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Surface Family with a Common Natural Geodesic Lift of a Spacelike Curve with Timelike Binormal in Minkowski 3-Space

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Abstract
In this work we aim to find a surface family possessing the natural lift of a given spacelike curve with timelike binormal as a geodesic in Minkowski 3-space. We express necessary and sufficient conditions for the given curve such that its natural lift is a geodesic on any member of the surface family. Finally, we illustrate the method with some examples.

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1. Introduction

Minkowski 3-space $\mathbb{R}^3_1$ is the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product g given by

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2$$

where $X = (x_1,x_2,x_3) \in \mathbb{R}^3$. A vector $X = (x_1,x_2,x_3) \in \mathbb{R}^3$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ or $X = 0$ and lightlike (or null) if $g(X,X) = 0$ and $X \neq 0$ [1]. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{R}^3_1$ can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are, respectively, timelike, spacelike or null (lightlike), for every $s \in I \subset \mathbb{R}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$) and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$). The norm of a vector X is defined by $\|X\| = \sqrt{|g(X,X)|}$ [1].

The vectors $X = (x_1,x_2,x_3)$, $Y = (y_1,y_2,y_3) \in \mathbb{R}^3_1$ are Lorentzian orthogonal if and only if $g(X,X) = 0$ [2].

Lemma 1. *Let X and Y be nonzero Lorentz orthogonal vectors in $\mathbb{R}^3_1$. If X is timelike, then Y is spacelike [2].*

Lemma 2. *Let X and Y be positive (negative) timelike vectors in $\mathbb{R}^3_1$. Then*

$$g(X,Y) \leq \|X\| \|Y\|$$

whit equality if and only if X and Y are linearly dependent [2].

Lemma 3. ***i)*** *Let X and Y be positive (negative) timelike vectors in $\mathbb{R}^3_1$. By Lemma 2, there is a unique nonnegative real number $\varphi(X,Y)$ such that*

$$g(X,Y) = \|X\| \|Y\| \cosh \varphi(X,Y).$$

The Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$ [2].
ii) *Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a spacelike vector subspace. Then we have*

$$|g(X,Y)| \leq \|X\| \|Y\|.$$

Hence, there is a unique real number $\varphi(X,Y)$ between 0 and π such that

$$g(X,Y) = \|X\| \|Y\| \cos\varphi(X,Y).$$

$\varphi(X,Y)$ is defined to be the Lorentzian spacelike angle between X and Y [2].
iii) *Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a timelike vector subspace. Then, we have*

$$g(X,Y) > \|X\| \|Y\|.$$

Hence, there is a unique positive real number $\varphi(X,Y)$ between 0 and π such that

$$|g(X,Y)| = \|X\| \|Y\| \cosh\varphi(X,Y).$$

$\varphi(X,Y)$ is defined to be the Lorentzian timelike angle between X and Y [2].
iv) *Let X be a spacelike vector and Y be a positive timelike vector in $\mathbb{R}_1^3$. Then there is a unique nonnegative real number $\varphi(X,Y)$ such that*

$$|g(X,Y)| = \|X\| \|Y\| \sinh\varphi(X,Y).$$

$\varphi(X,Y)$ is defined to be the Lorentzian timelike angle between X and Y [2].

Now, let X and Y be two vectors in $\mathbb{R}_1^3$, then the Lorentzian cross product is defined by [3]

$$\begin{aligned} X \times Y &= \begin{vmatrix} \overrightarrow{e_1} & -\overrightarrow{e_2} & -\overrightarrow{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\ &= (x_2y_3 - x_3y_2,\ x_1y_3 - x_3y_1,\ x_2y_1 - x_1y_2), \end{aligned}$$

where $\overrightarrow{e_1} = (1,0,0)$, $\overrightarrow{e_2} = (0,1,0)$, $\overrightarrow{e_3} = (0,0,1)$.

- We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve α, where T, N and B are the tangent, the principal normal and the binormal vector fields of the curve α, respectively.
- Let α be a unit speed timelike curve with curvature κ and torsion τ. So, T is a timelike vector field, N and B are spacelike vector fields. For these vectors, we can write

  $$T \times N = -B, \quad N \times B = T, \quad B \times T = -N,$$

  where $\times$ is the Lorentzian cross product in $\mathbb{R}_1^3$ [4]. The binormal vector field $B(s)$ is the unique spacelike unit vector fleld perpendicular to the timelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T, N, B\}$ has the same orientation as $\mathbb{R}_1^3$. Then, Frenet formulas are given by [4]

  $$T' = \kappa N,\ N' = \kappa T + \tau B,\ B' = -\tau N.$$

- Let α be a unit speed spacelike curve with spacelike binormal. Now, T and B are spacelike vector fields and N is a timelike vector field. In this situation, we have

