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Some extended trapezoid-type inequalities and applications

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Abstract

In this paper, we shall establish some extended trapezoid-type

$$\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \quad (a\le x\le c\le y\le b)$$

inequalities for differentiable convex functions and differentiable concave functions which are connected with Hermite-Hadamard inequality. Some error estimates for the midpoint, trapezoidal and Ostrowski formulae are also given.

Keywords: Hermite-Hadamard Inequality, Midpoint Inequality, Trapezoid Inequality, Ostrowski Inequality, Convex Function, Concave Functions, Special Means, Quadrature Rules.

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1. Introduction

Throughout in this paper, let $a<b$ in $\mathbb{R}$.

The inequality

$$f\left(\frac{a+b}{2}\right)\le(\ge)\frac{1}{b-a}\int_a^b f(t)\,dt\le(\ge)\frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex (concave) functions $f:[a,b]\to\mathbb{R}$, is known in the literature as Hermite-Hadamard inequality [7].

For some results which generalize, improve, and extend the inequality (1.1), see [1]-[6] and [8]-[15].

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In [11], Tseng *et al.* established the following Hermite-Hadamard-type inequality which refines the first inequality of (1.1).

A. Theorem. *Suppose that* $f:[a,b]\to\mathbb{R}$ *is a convex function on* $[a,b]$*. Then we have the inequality*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\leq\frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.2}$$

Using the similar proof of Theorem A, we also note that the inequalities in (1.2) are reversed when f is concave on $[a,b]$.

In [4], Dragomir and Agarwal established the following results connected with the second inequality in the inequality (1.1).

B. Theorem. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function on* (a,b) *with* $a<b$*. If* $|f'|$ *is convex on* $[a,b]$*, then we have*

$$\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \tag{1.3}$$

which is the trapezoid inequality provided $|f'|$ *is convex on* $[a,b]$*.*

C. Theorem. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function on* (a,b) *with* $a<b$ *and let* $p>1$*. If* $|f'|^{p/(p-1)}$ *is convex on* $[a,b]$*, then we have*

$$\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}} \tag{1.4}$$

which is the trapezoid inequality provided $|f'|^{p/(p-1)}$ *is convex on* $[a,b]$*.*

In [10], Pearce and Pečarić established the following theorems that improve Theorem C, generalize Theorem D and give similar results of Theorems B-C with a concavity property instead of convexity.

D. Theorem. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function on* (a,b) *with* $a<b$ *and* $q\geq1$*. If* $|f'|^q$ *is convex on* $[a,b]$*, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}} \tag{1.5}$$

which is the trapezoid inequality provided $|f'|^q$ *is convex on* $[a,b]$*.*

E. Theorem. *Under the assumptions of Theorem D. Then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}} \tag{1.6}$$

which is the midpoint inequality provided $|f'|^q$ *is convex on* $[a,b]$*.*

F. Theorem. *Under the assumptions of Theorem D and* $|f'|^q$ $(q\geq1)$ *is concave on* $[a,b]$*. Then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right| \tag{1.7}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right| \tag{1.8}$$

which are the trapezoid inequality and the midpoint inequality provided $|f'|^q$ *is concave on* $[a,b]$, *respectively.*

In [1], Alomari and Darus established the following Ostrowski-type inequalities.

G. Theorem. *Under the assumptions of Theorem B. Then, for all* $x\in[a,b]$, *we have*

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le (b-a)&\left[\left(\frac{1}{6}+\frac{1}{3}\left(\frac{b-x}{b-a}\right)^3\right)|f'(a)|\right.\\ &\left.+\left(\frac{1}{6}+\frac{1}{3}\left(\frac{x-a}{b-a}\right)^3\right)|f'(b)|\right].\end{aligned} \tag{1.9}$$

H. Theorem. *Under the assumptions of Theorem D. Then, for all* $x\in[a,b]$, *we have*

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{b-a}{2^{\frac{1}{q}}}&\left[\left(\frac{x-a}{b-a}\right)^2\left|f'\left(\frac{a+2x}{3}\right)\right|\right.\\ &\left.+\left(\frac{b-x}{b-a}\right)^2\left|f'\left(\frac{2x+b}{3}\right)\right|\right].\end{aligned} \tag{1.10}$$

From the above results, it is natural to consider the extended trapezoid-type formulae in the following lemma.

1.1. Lemma. *Let* $a\le x\le c\le y\le b$. *Then we have the extended trapezoid-type formula*

$$\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

as follows:

(1) *The trapezoid-type formula*

$$\begin{aligned}&\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ =&\left|\frac{f((1-\alpha)a+\alpha b)+f(\alpha a+(1-\alpha)b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\end{aligned}$$

as $0\le\alpha\le\frac{1}{2}, x=(1-\alpha)a+\alpha b, c=\frac{a+b}{2}$ *and* $y=\alpha a+(1-\alpha)b$.

