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Laguerre polynomial solution of high- order linear Fredholm integro-differential equations

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Abstract: In this paper, a Laguerre matrix method is developed to find an approximate solution of linear differential, integral and integro-differential equations with variable coefficients under mixed conditions in terms of Laguerre polynomials. For this purpose, Laguerre polynomials are used in the interval [0,b]. The proposed method converts these equations into matrix equations, which correspond to systems of linear algebraic equations with unknown Laguerre coefficients. The solution function is obtained easily by solving these matrix equations. The examples of these kinds of equations are solved by using this new method and the results are discussed and it is seen that the present method is accurate, efficient and applicable.

Keywords: Laguerre polynomials, high order differential equation, Fredholm integro-differential equation.

1 Introduction

In recent years, there has been a growing interest in integro differential equations (IDEs) which are a combination of differential and Fredholm- Volterra equations. This is an important branch of modern mathematics and arise frequently in many applied areas which include engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, etc. [15-17]. Many problems of physics and engineering lead naturally to the resolution of differential and integral equations in bounded or unbounded domains. For example, some problems arise in coastal hydrodynamics and in meteorology. Also integrals involving products of orthogonal polynomials or special functions arise in several physical contexts. For example, the wave functions of the hydrogen as well as the 2-, 3-, and in general N- dimension harmonic oscillator involve Laguerre polynomials and the evaluation of integrals involving the product of these polynomials is essential [6]. The mentioned IDEs are usually difficult to solve analytically; so a numerical method is required. Several numerical methods for the solution of linear and nonlinear Fredholm integro-differential equation (FIDE) and fractional integro-differential equations have been studied by some authors [1,2,7,8,14,16,21,22,23,24]. Additionally, the following methods for FIDEs have been presented:Adomian decomposition, Chebyshev and Taylor collocation, Haar Wavelet, Tau and Walsh series methods, etc. Similarly, since the beginning of 1994, Taylor and Chebyshev matrix methods have also been used by Sezer et al. [12,13,16,20] to solve linear differential, Fredholm integral and Fredholm integro-differential equations. And also approximate solution of Kuramoto–Sivashinsky equation using reduced differential transform method has been used by Acan et al. [18,19].

Laguerre polynomials are defined as solutions of Laguerre's differential equation [3,4].

$$xy'' + (1-x)y' + ny = 0.$$

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Solutions corresponding to the non-negative integer n can be expressed using Rodrigues' formula

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right).$$

The first few Laguerre polynomials are

$$\begin{gathered} L_0(x) = 1 \\ L_1(x) = -x+1 \\ L_2(x) = \tfrac{1}{2}x^2 - 2x + 1 \\ L_3(x) = -\tfrac{1}{6}x^3 + \tfrac{3}{2}x^2 - 3x + 1 \\ \vdots \\ L_n(x) = \sum_{i=0}^{n} \frac{(-1)^i}{i!}\binom{n}{n-i} x^i, \quad 0 \le x \le b < +\infty. \end{gathered} \tag{1}$$

These polynomials may be expressed in matrix form as

$$\mathbf{L}(x) = \left[L_0(x)\ L_1(x)\ \ldots\ L_N(x)\right].$$

This matrix can be written in the following convenient form

$$\mathbf{L}(x) = \mathbf{X}(x)\mathbf{H}^{\mathbf{T}} \tag{2}$$

where

$$\mathbf{X}(x) = \left[1\ x\ x^2\ \ldots\ x^N\right]$$

and

$$\mathbf{H} = \begin{bmatrix} \frac{(-1)^0}{0!}\binom{0}{0} & 0 & 0 & \cdots & 0 \\ \frac{(-1)^0}{0!}\binom{1}{0} & \frac{(-1)^1}{1!}\binom{1}{1} & 0 & \cdots & 0 \\ \frac{(-1)^0}{0!}\binom{2}{0} & \frac{(-1)^1}{1!}\binom{2}{1} & \frac{(-1)^2}{2!}\binom{2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^0}{0!}\binom{N}{0} & \frac{(-1)^1}{1!}\binom{N}{1} & \frac{(-1)^2}{2!}\binom{N}{1} & \cdots & \frac{(-1)^N}{N!}\binom{N}{N} \end{bmatrix}$$

