1 \tag{18}$$

and terminal conditions

$$v_t(x,t_f) - \alpha v(x,t_f) = 2\mu_1 u(x,t_f), \quad v(x,t_f) = -2\mu_2 u_t(x,t_f) \quad \text{at} \quad t_f = 1. \tag{19}$$

A maximum principle in terms of Hamiltonian functional is derived as a necessary condition for the optimal control function. It is proved in [5] that under some convexity assumption, which are satisfied by Eq.(15), on performance index function, maximum principle is also sufficient condition for the optimal control function. Note that u is unique solution to the system defined by Eqs.(1)-(3). By observing Eq.(6) in Lemma 1, it can be concluded that when u° is unique solution to the system, the corresponding control function C° must be unique. The maximum principle gives an explicit expression for the optimal control function and relates the optimal control to the state variable implicitly. Then, the maximum principle can be given as follows:

Theorem 1. *(Maximum principle) The maximization problem states that if*

$$\mathscr{H}[t;v^{\circ},C^{\circ}(x,t)] = \max_{C(x,t)\in C_{ad}} \mathscr{H}[t;v,C(x,t)] \tag{20}$$

in which $v = v(x,t)$ satisfies the adjoint system given by Eqs.(17)-(19) and the Hamiltonian function is defined by

$$\mathscr{H}[t;v,C(x,t)] = vC(x,t) - \mu_3 C^2(x,t), \tag{21}$$

then

$$\mathscr{J}[C^{\circ}(x,t)] \leq \mathscr{J}[C(x,t)], \quad \forall C(x,t) \in C_{ad} \tag{22}$$

where $C^{\circ}(x,t)$ is the optimal control function.

Proof. Before starting the proof, let us introduce an operator and its adjoint operator as follows:

$$\Upsilon(u) = u_{tt} + \alpha u_t + \beta u - u_{xx}, \quad \Upsilon^*(v) = v_{tt} - \alpha v_t + \beta v - v_{xx}. \tag{23}$$

The deviations are given by $\Delta u = u - u^{\circ}, \quad \Delta u_t = u_t - u_t^{\circ}$ in which u° is the optimal displacement. The operator $\Upsilon(\Delta u) = \Delta C(x,t)$ is subject to the following boundary conditions

$$\Delta u(x,t) = 0 \quad \text{at} \quad x = 0,1 \tag{24}$$

and initial conditions

$$\Delta u(x,t) = \Delta u_t(x,t) = 0 \quad \text{at} \quad t = 0. \tag{25}$$

Consider the following functional

$$\int_0^1\int_0^{t_f}\left\{v\Upsilon(\Delta u) - \Delta u\Upsilon^*(v)\right\}dtdx = \int_0^1\int_0^{t_f}\left\{v\Delta C(x,t)\right\}dtdx. \tag{26}$$

Integrating the left side of Eq.(26) twice integration by parts with respect to t and four times integration by parts with respect to x, using Eqs.(24)-(25), one observes the following relation:

$$\int_0^1\int_0^{t_f}\left\{v\Upsilon(\Delta u) - \Delta u\Upsilon^*(v)\right\}dtdx = \int_0^1\left(v(x,t_f)\Delta u_t(x,t_f) - \Delta u(x,t_f)\left[v_t(x,t_f) - \alpha v(x,t_f)\right]\right)dx. \tag{27}$$

In view of Eq.(19), Eq.(27) becomes

$$\int_0^1\int_0^{t_f}\left\{v\Upsilon(\Delta u) - \Delta u\Upsilon^*(v)\right\}dtdx = -2\int_0^1\left(\mu_1 u(x,t_f)\Delta u(x,t_f) + \mu_2 u_t(x,t_f)\Delta u_t(x,t_f)\right)dx. \tag{28}$$

Consider the difference of the performance index

$$\begin{aligned}\Delta \mathscr{J}[C(x,t)] &= \mathscr{J}[C(x,t)] - \mathscr{J}[C^{\circ}(x,t)] \\ &= \int_0^1 \left\{ \mu_1[u^2(x,t_f) - u^{\circ^2}(x,t_f)] + \mu_2[u_t^2(x,t_f) - u_t^{\circ^2}(x,t_f)] \right\} dx \\ &\quad + \int_0^{t_f}\int_0^1 \mu_3[C^2(x,t) - C^{\circ^2}(x,t)]dtdx\end{aligned} \tag{29}$$