(2) *The trapezoid formula*

$$\begin{aligned}&\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ =&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\end{aligned}$$

as $x=a, c=\frac{a+b}{2}$ *and* $y=b$.

(3) *The midpoint formula*

$$\begin{aligned}&\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ =&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\end{aligned}$$

as $x=c=y=\frac{a+b}{2}$.

(4) *The Ostrowski formula*

$$\begin{aligned}&\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ =&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\end{aligned}$$

as $x=c=y$.

In this paper, we establish some extended trapezoid-type inequalities which reduce the trapezoid-type, midpoint-type, Ostrowski-type inequalities, and improve Theorems B and D-H. Some applications to special means of real numbers are given. Finally, the approximations for quadrature formulae are also given.

2. Extended trapezoid-type Inequality

Throughout in this section, let $0\le\alpha\le\gamma\le 1-\beta\le 1, a\le x\le c\le y\le b$ and let $P_1, P_2, I_i\ (i=1,\cdots,4), h(t), h_1(t)\ (t\in[a,b])$ be defined as follows.

$$\begin{aligned}P_1=&\frac{1}{3(b-a)^3}\left[(x-a)^2\left(\frac{3b-a}{2}-x\right)+(c-x)^2\left(\frac{3b-c}{2}-x\right)\right.\\&\left.+(y-c)^2\left(\frac{3b-c}{2}-y\right)+(b-y)^3\right].\end{aligned}\tag{2.1}$$

$$\begin{aligned}P_2=&\frac{1}{3(b-a)^3}\left[(x-a)^3+(c-x)^2\left(\frac{c-3a}{2}+x\right)\right.\\&\left.+(y-c)^2\left(\frac{c-3a}{2}+y\right)+(b-y)^2\left(\frac{b-3a}{2}+y\right)\right].\end{aligned}\tag{2.2}$$

$$I_1=\frac{1}{3(b-a)^2(c-a)}\left[(x-a)^2\left(\frac{3c-a}{2}-x\right)+(c-x)^3\right],\tag{2.3}$$

$$I_2=\frac{1}{3(b-a)^2(c-a)}\left[(c-x)^2\left(\frac{c-3a}{2}+x\right)+(x-a)^3\right],\tag{2.4}$$

$$I_3=\frac{1}{3(b-a)^2(b-c)}\left[(y-c)^2\left(\frac{3b-c}{2}-y\right)+(b-y)^3\right],\tag{2.5}$$

and

$$I_4=\frac{1}{3(b-a)^2(b-c)}\left[(b-y)^2\left(\frac{b-3c}{2}+y\right)+(y-c)^3\right]\tag{2.6}$$

where $a<c<b$.

$$h(t)=\begin{cases}t-a, & a\le t<x\\ t-c & x\le t<y\\ t-b & y\le t\le b\end{cases}\quad\text{and } h_1(t)=\begin{cases}t-a, & a\le t<x\\ c-t, & x\le t<c\\ t-c, & c\le t<y\\ b-t, & y\le t\le b\end{cases}.$$

In order to prove our main results, we need the following lemma and remark whose proof can be obtained by simple computations and $r^2+s^2=(r+s)^2-2rs, r^2+s^2+t^2+u^2=(r+s+t+u)^2-[2(r+s)(t+u)+2rs+2tu]$ where $r,s,t,u\in\mathbb{R}$.

2.1. Lemma. *Let* $a, b, x, c, y, P_1, P_2, I_i \ (i = 1, \cdots, 4), h(t), h_1(t) \ (t \in [a, b])$ *be defined as above. Then we have*

$$|h(t)| = h_1(t) \, (t \in [a, b]),$$

$$P_1 = \frac{1}{(b-a)^3} \int_a^b (b-t) h_1(t) \, dt \text{ and } P_2 = \frac{1}{(b-a)^3} \int_a^b (t-a) h_1(t) \, dt,$$