In this study, we consider the mth order linear FIDE with variable coefficients

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \lambda \int_0^b K(x,t)y(t)dt \tag{3}$$

under the mixed conditions

$$\sum_{k=0}^{m-1}\left(a_{jk}y^{(k)}(0) + b_{jk}y^{(k)}(b)\right) = \lambda_j, \quad j = 0,1,...,m-1 \tag{4}$$

and assume that an approximation to the solution of (3) can be written in the following Laguerre polynomial form

$$y(x)=\sum_{n=0}^{N}a_nL_n(x) \tag{5}$$

where a_{jk}, b_{jk}, λ_j are suitable constants; a_n , $n=0,1,2,\ldots,N$ are the Laguerre coefficients to be determined, which are obtained by means of the proposed matrix method based on collocation points and the functions $L_n(x)$ are the Laguerre polynomials defined by (1).

2 Fundamental matrix relations

Eq. (3) may be written briefly in the form

$$D(x)=g(x)+\lambda F(x) \tag{6}$$

where

$$D(x)=\sum_{k=0}^{m}P_k(x)y^{(k)}(x) \tag{7}$$

Now, the solution $y(x)$ and its derivatives$y^{(k)}(x)$, the parts $D(x)$ and $F(x)$, and the mixed conditions (4) shall be converted into the matrix forms.

2.1 Matrix relations for $y(x)$ and $y^{(k)}(x)$

We assume that the solution function $y(x)$ can be expanded, as well, to the truncated Laguerrer series in the form

$$y(x)=\sum_{n=0}^{N}a_nL_n(x),\ 0\le x\ \le b<\infty. \tag{8}$$

The solutions (5) and (8), and their derivatives can be written in matrix forms, respectively, as

$$[y(x)]=\mathbf{L}(x)\mathbf{A},\ \left[y^{(k)}(x)\right]=\mathbf{L}^{(k)}(x)\mathbf{A} \tag{9}$$

$$[y(x)]=\mathbf{X}(x)\mathbf{Y},\ \left[y^{(k)}(x)\right]=\mathbf{X}^{(k)}(x)\mathbf{Y} \tag{10}$$

where $\mathbf{A}=\left[a_0\ a_1\ a_2\ \ldots\ a_N\right]^{\mathbf{T}}$are the unknown Laguerre coefficients and $\mathbf{Y}=\left[y_0\ y_1\ y_2\ \ldots\ y_N\right]^{\mathbf{T}}$ are the unknown Taylor coefficients.

On the other hand, the Laguerre polynomials satisfy the recurrence relation [19],

$$L'_n(x)=L'_{n-1}(x)-L_{n-1}(x). \tag{11}$$

Using (11), one may write

$$\begin{aligned} L'_1(x) &= L'_0(x) - L_0(x) = -L_0(x) \\ L'_2(x) &= L'_1(x) - L_1(x) = -L_0(x) - L_1(x) \\ L'_3(x) &= L'_2(x) - L_2(x) = -L_0(x) - L_1(x) - L_2(x)\,. \\ &\vdots \\ L'_N(x) &= -L_0(x) - L_1(x) - \ldots - L_{N-1}(x) \end{aligned} \tag{12}$$

It is obvious from (12) that the relation between the matrix $\mathbf{L}(x)$ and its derivative is

$$\left(\mathbf{L}'(x)\right)^{\mathbf{T}} = \mathbf{E}(\mathbf{L}(x))^{\mathbf{T}} \quad \text{or} \quad \mathbf{L}'(x) = \mathbf{L}(x)\mathbf{E}^{\mathbf{T}} \tag{13}$$

where

$$\mathbf{L}'(x) = \left[L'_0(x)\ L'_1(x)\ L'_2(x)\ \ldots\ L'_{N-1}(x)\ L'_N(x) \right]$$