  $$T \times N = -B, \quad N \times B = -T, \quad B \times T = N.$$

  The binormal vector field $B(s)$ is the unique spacelike unit vector fleld perpendicular to the timelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T, N, B\}$ has the same orientation as $\mathbb{R}_1^3$. Then, Frenet formulas are given by [4]

  $$T' = \kappa N,\ N' = \kappa T + \tau B,\ B' = \tau N.$$

- Let α be a unit speed spacelike curve with timelike binormal. In this case, T and N are spacelike vector fields and B is a timelike vector field and we have the following vectoral relation

$$T \times N = B, \quad N \times B = -T, \quad B \times T = -N,$$

The binormal vector field $B(s)$ is the unique timelike unit vector field perpendicular to the spacelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T, N, B\}$ has the same orientation as $\mathbb{R}_1^3$. Then, Frenet formulas are given by [4]

$$T' = \kappa N, \ N' = -\kappa T + \tau B, \ B' = \tau N.$$

Definition 4. *Let P be a surface and $\alpha : I \longrightarrow P$ be a parametrized curve in $\mathbb{R}_1^3$. α is called an integral curve of X if*

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)), \ (\text{for all } t \in I),$$

where X is a smooth tangent vector field on P [1]. We have

$$TP = \bigcup_{p \in P} T_p P = \chi(P),$$

where T_pP is the tangent space of the surface P at the point p and $\chi(P)$ is the space of tangent vector fields on P.

Definition 5. *For any parametrized curve $\alpha : I \longrightarrow P$, $\bar{\alpha} : I \longrightarrow TP$ is given by*

$$\bar{\alpha}(s) = \left(\alpha(s), \alpha'(s)\right) = \alpha'(s)|_{\alpha(s)}$$

is called the natural lift of the curve α on the space of tangent vector fields TP [5].

Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be an arc length timelike curve. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike or spacelike binormal. We have following relations between the Frenet frame $\{T(s), N(s), B(s)\}$ of α and the Frenet frame $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ of $\bar{\alpha}$.

a) Let the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike binormal.

i) If the Darboux vector W of the curve α is a timelike vector, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cosh\theta & 0 & \sinh\theta \\ -\sinh\theta & 0 & \cosh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \tag{2}$$

ii) If W is a spacelike vector, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\sinh\theta & 0 & \cosh\theta \\ -\cosh\theta & 0 & \sinh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \tag{3}$$

b) Let the natural lift $\bar{\alpha}$ of α is a spacelike curve with spacelike binormal.

i) If W is a timelike vector, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cosh\theta & 0 & \sinh\theta \\ \sinh\theta & 0 & -\cosh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \tag{4}$$

ii) If W is a spacelike vector, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\sinh\theta & 0 & \cosh\theta \\ \cosh\theta & 0 & -\sinh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \tag{5}$$

2. Surface family with a common natural geodesic lift of a spacelike curve with timelike binormal in Minkowski 3-space

This section is the original part of our study. Our purpose is to give a surface family which have a common geodesic lift of a spacelike curve with timelike binormal in Minkowski 3-space. Suppose we are given a 3-dimensional a spacelike curve with timelike binormal $\alpha(s)$, $L_1 \leq s \leq L_2$, in which s is the arc length and $\|\alpha''(s)\| \neq 0$, $L_1 \leq s \leq L_2$. Let $\bar{\alpha}(s)$, $L_1 \leq s \leq L_2$, be the natural lift of the given curve $\alpha(s)$. Now, $\bar{\alpha}$ is a spacelike curve with timelike or spacelike binormal.

Definition 6. *Surface family that interpolates $\bar{\alpha}(s)$ as a common curve is given in the parametric form as*

$$P(s,t) = \bar{\alpha}(s) + u(s,t)\bar{T}(s) + v(s,t)\bar{N}(s) + w(s,t)\bar{B}(s), \tag{6}$$

where $u(s,t)$, $v(s,t)$ and $w(s,t)$ are C^1 functions, called marching-scale functions, and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ is the Frenet frame of the curve $\bar{\alpha}$.