As $a < c < b$,

$$\begin{aligned} I_1 &= \frac{1}{(b-a)^2 (c-a)} \int_a^c (c-t) h_1(t) \, dt, \\ I_2 &= \frac{1}{(b-a)^2 (c-a)} \int_a^c (t-a) h_1(t) \, dt, \\ I_3 &= \frac{1}{(b-a)^2 (b-c)} \int_c^b (b-t) h_1(t) \, dt, \\ I_4 &= \frac{1}{(b-a)^2 (b-c)} \int_c^b (t-c) h_1(t) \, dt, \end{aligned}$$

$$I_1 + \frac{b-c}{b-a} (I_2 + I_3) = P_1, \tag{2.7}$$

$$I_4 + \frac{c-a}{b-a} (I_2 + I_3) = P_2, \tag{2.8}$$

$$\begin{aligned} I_1 + I_2 &= \frac{1}{(b-a)^2} \int_a^c h_1(t) \, dt = \frac{1}{2(b-a)^2} \left[(x-a)^2 + (c-x)^2 \right] \\ &= \frac{1}{2} \left(\frac{c-a}{b-a} \right)^2 - \frac{(x-a)(c-x)}{(b-a)^2}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} I_3 + I_4 &= \frac{1}{(b-a)^2} \int_c^b h_1(t) \, dt = \frac{1}{2(b-a)^2} \left[(y-c)^2 + (b-y)^2 \right] \\ &= \frac{1}{2} \left(\frac{b-c}{b-a} \right)^2 - \frac{(y-c)(b-y)}{(b-a)^2}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} & I_1 + I_2 + I_3 + I_4 = P_1 + P_2 = \frac{1}{(b-a)^2} \int_a^b h_1(t) \, dt \\ &= \frac{1}{2(b-a)^2} \left[(x-a)^2 + (c-x)^2 + (y-c)^2 + (b-y)^2 \right] \\ &= \frac{1}{2} - \left[\frac{(c-a)(b-c)}{(b-a)^2} + \frac{(x-a)(c-x)}{(b-a)^2} + \frac{(y-c)(b-y)}{(b-a)^2} \right], \end{aligned} \tag{2.11}$$

$$\begin{aligned} & aI_1 + cI_2 \\ &= \frac{1}{(b-a)^2} \int_a^c h_1(t) \, t dt \\ &= \frac{1}{6} \left(\frac{x-a}{b-a} \right)^2 (2x+a) + \frac{1}{6} \left(\frac{c-x}{b-a} \right)^2 (2x+c), \end{aligned}$$

$$
\begin{aligned}
& cI_3 + bI_4 \\
= \; & \frac{1}{(b-a)^2} \int_c^b h_1(t)\, tdt \\
= \; & \frac{1}{6}\left(\frac{y-c}{b-a}\right)^2 (2y+c) + \frac{1}{6}\left(\frac{b-y}{b-a}\right)^2 (2y+b)
\end{aligned}
$$

and

$$0 < P_1, P_2, I_i \leq I_1 + I_2 + I_3 + I_4 \leq \frac{1}{2} \;\; (i = 1 \cdots 4).$$

2.2. Remark. Let $\alpha \in [0,1]$, $x = (1-\alpha)a + \alpha b, c = \frac{a+b}{2}$ and $y = \alpha a + (1-\alpha) b$ in the identities $(2.1) - (2.11)$. Then we have the identities

$$I_1 = I_4 = \frac{1}{3}\left[\alpha^2\left(\frac{3}{2}\gamma - 2\alpha\right) + 2\left(\frac{1}{2} - \alpha\right)^3\right] \quad \text{as} \;\; 0 < \gamma \leq 1, \tag{2.12}$$

$$I_2 = I_3 = \frac{1}{3}\left[\left(\frac{1}{2} - \alpha\right)^2\left(\frac{1}{2} + 2\alpha\right) + 2\alpha^3\right] \quad \text{as} \;\; 0 \leq \gamma < 1,$$

$$P_1 = P_2 = I_1 + I_2 = I_3 + I_4 = \frac{1}{8} - \alpha\left(\frac{1}{2} - \alpha\right), \tag{2.13}$$

and

$$I_1 + I_2 + I_3 + I_4 = P_1 + P_2 = \frac{1}{4} - \alpha(1 - 2\alpha). \tag{2.14}$$

2.3. Remark. In Theorem 2.4, Let $x = c = y$ in the identities $(2.1) - (2.6)$ and (2.11). Then we have the identities

$$
\begin{aligned}
P_1 &= \frac{(x-a)^2(3b-a-2x)}{6(b-a)^3} + \frac{1}{3}\left(\frac{b-x}{b-a}\right)^3, \\
P_2 &= \frac{(b-x)^2(b-3a+2x)}{6(b-a)^3} + \frac{1}{3}\left(\frac{x-a}{b-a}\right)^3, \\
I_1 &= \frac{1}{2}I_2 = \frac{1}{6}\left(\frac{x-a}{b-a}\right)^2, \qquad I_4 = \frac{1}{2}I_3 = \frac{1}{6}\left(\frac{b-x}{b-a}\right)^2, \\
I_1 &+ I_2 + I_3 + I_4 = P_1 + P_2 = \frac{1}{2} - \frac{(x-a)(b-x)}{(b-a)^2}
\end{aligned}
$$

Now, we are ready to state and prove the main results.