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}.$$

From the matrix equations (9) and (13), it follows that

$$y'(x) = \mathbf{L}(x)\mathbf{E}^{\mathbf{T}}\mathbf{A}. \tag{14}$$

Using the relations (13) and (14), the recurrence relation can be written as

$$y^{(k)}(x) = \mathbf{L}(x)\left(\mathbf{E}^{\mathbf{T}}\right)^{k}\mathbf{A}. \tag{15}$$

2.2 Matrix relation for collocations points

A Laguerre matrix method based on collocation points is proposed to solve (3), and (4) in terms of Laguerre polynomials

$$x_p = \frac{b}{N}p,\ \ p = 0,1,2,\ldots,N \tag{16}$$

Substitution of the expression (16) into the part $D(x)$ of Eq.(7) yields

$$D(x_P) = g(x_P) + \lambda F(x_P)$$

or

$$\mathbf{D} = \mathbf{G} + \lambda\mathbf{F} \tag{17}$$

where

$$\mathbf{D}=\begin{bmatrix} D(x_0)\\ D(x_1)\\ \vdots \\ D(x_N)\end{bmatrix},\mathbf{G}=\begin{bmatrix} g(x_0)\\ g(x_1)\\ g(x_2)\\ \vdots \\ g(x_N)\end{bmatrix},\mathbf{F}=\begin{bmatrix} F(x_0)\\ F(x_1)\\ \vdots \\ F(x_N)\end{bmatrix}.$$

In order to construct the fundamental matrix equation, the matrix relation (17) is substituted into (6), and simplified to yield.

$$\sum_{k=0}^{m} P_k(x_p)\mathbf{L}(x_p)\left(\mathbf{E}^{\mathbf{T}}\right)^k\mathbf{A}=g(x_p)+\lambda\mathbf{L}(x_p)\mathbf{KQA}. \tag{18}$$

2.3 Matrix relation for the differential part $D(x)$

Substitution of the expression (15) into the part $D(x)$ of Eq.(7) yields

$$[D(x)]=\sum_{k=0}^{m} P_k(x)\mathbf{L}(x)\left(\mathbf{E}^{\mathbf{T}}\right)^k\mathbf{A}. \tag{19}$$

2.4 Matrix relation for the integral part $F(x)$

The kernel function $K(x,t)$ can be approximated by the truncated Laguerre series

$$K_f(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N} k_{mn}^{f}L_m(x)L_n(t) \tag{20}$$

where k_{mn} are Laguerre coefficients which shall be derived using the relation between the truncated Laguerre and Taylor series. We can write Eq. (20) in the following matrix form

$$\left[K_f(x,t)\right]=\mathbf{L}(x)\mathbf{K}_f\mathbf{L}^{\mathbf{T}}(t). \tag{21}$$

Substituting (3) into (21) yields the Laguerre matrix form of the kernel function

$$\left[K_f(x,t)\right]=\mathbf{X}(x)\mathbf{H}^{\mathbf{T}}\mathbf{K}_f\mathbf{H}\mathbf{X}^{\mathbf{T}}(t). \tag{22}$$

On the other hand, the Taylor matrix form of the kernel can be written as

$$[K_t(x,t)]=\mathbf{X}(x)\mathbf{K_t}\mathbf{X}^{\mathbf{T}}(t) \tag{23}$$

where $\mathbf{K_t}$ are the Taylor coefficients [6] given by

$$\mathbf{K_t}=\left[k_{mn}^{t}\right],\; k_{mn}^{t}=\frac{1}{m!n!}\frac{\partial^{m+n}K(0,0)}{\partial x^m\partial t^n},$$