Remark 7. *Observe that choosing different marching-scale functions yields different surfaces possessing $\bar{\alpha}(s)$ as a common curve.*

Our goal is to find the necessary and sufficient conditions for which the curve $\bar{\alpha}(s)$ is isoparametric and geodesic on the surface $P(s,t)$. Firstly, as $\bar{\alpha}(s)$ is an isoparametric curve on the surface $P(s,t)$, there exists a parameter $t_0 \in [T_1, T_2]$ such that

$$u(s,t_0) = v(s,t_0) = w(s,t_0) \equiv 0,\ L_1 \leq s \leq L_2,\ T_1 \leq t_0 \leq T_2. \tag{7}$$

Secondly the curve $\bar{\alpha}$ is a geodesic on the surface $P(s,t)$ if and only if along the curve the normal vector field $n(s,t_0)$ of the surface is parallel to the principal normal vector field $\bar{N}$ of the curve $\bar{\alpha}$. The normal vector $n(s,t)$ of $P(s,t)$ can be written as

$$n(s,t) = \frac{\partial P(s,t)}{\partial s} \times \frac{\partial P(s,t)}{\partial t}.$$

Along the curve $\bar{\alpha}$, one can obtain the normal vector $n(s,t_0)$ using Eqns. $(6-7)$ with an appropriate equation in Eqns. $(2-5)$. It has one of the following forms:

i) if $\bar{\alpha}$ is a spacelike curve with timelike binormal and the Darboux vector W is spacelike or timelike, then we have

$$n(s,t_0) = \kappa\left[\frac{\partial w}{\partial t}(s,t_0)\bar{N}(s) + \frac{\partial v}{\partial t}(s,t_0)\bar{B}(s)\right], \tag{8}$$

ii) if $\bar{\alpha}$ is a spacelike curve with spacelike binormal and the Darboux vector W is spacelike, then we have

$$n(s,t_0) = -\kappa\left[\frac{\partial w}{\partial t}(s,t_0)\bar{N}(s) + \frac{\partial v}{\partial t}(s,t_0)\bar{B}(s)\right], \tag{9}$$

where κ is the curvature of the curve α.

Since $\kappa(s) \neq 0$, $L_1 \leq s \leq L_2$, the curve $\bar{\alpha}$ is a geodesic on the surface $P(s,t)$ if and only if

$$\frac{\partial w}{\partial t}(s,t_0) \neq 0,\ \frac{\partial v}{\partial t}(s,t_0) = 0.$$

So, we give the following theorem and corollary :

Theorem 8. *Let $\alpha(s)$ be a unit speed a spacelike curve with timelike binormal with nonvanishing curvature and $\bar{\alpha}(s)$ be its natural lift. $\bar{\alpha}$ is a geodesic on the surface in Eqn. (6) if and only if*

$$\begin{cases} u(s,t_0) = v(s,t_0) = w(s,t_0) = \frac{\partial v}{\partial t}(s,t_0) \equiv 0, \\ \frac{\partial w}{\partial t}(s,t_0) \neq 0, \end{cases} \tag{10}$$

where $L_1 \leq s \leq L_2$, $T_1 \leq t$, $t_0 \leq T_2$ (t_0 fixed).

Corollary 9. *Let $\alpha(s)$ be a unit speed a spacelike curve with timelike binormal with nonvanishing curvature and $\bar{\alpha}(s)$ be its natural lift. If*

$$\begin{aligned} u(s,t) &= w(s,t) = t - t_0,\ v(s,t) \equiv 0 \\ &\quad or \\ u(s,t) &= v(s,t) \equiv 0,\ w(s,t) = t - t_0, \end{aligned} \tag{11}$$

where $L_1 \leq s \leq L_2,\ T_1 \leq t,\ t_0 \leq T_2$ (t_0 fixed) then (6) is a ruled surface possessing $\bar{\alpha}$ as a geodesic.

Proof. By taking marching scale functions as $u(s,t) = w(s,t) = t - t_0,\ v(s,t) \equiv 0$ or $u(s,t) = v(s,t) \equiv 0,\ w(s,t) = t - t_0$, the surface (6) takes the form

$$\begin{aligned} P(s,t) &= \bar{\alpha}(s) + (t - t_0)\left[\bar{T}(s) + \bar{B}(s)\right] \\ &\quad or \\ P(s,t) &= \bar{\alpha}(s) + (t - t_0)\bar{B}(s), \end{aligned}$$

which is a ruled surface satisfying Eqn. (10). ■

3. Examples

Example 1. *Let $\alpha(s) = (0, \cos s, \sin s)$ be a spacelike curve with timelike binormal. It is easy to show that the Frenet frame of the curve α is*

$$\begin{aligned} T(s) &= (0, -\sin s, \cos s), \\ N(s) &= (0, -\cos s, -\sin s), \\ B(s) &= (1, 0, 0). \end{aligned}$$

The natural lift $\bar{\alpha}(s) = (0, -\sin s, \cos s)$ of α is a spacelike curve with timelike binormal and its Frenet vectors can be given as follows

$$\begin{aligned} \bar{T}(s) &= (0, -\cos s, -\sin s), \\ \bar{N}(s) &= (0, \sin s, -\cos s), \\ \bar{B}(s) &= (1, 0, 0). \end{aligned}$$