2.4. Theorem. *Let* $a, b, x, c, y, P_1, P_2, I_i \;(i = 1, \cdots, 4), h(t), h_1(t) \;(t \in [a,b])$ *be defined as above and let* q, f *be defined as in Theorem D. Then we have the following extended trapezoid-type inequalities.*

(1) The following inequality holds:

$$
\begin{aligned}
& \left|\frac{c-a}{b-a}f(x) + \frac{b-c}{b-a}f(y) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq \; & (P_1 + P_2)(b-a)\left(\frac{P_1|f'(a)|^q + P_2|f'(b)|^q}{P_1 + P_2}\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.15}
$$

(2) As $a < c < b$, we have the inequality

$$\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.16}$$
$$\leq (I_1+I_2+I_3+I_4)(b-a)\left(\frac{I_1|f'(a)|^q+(I_2+I_3)|f'(c)|^q+I_4|f'(b)|^q}{I_1+I_2+I_3+I_4}\right)^{\frac{1}{q}}$$
$$\leq (P_1+P_2)(b-a)\left(\frac{P_1|f'(a)|^q+P_2|f'(b)|^q}{P_1+P_2}\right)^{\frac{1}{q}}$$

which refines the inequality (2.15).

Proof. Using the integration by parts and simple computation, we have the following identity:

$$\frac{1}{b-a}\int_a^b h(t)\,f'(t)\,dt \tag{2.17}$$
$$=\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt.$$

(1) Now, using Hölder's inequality, the convexity of $|f'|^q$ and Lemma 2.1, we have the inequality

$$\left|\frac{1}{b-a}\int_a^b h(t)\,f'(t)\,dt\right| \tag{2.18}$$
$$\leq \frac{1}{b-a}\int_a^b |h(t)|\left|f'(t)\right|dt$$
$$= \frac{1}{b-a}\int_a^b h_1(t)\left|f'(t)\right|dt$$
$$\leq \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\int_a^b h_1(t)\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\int_a^b h_1(t)\left|f'\left(\frac{b-t}{b-a}\cdot a+\frac{t-a}{b-a}\cdot b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left[\int_a^b h_1(t)\frac{b-t}{b-a}\left|f'(a)\right|^q+h_1(t)\frac{t-a}{b-a}\left|f'(b)\right|^q dt\right]^{\frac{1}{q}}$$
$$= \left(\frac{1}{(b-a)^2}\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\frac{1}{(b-a)^3}\int_a^b h_1(t)(b-t)\,dt\cdot\left|f'(a)\right|^q\right.$$
$$\left.+\frac{1}{(b-a)^3}\int_a^b h_1(t)(t-a)\,dt\cdot\left|f'(b)\right|^q\right)^{\frac{1}{q}}\cdot(b-a)$$
$$= (P_1+P_2)^{1-\frac{1}{q}}\left(P_1\left|f'(a)\right|^q+P_2\left|f'(b)\right|^q\right)^{\frac{1}{q}}(b-a)$$
$$= (P_1+P_2)(b-a)\left(\frac{P_1|f'(a)|^q+P_2|f'(b)|^q}{P_1+P_2}\right)^{\frac{1}{q}}.$$

The inequality (2.15) follows from the identity (2.17) and the inequality (2.18).

(2) Let $a<c<b$. Using the inequality (2.18), the convexity of $|f'|^q$ and Lemma 2.1, we have the inequalities