Equating (22) to (23) gives the matrix relation between the truncated Laguerre series and the truncated Taylor series:

$$\mathbf{K}_f=\left(\mathbf{H}^{\mathbf{T}}\right)^{-\mathbf{1}}\mathbf{K_t}\mathbf{H}^{-\mathbf{1}}=[k_{mn}].$$

Finally, by substituting the matrix forms (9) and (21) into $F(x)$ of (6) and using Eq.(2), we have the matrix relation

$$\mathbf{F}(x)=\mathbf{L}(x)\mathbf{K}_f\left[\int_0^b \mathbf{L}^{\mathbf{T}}(t)\mathbf{L}(t)dt\right]\mathbf{A}=\mathbf{L}(x)\left(\mathbf{H}^{\mathbf{T}}\right)^{-\mathbf{1}}\mathbf{K_tQH^TA} \tag{24}$$

where

$$\mathbf{Q}=\int_0^b \mathbf{L}^{\mathbf{T}}(t)\mathbf{L}(t)dt=\int_0^b \mathbf{H}\mathbf{X}^{\mathbf{T}}(t)\mathbf{X}(t)\mathbf{H}^{\mathbf{T}}dt,\quad \mathbf{Q}^*=\int_0^b \mathbf{X}^{\mathbf{T}}(t)\mathbf{X}(t)dt=[q_{ij}]$$

$$q_{ij}=\frac{b^{i+j+1}}{i+j+1},\qquad i,j=0,1,2,...,N$$

and

$$\mathbf{Q}=\mathbf{H}\mathbf{Q}^*\mathbf{H}^{\mathbf{T}}.$$

2.5 Matrix relation for the mixed conditions

The corresponding matrix form of the conditions (4) can be obtained by means of (15) as

$$\sum_{k=0}^{m-1}\left(a_{jk}\mathbf{L}(0)+b_{jk}\mathbf{L}(b)\right)\left(\mathbf{E}^{\mathbf{T}}\right)^k\mathbf{A}=\sum_{k=0}^{m-1}\left(a_{jk}\mathbf{X}(0)+b_{jk}\mathbf{X}(b)\right)\mathbf{H}^{\mathbf{T}}\left(\mathbf{E}^{\mathbf{T}}\right)^k\mathbf{A}=[\lambda_{ij}] \tag{25}$$

where $j=0,1,2,...,m-1$.

3 Method of solution

In order to construct the fundamental matrix equation, the matrix relations (19), (24) and (26) are substituted into (6), and simplified to yield

$$\left[\sum_{k=0}^{m}P_k(x)\left(\mathbf{E}^{\mathbf{T}}\right)^k-\lambda\left(\mathbf{H}^{\mathbf{T}}\right)^{-\mathbf{1}}\mathbf{K_tQH^T}\right]\mathbf{A}=\mathbf{G} \tag{26}$$

which corresponds to a system of $(N+1)$ algebraic equations for the $(N+1)$ unknown Laguerre coefficients $a_0,a_1,...,a_n$. Eq. (26) can be written briefly in the form

$$\mathbf{WA}=\mathbf{G}\quad \text{or}\quad [\mathbf{W};\mathbf{G}] \tag{27}$$

where

$$\mathbf{W}=[w_{pq}]=\left[\sum_{k=0}^{m}P_k(x)\left(\mathbf{E}^{\mathbf{T}}\right)^k-\lambda\left(\mathbf{H}^{\mathbf{T}}\right)^{-\mathbf{1}}\mathbf{K_tQH^T}\right],\quad p,q=0,1,...,N$$

On the other hand, the matrix form (25) of the conditions (4) can be written as

$$\mathbf{U_jA}=[\lambda_j]\,,or\,\left[\mathbf{U_j};\lambda_j\right],j=0,1,2,...,m-1 \tag{28}$$

where

$$\mathbf{U_j}=\sum_{k=0}^{m-1}\left(a_{jk}\mathbf{L}(0)+b_{jk}\mathbf{L}(b)\right)\left(\mathbf{E}^{\mathbf{T}}\right)^k\equiv\left[u_{j0}\;u_{j1}\;\ldots\;u_{jN}\right],\qquad j=0,1,2,...,m-1$$

to obtain the general solution of Eq. (3) under the conditions (4), we replace the last m rows of the matrix (27) by the m-row matrix (28), and construct the new augmented matrix [15,16].