Let us expand the $u^2(x,t_f)$ and $u_t^2(x,t_f)$ to Taylor series around $u^{\circ^2}(x,t_f)$ and $u_t^{\circ^2}(x,t_f)$, respectively. Then, one observes the following

$$u^2(x,t_f) - u^{\circ^2}(x,t_f) = 2u^{\circ}(x,t_f)\Delta u(x,t_f) + r, \tag{30a}$$

$$u_t^2(x,t_f) - u_t^{\circ^2}(x,t_f) = 2u_t^{\circ}(x,t_f)\Delta u_t(x,t_f) + r_t \tag{30b}$$

where $r = 2(\Delta u)^2$ + higher order terms > 0 and $r_t = 2(\Delta u_t)^2$ + higher order terms > 0. Substituting Eq.(30) into Eq.(29) gives

$$\begin{aligned}\Delta \mathscr{J}[C(x,t)] = \int_0^1 &\left\{ \mu_1[2u^{\circ}(x,t_f)\Delta u(x,t_f) + r] \right. \\ &\left. + \mu_2[2u_t^{\circ}(x,t_f)\Delta u_t(x,t_f) + r_t] \right\} dx + \int_0^{t_f}\int_0^1 \mu_3[C(x,t)^2 - C(x,t)^{\circ^2}]dxdt.\end{aligned} \tag{31}$$

From Eq. (28) and because of $\mu_1 r + \mu_2 r_t > 0$, one obtains

$$\Delta \mathscr{J}[C(x,t)] \geq \int_0^1 \int_0^{t_f} \left\{ -\nu\Delta C(x,t) + \mu_3 C^2(x,t) - \mu_3 C^{\circ^2}(x,t) \right\} dt \geq 0 \tag{32}$$

which leads to

$$\Delta \mathscr{J}[C(x,t)] \geq \int_0^1 \int_0^{t_f} \left\{ [\mu_3 C^2(x,t) - \nu C(x,t)] - [\mu_3 C^{\circ^2}(x,t) - \nu C^{\circ}(x,t] \right\} dxdt \geq 0 \tag{33}$$

that is,

$$\mathscr{H}[t;\nu^{\circ},C] \geq \mathscr{H}[t;\nu,C].$$

Hence, we obtain

$$\mathscr{J}[C] \geq \mathscr{J}[C^{\circ}], \quad \forall C \in C_{ad}$$

Therefore, the optimal control function is given by

$$C(x,t) = \frac{\nu^{\circ}(x,t)}{2\mu_3}. \tag{34}$$

The existence and uniqueness of the solution to adjoint system, defined by Eqs.(17)-(19), can be obtained by similar way to Eqs.(1)-(3). Then, the state system given by Eqs.(1)-(3) is controllable.

4 Solution method

The solution of the optimal control problem is sought as follows: Let the adjoint variable $v(x,t)$ satisfying Eqs.(17)-(19) be expanded in Fourier sine series as

$$v(x,t)=\sum_{n=1}^{\infty}\theta_n(t)\varphi_n(x) \tag{35}$$

where the orthonormal eigenfunctions

$$\varphi_n(x)=\sqrt{2}\sin(\lambda_n x),\quad \lambda_n=n\pi \tag{36}$$

satisfy boundary conditions given by Eq.(18). Substituting Eq.(35) into Eq.(17), multiplying both sides with $\varphi_n(x)$ and integrating both sides over $(0,1)$ lead to the following lumped parameter system(LPS) in time

$$\ddot{\theta}_n-\alpha\dot{\theta}_n+\beta\theta_n-\theta\lambda_n^2=0,\quad \text{for}\ \ n=1,2,\dots. \tag{37}$$

The general solution of LPS given by Eq. (37) is given by

$$\theta_n(t)=a_n\kappa_n(t)+b_n\iota_n(t), \tag{38}$$

where

$$\kappa_n(t)=\exp\left((\alpha+\sqrt{\alpha^2-4(\beta^2-\lambda_n^2)})t/2\right),\quad \iota_n(t)=\exp\left((\alpha-\sqrt{\alpha^2-4(\beta^2-\lambda_n^2)})t/2\right) \tag{39}$$

and a_n and b_n are constants to be determined. Next, we solve the equation of the optimal motion. In order to convert the nonhomogeneous boundary conditions to homogeneous ones, let us define following relation

$$\varpi=u-xT_2(t)+(1-x)T_1(t). \tag{40}$$

Then, the system given by Eqs.(1)-(3) becomes

$$\varpi_{tt}+\alpha\varpi_t+\beta\varpi-\varpi_{xx}=C(x,t)-\sum_{i=1}^{3}\gamma_i(x,t) \tag{41}$$

in which

$$\gamma_1(x,t)=xT_2''(t)+(1-x)T_1''(t), \tag{42}$$

$$\gamma_2(x,t)=\alpha xT_2'(t)+\alpha(1-x)T_1'(t), \tag{43}$$

$$\gamma_3(x,t)=\beta xT_2(t)+\beta(1-x)T_1(t). \tag{44}$$

Eq.(41) subject to the new homogeneous boundary conditions

$$\varpi(x,t)=0\quad \text{at}\quad x=0,1 \tag{45}$$

and initial conditions

$$\varpi(x,0)=u_0(x)-T_2(0)-(1-x)T_1(0),\quad \varpi_t(x,0)=u_1(x)-T_2'(0)-(1-x)T_1'(0) \tag{46}$$