$$\begin{aligned}
(2.19)\quad & \left|\frac{1}{b-a}\int_a^b h(t)\,f'(t)\,dt\right| \\
\le\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\int_a^b h_1(t)\,|f'(t)|^q\,dt\right)^{\frac{1}{q}} \\
=\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\int_a^c h_1(t)\,|f'(t)|^q\,dt+\int_a^c h_1(t)\,|f'(t)|^q\,dt\right)^{\frac{1}{q}} \\
=\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left[\int_a^c h_1(t)\left|f'\left(\frac{c-t}{c-a}\cdot a+\frac{t-a}{c-a}\cdot c\right)\right|^q dt\right. \\
& \left.+\int_c^b h_1(t)\left|f'\left(\frac{b-t}{b-c}\cdot c+\frac{t-c}{b-c}\cdot b\right)\right|^q\right]^{\frac{1}{q}} \\
\le\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left[\int_a^c h_1(t)\left(\frac{c-t}{c-a}\,|f'(a)|^q+\frac{t-a}{c-a}\,|f'(c)|^q\right)dt\right. \\
& \left.+\int_c^b h_1(t)\left(\frac{b-t}{b-c}\,|f'(c)|^q+\frac{t-c}{b-c}\,|f'(b)|^q\right)\right]^{\frac{1}{q}} \\
=\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left[\int_a^c h_1(t)\left(\frac{c-t}{c-a}\,|f'(a)|^q+\frac{t-a}{c-a}\,|f'(c)|^q\right)dt\right. \\
& \left.+\int_c^b h_1(t)\left(\frac{b-t}{b-c}\,|f'(c)|^q+\frac{t-c}{b-c}\,|f'(b)|^q\right)\right]^{\frac{1}{q}} \\
=\ & \left(\frac{1}{(b-a)^2}\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\frac{|f'(a)|^q}{(b-a)^2(c-a)}\int_a^c (c-t)\,h_1(t)\,dt\right. \\
& +\frac{|f'(c)|^q}{(b-a)^2(c-a)}\int_a^c (t-a)\,h_1(t)\,dt+\frac{|f'(c)|^q}{(b-a)^2(b-c)}\int_c^b (b-t)\,h_1(t)\,dt \\
& \left.+\frac{|f'(b)|^q}{(b-a)^2(b-c)}\int_c^b h_1(t)\,(t-c)\,dt\right)^{\frac{1}{q}}\cdot(b-a) \\
=\ & (I_1+I_2+I_3+I_4)^{1-\frac{1}{q}}\left(I_1\,|f'(a)|^q+(I_2+I_3)\,|f'(c)|^q+I_4\,|f'(b)|^q\right)^{\frac{1}{q}}(b-a) \\
=\ & (I_1+I_2+I_3+I_4)\,(b-a)\left(\frac{I_1\,|f'(a)|^q+(I_2+I_3)\,|f'(c)|^q+I_4\,|f'(b)|^q}{I_1+I_2+I_3+I_4}\right)^{\frac{1}{q}}
\end{aligned}$$

and

$$\begin{aligned}
(2.20)\quad & \frac{I_1\,|f'(a)|^q+(I_2+I_3)\,|f'(c)|^q+I_4\,|f'(b)|^q}{I_1+I_2+I_3+I_4} \\
=\ & \frac{I_1\,|f'(a)|^q+I_4\,|f'(b)|^q}{P_1+P_2}+\frac{I_2+I_3}{P_1+P_2}\left|f'\left(\frac{b-c}{b-a}c+\frac{c-a}{b-a}b\right)\right|^q \\
\le\ & \frac{\left[I_1+\frac{b-c}{b-a}(I_2+I_3)\right]|f'(a)|^q+\left[I_4+\frac{c-a}{b-a}(I_2+I_3)\right]|f'(b)|^q}{P_1+P_2} \\
=\ & \frac{P_1\,|f'(a)|^q+P_2\,|f'(b)|^q}{P_1+P_2}.
\end{aligned}$$

The inequality (2.16) follows from the identities (2.11), (2.17) and the inequalities (2.19)−(2.20). This completes the proof. ∎

Under the conditions of Theorem 2.4, Remark 2.2 and the identities (2.11), (2.1) − (2.6), we have the following corollaries and remarks.

2.5. Corollary. *Let $0 \leq \alpha \leq 1, x = (1-\alpha)a + \alpha b$, $c = \frac{a+b}{2}$ and $y = \alpha a + (1-\alpha)b$. Then, using Theorem 2.4 and Remark 2.2, we have the trapezoid-type inequality*

$$\begin{aligned} &\left|\frac{f((1-\alpha)a+\alpha b)+f(\alpha a+(1-\alpha)b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \leq\ & \left[\frac{1}{4}-\alpha(1-2\alpha)\right](b-a)\left(\frac{I_1|f'(a)|^q+2I_2\left|f'\left(\frac{a+b}{2}\right)\right|^q+I_1|f'(b)|^q}{2(I_1+I_2)}\right)^{\frac{1}{q}} \\ \leq\ & \left[\frac{1}{4}-\alpha(1-2\alpha)\right](b-a)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

which is provided $|f'|^q$ is convex on $[a,b]$.

2.6. Remark. Let $\alpha = 0$. Then, using Corollary 2.5 and Remark 2.2, we have $I_1 = \frac{1}{12}$, $I_2 = \frac{1}{24}$ and the trapezoid inequality

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \leq\ & \frac{b-a}{4}\left(\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}} \\ \leq\ & \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

which improves Theorems B and D.

2.7. Remark. Let $\alpha = \frac{1}{2}$. Then, using Corollary2.5 and Remark 2.2, we have $I_1 = \frac{1}{24}$, $I_2 = \frac{1}{12}$ and the midpoint inequality

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \leq\ & \frac{b-a}{4}\left(\frac{|f'(a)|^q+4\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{6}\right)^{\frac{1}{q}} \\ \leq\ & \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

which improves Theorem E.