$$\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0)\\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(x_1)\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & g(x_{N-m})\\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0\\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix}. \tag{29}$$

If rank $\tilde{\mathbf{W}}$ =rank$\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right]=N+1$, then

$$\mathbf{A}=\left(\tilde{\mathbf{W}}\right)^{-1}\tilde{\mathbf{G}}. \tag{30}$$

Note that, if, $\tilde{\mathbf{W}}$ comes out to be a singular matrix, then any other m rows of $\tilde{\mathbf{W}}$ can be replaced by the m-row matrix (28) in order that the singularity is eliminated. Therefore, the unknown Laguerre coefficients a_n , $(n=0,1,...,N)$ are uniquely determined by (30). If $\lambda=0$ in Eq. (3), the equation becomes a high-order linear differential equation; if $P_k=0$ for $k\neq 0$, then the equation becomes a Fredholm integral equation. The present solution is valid also for these cases.

4 Numerical examples

Now we apply the method to the above numerical examples with initial conditions. All computations were carried out using MathCAD 14 [9] and Mathlab 6.5.1.[10].

Example 1. Let us consider the eighth-order linear differential equation given in Ref. [11]

$$y^{(viii)}(x)-y(x)=-8e^{x} \quad , \quad 0\leq x\leq 1$$

with the initial conditions

$$y(0)=1\,,\quad y'(0)=0\,,\quad y''(0)=-1\,,\quad y'''(0)=-2\,,\quad y^{(iv)}(0)=-3$$
$$y^{(v)}(0)=-4\,,\quad y^{(vi)}(0)=-5\,,\quad y^{(vii)}(0)=-6$$

and approximate the solution $y(x)$ by the Laguerre polynomial of degree $N=8$. In this problem $P_0=-1$, $P_1=P_2=P_3=P_4=P_5=P_6=P_7=0$, $P_8=1$, $g(x)=-8e^{x}$ for N=8, the Laguerre collocation points are $x_p=\frac{b}{N}p$ $i=0,1,2,3,4,5,6,7,8$ $x_0=0$, $x_1=\frac{1}{8}$, $x_2=\frac{2}{8}$, $x_3=\frac{3}{8}$, $x_4=\frac{4}{8}$, $x_5=\frac{5}{8}$, $x_6=\frac{6}{8}$, $x_7=\frac{7}{8}$, $x_8=1$. Following the procedure in Section 3, the augmented matrix of the given differential equation becomes

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & ; & -8\\ 1 & 0.875 & 0.758 & 0.648 & 0.546 & 0.45 & 0.361 & 0.278 & 1.201 & ; & -7.06\\ 1 & 0.75 & 0.531 & 0.341 & 0.177 & 0.037 & -0.081 & -0.179 & 0.74 & ; & -6.23\\ 1 & 0.625 & 0.32 & 0.077 & -0.112 & -0.256 & -0.359 & -0.428 & 0.531 & ; & -5.498\\ 1 & 0.5 & 0.125 & -0.146 & -0.331 & -0.446 & -0.505 & -0.518 & 0.502 & ; & -4.852\\ 1 & 0.375 & -0.055 & -0.33 & -0.485 & -0.548 & -0.543 & -0.491 & 0.593 & ; & -4.282\\ 1 & 0.25 & -0.219 & -0.477 & -0.581 & -0.577 & -0.501 & -0.383 & 0.756 & ; & -3.779\\ 1 & 0.125 & -0.367 & -0.588 & -0.625 & -0.546 & -0.4 & -0.224 & 0.953 & ; & -3.335\\ 1 & 0 & -0.5 & -0.667 & -0.625 & -0.467 & -0.257 & -0.04 & 1.154 & ; & -2.943\end{bmatrix}$$