Choosing marching scale functions as $u(s,t) = v(s,t) \equiv 0,\ w(s,t) = t$, Eqn. 11 is satisfied and we obtain the ruled surface

$$P_1(s,t) = (t, -\sin s, \cos s),$$

$-4 \leq s \leq 4,\ -1 \leq t \leq 1$, *possessing $\bar{\alpha}$ as a common natural geodesic lift (Fig. 1).*

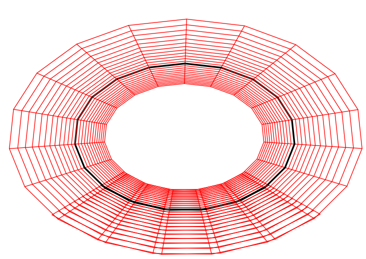

Fig. 1. Ruled surface $P_1(s,t)$ as a member of the surface family with a common natural geodesic lift $\bar{\alpha}$.

For the same curve, if we choose $u(s,t) \equiv 0,\ v(s,t) = t - \sinh t,\ w(s,t) = (\sinh s)\sinh t$ then we get the surface

$$P_2(s,t) = ((\sinh t)(\sinh s), (\sinh t - t)\cos s - \sin s, (\sinh t - t)\sin s + \cos s),$$

$0 < s \leq 1,\ -1 \leq t \leq 1$, *satisfying Eqn. 10 and accepting $\bar{\alpha}$ as a common natural geodesic lift (Fig. 2).*

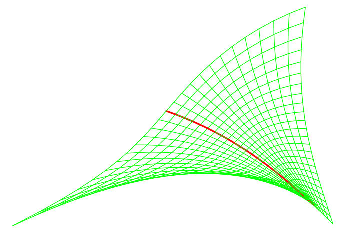

Fig. 2. $P_2(s,t)$ as a member of the surface family with a common natural geodesic lift $\bar{\alpha}$.

Example 2. *The Frenet apparatus of the arc length spacelike curve with timelike binormal* $\alpha(s) = \left(\frac{4}{9}\sinh 3s, \frac{4}{9}\cosh 3s, \frac{5}{3}s\right)$ *are*

$$\begin{aligned} T(s) &= \left(\frac{4}{3}\cosh 3s, \frac{4}{3}\sinh 3s, \frac{5}{3}\right), \\ N(s) &= (\sinh 3s, \cosh 3s, 0), \\ B(s) &= \left(-\frac{5}{3}\cosh, \frac{5}{3}\sinh 3s, -\frac{4}{3}\right). \end{aligned}$$

The natural lift $\bar{\alpha}(s) = \left(\frac{4}{3}\cosh 3s, \frac{4}{3}\sinh 3s, \frac{5}{3}\right)$ *of* α *is a spacelike curve with spacelike binormal and its Frenet vectors are*

$$\begin{aligned} \bar{T}(s) &= (\sinh 3s, \cosh 3s, 0), \\ \bar{N}(s) &= (\cosh 3s, \sinh 3s, 0), \\ \bar{B}(s) &= (0, 0, -1). \end{aligned}$$

If we let marching scale functions as $u(s,t) \equiv 0,\ v(s,t) = t^2 e^s,\ w(s,t) = t\ln s$ *we get the ruled surface*

$$P_3(s,t) = \left(\left(\frac{4}{3} + t^2 e^s\right)\cosh 3s, \left(\frac{4}{3} + t^2 e^s\right)\sinh 3s, \frac{5}{3} - t\ln s\right),$$

$1 < s \leq 2,\ 0 \leq t \leq 1,$ *satisfying Eqn. 10 and passing through* $\bar{\alpha}$ *as a common natural asymptotic lift (Fig. 3).*

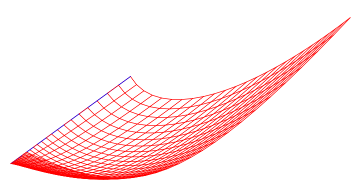

Fig. 3. $P_3(s,t)$ as a member of the surface family with a common natural geodesic lift $\bar{\alpha}$.

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5. Conclusions

We obtain necessary and sufficient conditions for a given spacelike curve with timelike binormal such that its natural lift is a common geodesic on every member of the surface family. Choosing different marching scale functions satisfying the conditions yields different surfaces possessing the natural lift of the given curve as a common geodesic. Constraints for a ruled surface are given. There are lots of problem to study related with surface families. One of them is to consider the construction of implicitly defined surfaces.

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