2.8. Remark. Let $\alpha = \frac{1}{4}$. Then, using Corollary 2.5 and Remark 2.2, we have $I_1 = I_2 = \frac{1}{32}$ and the inequality

$$\begin{aligned} &\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \leq\ & \frac{b-a}{8}\left(\frac{|f'(a)|^q+2\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}} \\ \leq\ & \frac{b-a}{8}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

which is the second inequality in (1.2) provided $|f'|^q$ is convex on $[a, b]$.

2.9. Corollary. *Let P_1, P_2, I_i $(i = 1, \cdots, 4)$ be defined as in Remark 2.3. Then, using Theorem 2.4 and Remark 2.3, we have the following Ostrowski-type inequalities which are provided $|f'|^q$ is convex on $[a, b]$.*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ \leq \quad & (P_1 + P_2)(b-a) \left(\frac{P_1 |f'(a)|^q + P_2 |f'(b)|^q}{P_1 + P_2} \right)^{\frac{1}{q}} \end{aligned}$$

as $x = c = y$ and $a \leq x \leq b$.

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ \leq \quad & 3(I_1 + I_4)(b-a) \left[\frac{I_1 |f'(a)|^q + 2(I_1 + I_4)|f'(x)|^q + I_4 |f'(b)|^q}{3(I_1 + I_4)} \right]^{\frac{1}{q}} \\ \leq \quad & (P_1 + P_2)(b-a) \left(\frac{P_1 |f'(a)|^q + P_2 |f'(b)|^q}{P_1 + P_2} \right)^{\frac{1}{q}} \end{aligned}$$

as $x = c = y$ and $a < x < b$.

2.10. Remark. Let $k_1(t) = (t-a)^2 (3b - a - 2t)$ and $k_2(t) = (b-t)^2 (b - 3a + 2t)$ be defined on $[a, b]$. By simple computations,we obtain that k_1 is strictly increasing on $[a, b]$, k_2 is strictly decreasingon $[a, b]$ and $k_1(t), k_2(t) \leq (b-a)^3$ $(t \in [a, b])$. The , using the above inequalities, Corollary 2.9 improves Theorem G as $q = 1$.

2.11. Theorem. *Let $a, b, x, c, y, P_1, P_2, I_i$ $(i = 1, \cdots, 4), h(t), h_1(t)$ $(t \in [a, b])$ be defined as above and let q, f be defined as in Theorem F. Then we have the following extended trapezoid-type inequalities.*

(1) *The following inequality holds:*

$$\begin{aligned} & \left| \frac{c-a}{b-a} f(x) + \frac{b-c}{b-a} f(y) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ \leq \quad & (P_1 + P_2)(b-a) \left| f'\left(\frac{P_1 a + P_2 b}{P_1 + P_2} \right) \right|. \end{aligned} \tag{2.21}$$

(2) *As $a < c < b$, we have the inequality*

$$\begin{aligned} & \left| \frac{c-a}{b-a} f(x) + \frac{b-c}{b-a} f(y) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ \leq \quad & (b-a) \left[(I_1 + I_2) \left| f'\left(\frac{I_1 a + I_2 c}{I_1 + I_2} \right) \right| + (I_3 + I_4) \left| f'\left(\frac{I_3 c + I_4 b}{I_3 + I_4} \right) \right| \right] \\ \leq \quad & (P_1 + P_2)(b-a) \left| f'\left(\frac{P_1 a + P_2 b}{P_1 + P_2} \right) \right| \end{aligned} \tag{2.22}$$

which refines the inequality (2.21).

Proof. We observe that $|f'|^q$ is concave on $[a, b]$ implies $|f'| = \left(|f'|^q \right)^{\frac{1}{q}}$ is also concave on $[a, b]$. Using the inequality (2.18), the Jensen's integral inequality and Lemma 2.1, we have the following inequalities:

$$
\begin{aligned}
(2.23)\qquad & \left|\frac{1}{b-a}\int_a^b h(t)\, f'(t)\, dt\right| \\
&\leq \frac{1}{b-a}\int_a^b h_1(t)\left|f'(t)\right| dt \\
&\leq \frac{1}{b-a}\left(\int_a^b h_1(t)\, dt\right)\left|f'\left(\frac{\int_a^b h_1(t)\, tdt}{\int_a^b h_1(t)\, dt}\right)\right| \\
&= (b-a)\left(\frac{1}{(b-a)^2}\int_a^b h_1(t)\, dt\right)\left|f'\left(\frac{\frac{1}{(b-a)^2}\int_a^b h_1(t)\, tdt}{\frac{1}{(b-a)^2}\int_a^b h_1(t)\, dt}\right)\right| \\
&= (P_1+P_2)(b-a)\left|f'\left(\frac{P_1 a+P_2 b}{P_1+P_2}\right)\right|.
\end{aligned}
$$