from Eq. (22), the matrix forms for initial conditions are

$$\begin{aligned}\mathbf{L}(0).\left(\mathbf{E}^{\mathbf{T}}\right)^0&=\begin{bmatrix}1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\end{bmatrix}\\ \mathbf{L}(0).\left(\mathbf{E}^{\mathbf{T}}\right)^1&=\begin{bmatrix}0\ -1\ -2\ -3\ -4\ -5\ -6\ -7\ -8\end{bmatrix}\\ \mathbf{L}(0).\left(\mathbf{E}^{\mathbf{T}}\right)^2&=\begin{bmatrix}0\ 0\ 1\ 3\ 6\ 10\ 15\ 21\ 28\end{bmatrix}\\ \mathbf{L}(0).\left(\mathbf{E}^{\mathbf{T}}\right)^3&=\begin{bmatrix}0\ 0\ 0\ -1\ -4\ -10\ -20\ -35\ -56\end{bmatrix}\\ \mathbf{L}(0).\left(\mathbf{E}^{\mathbf{T}}\right)^4&=\begin{bmatrix}0\ 0\ 0\ 0\ 1\ 5\ 15\ 35\ 70\end{bmatrix}\\ \mathbf{L}(0).\left(\mathbf{E}^{\mathbf{T}}\right)^5&=\begin{bmatrix}0\ 0\ 0\ 0\ 0\ -1\ -6\ -21\ -56\end{bmatrix}\\ \mathbf{L}(0).\left(\mathbf{E}^{\mathbf{T}}\right)^6&=\begin{bmatrix}0\ 0\ 0\ 0\ 0\ 0\ 1\ 7\ 28\end{bmatrix}\\ \mathbf{L}(0).\left(\mathbf{E}^{\mathbf{T}}\right)^7&=\begin{bmatrix}0\ 0\ 0\ 0\ 0\ 0\ 0\ -1\ -8\end{bmatrix}.\end{aligned}$$

The new augmented matrix (31) based on initial conditions is

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]=\begin{bmatrix}0 & 1 & -0.5 & -0.66666667 & -0.625 & -0.46666667 & -0.25694444 & -0.04047619 & 1.15399306 & ; & -8\\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & ; & 1\\ 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & ; & 0\\ 0 & 0 & 1 & 3 & 6 & 10 & 15 & 21 & 28 & ; & -1\\ 0 & 0 & 0 & -1 & -4 & -10 & -20 & -35 & -56 & ; & -2\\ 0 & 0 & 0 & 0 & 1 & 5 & 15 & 35 & 70 & ; & -3\\ 0 & 0 & 0 & 0 & 0 & -1 & -6 & -21 & -56 & ; & -4\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 28 & ; & -5\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -8 & ; & -6\end{bmatrix}$$

solving this system yields the unknown Laguerre coefficients

$$\mathbf{A}=\begin{bmatrix}-28\ 176\ -490\ 812\ -868\ 608\ -271\ 70\ -8\end{bmatrix}^{\mathbf{T}}$$

thus the solution function is

$$y(x)=-28L_0+176L_1-490L_2+812L_3-868L_4-608L_5-271L_6+70L_7-8L_8.$$

x	Exact solution	N = 8 [16]	Present N=8
0	1	1	1
0.1	0.9946538	0.9946538	0.9946538
0.2	0.9771222	0.9771222	0.9771222
0.3	0.9449012	0.9449011	0.9449012
0.4	0.8950948	0.8950942	0.8950948
0.5	0.8243606	0.8243574	0.8243606
0.6	0.7288475	0.7288360	0.7288473
0.7	0.6041258	0.6040917	0.6041253
0.8	0.4451082	0.4450208	0.4451073
0.9	0.2459603	0.2457599	0.2459591
1	0	-4.212943×10^{-4}	-4.9359339×10^{-9}

Table 1: Comparison of numerical results for Example 1.