Due to Eq.(40), one observes the terminal conditions of adjoint equation Eq.(19) as follows:

$$v_t(x,t_f)-\alpha v(x,t_f)=2\mu_1[\varpi(x,t_f)+xT_2(t_f)+(1-x)T_1(t_f)], \tag{47a}$$

$$v(x,t_f) = -2\mu_2[\varpi_t(x,t_f) + xT_2'(t_f) + (1-x)T_1'(t_f)]. \tag{47b}$$

Now, let us obtain the solution of the motion of equation by using Fourier sine series

$$\varpi(x,t) = \sum_{n=1}^{\infty} \psi_n(t)\varphi_n(x) \tag{48}$$

in which $\Omega_n(t)$ satisfies the following LPS

$$\ddot{\psi}_n(t) - \alpha\dot{\psi}_n(t) + \beta\psi_n(t) - \psi_n(t)\lambda_n^2 = \gamma_4^* - \sum_{i=1}^{3}\gamma_i^*(t), \quad \text{for } n = 1,2,\ldots. \tag{49}$$

where

$$\gamma_1^*(t) = \int_0^1 \gamma_1(x,t)\varphi_n(x)dx, \quad \gamma_2^*(t) = \int_0^1 \gamma_2(x,t)\varphi_n(x)dx$$

$$\gamma_3^*(t) = \int_0^1 \gamma_3(x,t)\varphi_n(x)dx, \quad \gamma_4^*(t) = \int_0^1 C(x,t)\varphi_n(x)dx$$

The general solution of Eq.(49) is given by

$$\psi_n(t) = c_n\kappa_n(-t) + d_n\iota_n(t) + \frac{1}{2\sqrt{(\beta^2-\lambda_n^2)}}\int_0^t (\iota_n(t-s) - \kappa_n(s-t))(\gamma_4^*(s) - \sum_{i=1}^{3}\gamma_i^*(s))ds \tag{50}$$

in which c_n and d_n are constants to be determined by means of Eq.(46). Remaining unknown constants a_n and b_n appearing in Eq.(50) are evaluated by using the terminal conditions given by Eq.(47).

5 Numerical results and discussions

In this section, the theoretical results obtained in the previous sections are illustrated to show the effectiveness and capability of the proposed control algorithm for the beam equation in one space dimension. In the tables and graphics, following cases are taken into account;

For the case a, $T_1(t) = e^{-t}$, $T_2(t) = e^{-t}$, $u_0(x) = \sqrt{2}\sin(\pi x)$ and $u_1(x) = e^{-x}$,
For the case b, $T_1(t) = 0, T_2(t) = e^{t}$, $u_0(x) = e^{-x}$, and $u_1(x) = \pi\sqrt{2}\sin(\pi x)$.

Also, in the numerical calculations $\mu_3 = 10^{-4}$ for controlled case, $\mu_3 = 10^6$ for uncontrolled case, $\mu_1 = \mu_2 = 1$, $\alpha = 0.2$, $\beta = 0.1$. Let us define the dynamic response of the system and spend control in control process as follows, respectively;

$$\mathscr{D}(t_f) = \int_0^1 [u^2(x,t_f) + u_t^2(x,t_f)]dx, \quad \mathscr{C} = \int_0^{t_f}\int_0^1 C^2(x,t)dxdt. \tag{51}$$