The inequality (2.21) follows from the identity (2.17) and the inequality (2.23). Now, let $a<c<b$. Then we have the inequality

$$
\begin{aligned}
(2.24)\qquad & \left|\frac{1}{b-a}\int_a^b h(t)\, f'(t)\, dt\right| \\
&\leq \frac{1}{b-a}\int_a^b h_1(t)\left|f'(t)\right| dt \\
&= \frac{1}{b-a}\left(\int_a^c h_1(t)\left|f'(t)\right| dt+\int_c^b h_1(t)\left|f'(t)\right| dt\right) \\
&\leq \frac{1}{b-a}\left[\int_a^c h_1(t)\, dt\left|f'\left(\frac{\frac{1}{(b-a)^2}\int_a^c h_1(t)\, tdt}{\frac{1}{(b-a)^2}\int_a^c h_1(t)\, dt}\right)\right|\right. \\
&\qquad\qquad \left.+\int_c^b h_1(t)\, dt\left|f'\left(\frac{\int_c^b h_1(t)\, tdt}{\int_c^b h_1(t)\, dt}\right)\right|\right] \\
&= (b-a)\left[\frac{1}{(b-a)^2}\int_a^c h_1(t)\, dt\left|f'\left(\frac{\frac{1}{(b-a)^2}\int_a^c h_1(t)\, tdt}{\frac{1}{(b-a)^2}\int_a^c h_1(t)\, dt}\right)\right|\right. \\
&\quad \left.+\frac{1}{(b-a)^2}\int_c^b h_1(t)\, dt\left|f'\left(\frac{\frac{1}{(b-a)^2}\int_c^b h_1(t)\, tdt}{\frac{1}{(b-a)^2}\int_c^b h_1(t)\, dt}\right)\right|\right] \\
&= (b-a)\left[(I_1+I_2)\left|f'\left(\frac{I_1 a+I_2 c}{I_1+I_2}\right)\right|+(I_3+I_4)\left|f'\left(\frac{I_3 c+I_4 b}{I_3+I_4}\right)\right|\right].
\end{aligned}
$$

Using the inequality (2.18), the convcavity of $|f'|^q$ and Lemma 2.1, we have the inequalitiy

$$\begin{aligned}
(2.25)\qquad & (I_1+I_2)\left|f'\left(\frac{I_1a+I_2c}{I_1+I_2}\right)\right|+(I_3+I_4)\left|f'\left(\frac{I_3c+I_4b}{I_3+I_4}\right)\right| \\
&= (I_1+I_2+I_3+I_4)\left[\frac{I_1+I_2}{I_1+I_2+I_3+I_4}\left|f'\left(\frac{I_1a+I_2c}{I_1+I_2}\right)\right|\right. \\
&\qquad \left.+\frac{I_3+I_4}{I_1+I_2+I_3+I_4}\left|f'\left(\frac{I_3c+I_4b}{I_3+I_4}\right)\right|\right] \\
&\leq (I_1+I_2+I_3+I_4)\left|f'\left(\frac{I_1a+(I_2+I_3)c+I_4b}{I_1+I_2+I_3+I_4}\right)\right| \\
&= (I_1+I_2+I_3+I_4)\left|f'\left(\frac{I_1a+(I_2+I_3)\left(\frac{b-c}{b-a}a+\frac{c-a}{b-a}b\right)+I_4b}{I_1+I_2+I_3+I_4}\right)\right| \\
&= (I_1+I_2+I_3+I_4)\left|f'\left(\frac{\left(I_1+(I_2+I_3)\frac{b-c}{b-a}\right)a+\left(I_4+(I_2+I_3)\frac{c-a}{b-a}\right)b}{I_1+I_2+I_3+I_4}\right)\right| \\
&= (P_1+P_2)\left|f'\left(\frac{P_1a+P_2b}{P_1+P_2}\right)\right|.
\end{aligned}$$

The inequality (2.22) follows from the identity (2.17) and the inequalities (2.24) − (2.25). This completes the proof. ∎

Under the conditions of Theorem 2.11 nad Remark 2.2, we have the following corollaries and remarks.

2.12. Corollary. *Let $0\leq\alpha\leq 1, x=(1-\alpha)a+\alpha b$, $c=\frac{a+b}{2}$ and $y=\alpha a+(1-\alpha)b$. Then, using Theorem 2.11 and Remark 2.2, we have the trapezoid-type inequality*

$$\begin{aligned}
&\left|\frac{f((1-\alpha)a+\alpha b)+f(\alpha a+(1-\alpha)b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq\ & \left[\frac{1}{8}-\alpha\left(\frac{1}{2}-\alpha\right)\right](b-a)\left(\left|f'\left(\frac{I_1a+I_2\frac{a+b}{2}}{I_1+I_2}\right)\right|+\left|f'\left(\frac{I_2\frac{a+b}{2}+I_1b}{I_1+I_2}\right)\right|\right) \\
\leq\ & \left[\frac{1}{4}-\alpha(1-2\alpha)\right](b-a)\left|f'\left(\frac{a+b}{2}\right)\right|
\end{aligned}$$

which is provided $|f'|^q$ is convex on $[a,b]$.