The exact solution of this problem is $y(x)=(1-x)e^x$. In Table 1, the numerical results obtained by the present method are compared with the results of exact solution and those given in [11] for the interval $[0,1]$. It is seen from Table 1 that the results obtained by the present method are in good agreement with the results of the exact solution and of [11]. Moreover, the present method is very effective and convenient. The numerical computations of this example have been carried out by MathCAD 14 [9] software.

Example 2.

$$y''+xy'-xy=-2x^4+11x^3-11x^2+\frac{47}{3}x-10+\int_0^1 xy(t)\,dt,$$

with the initial conditions

$$y(0)=-1,\quad y'(0)=1$$

and approximate the solution $y(x)$ by the Laguerre polynomial of degree $N=8$. In this problem

$$F_2(x)=1,\quad F_1(x)=x,\quad F_0(x)=-x,\qquad g(x)=-2x^4+11x^3-11x^2+\frac{47}{3}x-10,\qquad \lambda=1,\qquad K(x,t)=x$$

for $N=3$, the Laguerre collocation points are $x_p=\frac{b}{N}p,\quad i=0,1,2,3;\ x_0=0,\quad x_1=\frac{1}{3},\quad x_2=\frac{2}{3},\quad x_3=1$ and approximate the solution $y(x)$,

$$y(x)=\sum_{n=0}^{N}a_nL_n(x)$$

from Eq.(28) the matrix representation of the equation is

$$\mathbf{W}=\mathbf{P_0}.\mathbf{L}.\left(\mathbf{E^T}\right)^{\mathbf{0}}+\mathbf{P_1}.\mathbf{L}.\left(\mathbf{E^T}\right)^{\mathbf{1}}+\mathbf{P_2}.\mathbf{L}.\left(\mathbf{E^T}\right)^{\mathbf{2}}-\mathbf{L}.\mathbf{K}.\mathbf{Q}=\begin{bmatrix}0 & 0 & 1 & 3\\ -0.66666667 & -0.72222222 & 0.25925926 & 1.94187243\\ -1.33333333 & -1.22222222 & 0.07407407 & 1.80144033\\ -2 & -1.5 & 0.3333333 & 2.20833333\end{bmatrix}$$

where

$$\mathbf{P_0}=\begin{bmatrix}0&0&0&0\\0&\frac{-1}{3}&0&0\\0&0&\frac{-2}{3}&0\\0&0&0&-1\end{bmatrix},\quad \mathbf{P}_1=\begin{bmatrix}0&0&0&0\\0&\frac{1}{3}&0&0\\0&0&\frac{2}{3}&0\\0&0&0&1\end{bmatrix},\quad \mathbf{P}_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\mathbf{X}=\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & \frac{1}{3} & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3\\ 1 & \frac{2}{3} & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3\\ 1 & 1 & 1 & 1\end{bmatrix},\quad \mathbf{G}=\begin{bmatrix}-10\\ -5.61728395\\ -1.58024691\\ 3.66666667\end{bmatrix},\quad \mathbf{L}=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 0.6666666 & 0.38888889 & 0.16049383\\ 1 & 0.33333333 & -0.11111111 & -0.38271605\\ 1 & 0 & -0.5 & -0.66666667\end{bmatrix},$$

$$\mathbf{K}=\begin{bmatrix}1 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

for the conditions $y(0)=-1, \quad y'(0)=1$, the augmented matrices are obtained, as

$$\begin{bmatrix}U_0\ ;\ \lambda_0\\ U_1\ ;\ \lambda_1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & 1 & ; & -1\\ 0 & -1 & -2 & -3 & ; & 1\end{bmatrix}$$

by replacing the rows matrices by the last two rows of the matrix we have the required augmented matrix

$$\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right]=\begin{bmatrix}0 & 0 & 1 & 3 & ; & -10\\ -0.66666667 & -0.72222222 & 0.25925926 & 1.94187243 & ; & -5.61728395\\ 1 & 1 & 1 & 1 & ; & -1\\ 0 & -1 & -2 & -3 & ; & 1\end{bmatrix}.$$