The dynamic response of the system is presented in table 1 for the case a and b. Observing the table 1, it is concluded for the case a and b that as the penalty μ_3 on the expenditure of control decreases, the dynamic response of the beam decreases corresponding to an increase in the control function. In the case b, the vibrations in the system are induced by larger displacement/initial excitations than the case a. Therefore, it is seemed from table 1 that the value of the dynamic response of the system corresponding to the case a is less than the dynamic response of the beam corresponding to case b. Also, as a parallel result to this, the difference between dynamic responses for the case b is larger than the corresponding difference for the case a. A comparison of the controlled and uncontrolled dynamic responses in table 1 presents substantial decreasing as a result of the proposed control method. The un/controlled displacements and velocities are plotted at the midpoint of the beam at $x = 0.5$ since their maximum occurs at this point at $t = 0$ owing to displacement and initial conditions. Therefore, the midpoint gives an idea about the transient behavior of the system. By observing the Fig.1, it seems that the uncontrolled displacement displays a growing motion while the controlled displacement gradually decreases in case of a. Same observation is valid for the uncontrolled velocity plotted in Fig.2 and the velocity is effectively suppressed because of control. For the case b, the displacement and velocity of the beam is plotted in Fig.3 and Fig.4, respectively. Let us focus on the band-width in Fig.1-2 and Figs.3-4. Because the displacement/inital conditions effect in case b is larger than corresponding to the case a, the band-with of the Figs.3-4 is larger than the band-width of Figs.1-2. By taking into consideration the all tables and figures, it is concluded that introduced control method for the system is effective and applicable for the these kind of the systems.

6 Conclusion

In this paper, existence and uniqueness of the solution to the beam equation is presented. Also, the controllability of the system is discussed. Moreover, optimal control function for beam equation system in one space dimension is obtained by employing Maximum principle. Performance index functional of the control problem consists of a weighted quadratic functional of the dynamic responses of the system to be minimized and a penalty term defined as the control spent in the control process. By means of the maximum principle, the optimal control problem is transformed to the a coupled system of partial differential equations in terms of state, adjoint and control variables subject to the boundary, initial and terminal conditions. The explicit solution of the problem is sought by Galerkin expansion method. By using MATLAB, numerical results are given demonstrate the robustness and efficiency of the proposed control algorithm.

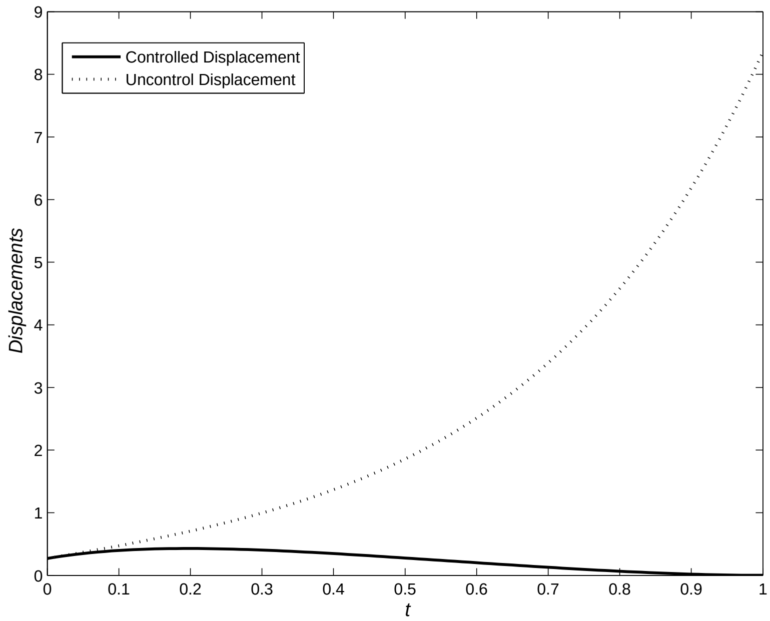

Fig. 1: Uncontrolled and controlled displacements at $x = (0.5)$ for case a.

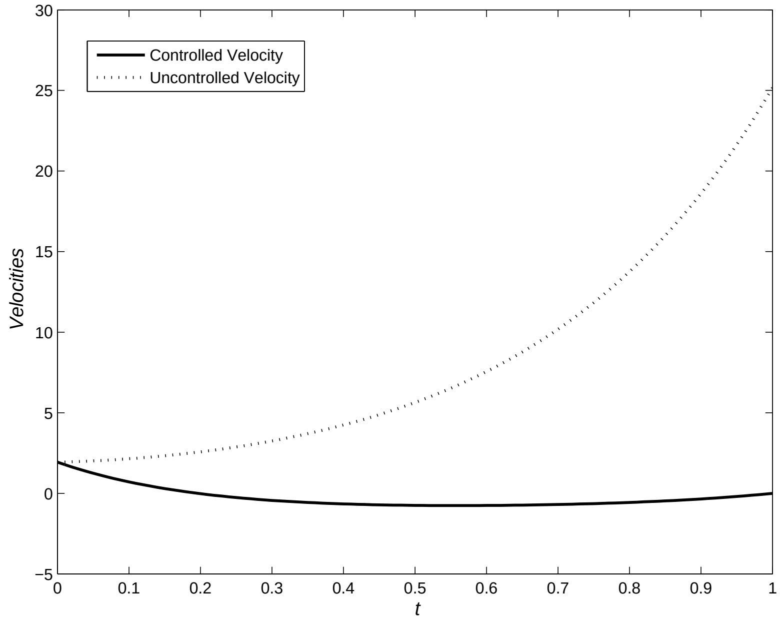

Fig. 2: Uncontrolled and controlled velocities at $x = (0.5)$ for case a.