Solving this system yields the unknown Laguerre coefficients

$$\mathbf{A}=\left(\tilde{\mathbf{W}}\right)^{-1}.\tilde{\mathbf{G}}=\begin{bmatrix}2.00000015\\ -17.00000045\\ 26.00000045\\ -12.00000015\end{bmatrix}.$$

Thus the solution function is

$$\begin{aligned}y(x)&=\sum_{n=0}^{N}a_nL_n(x)=a_0L_0(x)+a_1L_1(x)+a_2L_2(x)+a_3L_3(x)\\ &=2x^3-5x^2+x-1\end{aligned}$$

which is precisely equal to the exact solution. Hence, it is seen that present method is accurate, efficient and applicable.

Example 3. As a last example, the fifth order differential equation

$$y^{(5)}(x)-x^2y^{(3)}(x)-y^{(1)}(x)-xy(x)=x^2\cos x-x\sin x+\int_0^{\pi/2}y(t)\,dt\ ,$$

with the initial conditions

$$y(0)=y''(0)=y^{(4)}(0)=0\ ,\quad y'(0)=1\ ,\quad y'''(0)=-1$$

the new augmented matrix (28) for this problem is

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]=\begin{bmatrix}-1.571 & -0.337 & -0.749 & -2.554 & -5.577 & -10.384 & ; & 0\\ 1 & 1 & 1 & 1 & 1 & 1 & ; & 0\\ 0 & -1 & -2 & -3 & -4 & -5 & ; & 1\\ 0 & 0 & 1 & 3 & 6 & 10 & ; & 0\\ 0 & 0 & 0 & -1 & -4 & -10 & ; & -1\\ 0 & 0 & 0 & 0 & 1 & 5 & ; & 0\end{bmatrix}$$

in which the last five rows represent the initial values. Solving this system, Laguerre coefficients are determined and thus the solution function is

$$y(x)=1.48L_0-5.398L_1+11.796L_2-13.796L_3+7.398L_4-1.48L_5.$$

In Figure 1, the Laguerre polynomial solution is compared with the exact solution $y(x)=\sin(x)$.

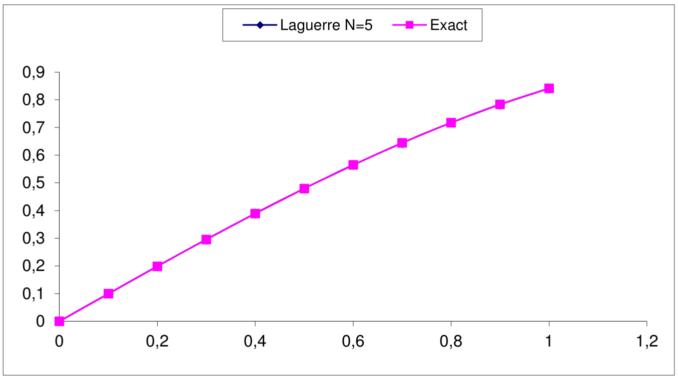

Fig. 1: Comparison of the present solution with the exact solution for Example 3.

5 Conclusion and discussions

The proposed practical matrix method is used to solve linear differential, integral and integro-differential equations with constant coefficients. The proposed method converts these equations into matrix equations, which correspond to systems of linear algebraic equations with unknown Laguerre coefficients. The solution function is obtained easily by solving these matrix equations. The method is illustrated by numerical applications. Comparison of the results obtained by the present method with those obtained by other methods reveals that the present method is very effective and convenient. The accuracy of the solution improves with increasing N. The Laguerre matrix method can be applied also to the variable coefficient differential, integral, integro-differential and differential-difference equations, and to the system of these equations.

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