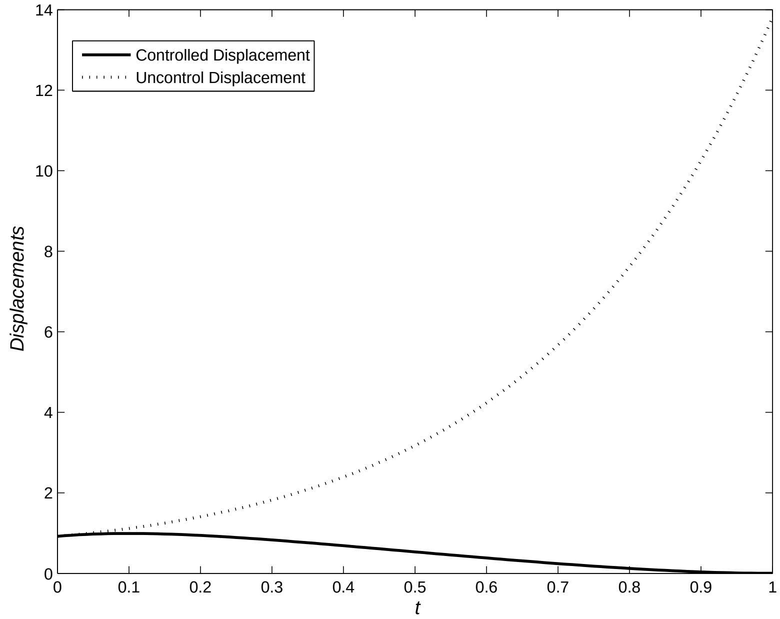

Fig. 3: Uncontrolled and controlled displacements at $x = (0.5)$ for case b.

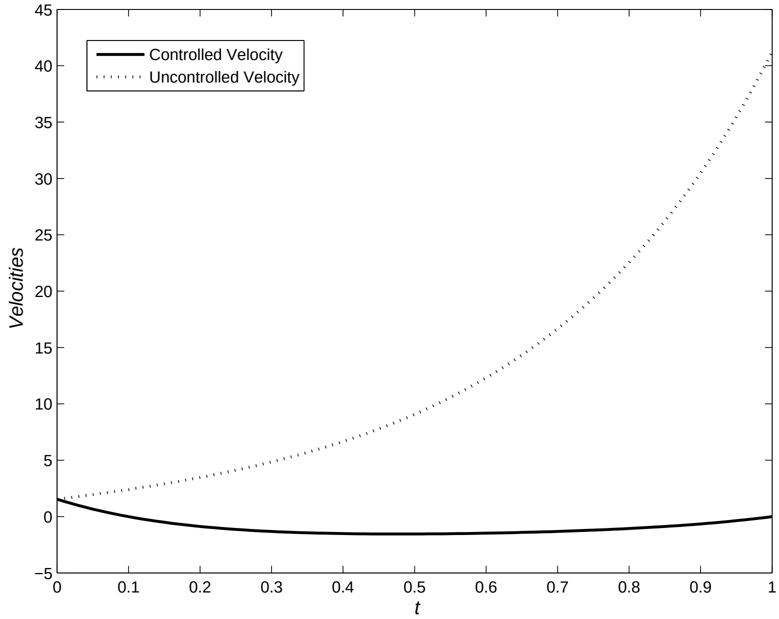

Fig. 4: Uncontrolled and controlled velocities at $x = (0.5)$ for case b.

Table 1: The values of $\mathscr{D}(t_f)$ at $\xi = 0$ for different values of μ_3 in case a and b.

μ_3	$\mathscr{D}^a$	$\mathscr{C}^a$
10^6	352	6.6×10^{-9}
10^4	350	6.5×10^{-5}
10^2	257	0.5
10^0	1.15	21
10^{-2}	7.7×10^{-3}	24
10^{-4}	2.4×10^{-5}	26

μ_3	$\mathscr{D}^b$	$\mathscr{C}^b$
10^6	945	1.7×10^{-8}
10^4	942	1.8×10^{-4}
10^2	692	1.3
10^0	3.13	56
10^{-2}	0.03	67
10^{-4}	1.0×10^{-5}	76

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