2.13. Remark. Let $\alpha=0$. Then, using Corollary 2.12 and Remark 2.2, we have $I_1=\frac{1}{12}$, $I_2=\frac{1}{24}$ and the trapezoid inequality

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq\ & \frac{b-a}{8}\left(\left|f'\left(\frac{5a+b}{6}\right)\right|+\left|f'\left(\frac{a+5b}{6}\right)\right|\right) \\
\leq\ & \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|
\end{aligned}$$

which refines the inequality (1.7).

2.14. Remark. Let $\alpha = \frac{1}{2}$. Then, using Corollary2.12 and Remark 2.2, we have $I_1 = \frac{1}{24}$, $I_2 = \frac{1}{12}$ and the midpoint inequality

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq & \frac{b-a}{8}\left(\left| f'\left(\frac{2a+b}{3}\right) \right| + \left| f'\left(\frac{a+2b}{3}\right) \right| \right) \\ \leq & \frac{b-a}{4}\left| f'\left(\frac{a+b}{2}\right) \right| \end{aligned}$$

which refines the inequality (1.8).

2.15. Remark. Let $\alpha = \frac{1}{4}$. Then, using Corollary 2.12 and Remark 2.2, we have $I_1 = I_2 = \frac{1}{32}$ and the inequality

$$\begin{aligned} & \left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq & \frac{b-a}{16}\left(\left| f'\left(\frac{3a+b}{4}\right) \right| + \left| f'\left(\frac{a+3b}{4}\right) \right| \right) \\ \leq & \frac{b-a}{8}\left| f'\left(\frac{a+b}{2}\right) \right| \end{aligned}$$

which is the second inequality in (1.2) provided $|f'|^q$ is concave on $[a,b]$.

2.16. Corollary. *Let P_1, P_2, I_i $(i = 1, \cdots, 4)$ be defined as in Remark 2.3. Then, using Theorem 2.11 and Remark 2.3, we have the following Ostrowski-type inequalities which are provided $|f'|^q$ is convex on $[a,b]$.*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq & (P_1+P_2)(b-a)\left| f'\left(\frac{P_1 a + P_2 b}{P_1+P_2}\right) \right| \end{aligned}$$

as $x = c = y$ and $a \leq x \leq b$.

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq & \frac{b-a}{2}\left[\left(\frac{x-a}{b-a}\right)^2 \left| f'\left(\frac{a+2x}{3}\right) \right| + \left(\frac{b-x}{b-a}\right)^2 \left| f'\left(\frac{2x+b}{3}\right) \right| \right] \\ \leq & (P_1+P_2)(b-a)\left| f'\left(\frac{P_1 a + P_2 b}{P_1+P_2}\right) \right| \end{aligned}$$

as $x = c = y$ and$a < x < b$.

2.17. Remark. Using the fact that $2 \geq 2^{\frac{1}{q}}$ as $q \geq 1$, Corollary 3.3 improves Theorem H.

3. Applications for Special Means

In the literature, let us recall the following special means:

(1) The weighted arithmetic mean

$A_\alpha(u,v) = \alpha u + (1-\alpha)v,\ 0 \leq \alpha \leq 1,\ u, v \in \mathbb{R}.$

(2) The unweighted arithmetic mean

$$A(u,v)=\frac{u+v}{2},\ u,v\in\mathbb{R}.$$

(3) The harmonic mean

$$H(u,v)=\frac{2}{\frac{1}{u}+\frac{1}{v}},\ u,v>0.$$

(4) The identric mean

$$I(u,v)=\begin{cases}\frac{1}{e}\left(\frac{v^v}{u^u}\right)^{\frac{1}{v-u}} & \text{if } u\neq v\\ u & \text{if } u=v\end{cases},\ u,v>0.$$

(5) The logarithmic mean

$$L(u,v)=\begin{cases}\frac{v-u}{\ln v-\ln u} & \text{if } u\neq v\\ u & \text{if } u=v\end{cases},\ u,v>0.$$

(6) The p-logarithmic mean

$$L_p(u,v)=\begin{cases}\left[\frac{v^{p+1}-u^{p+1}}{(p+1)(v-u)}\right]^{\frac{1}{p}} & \text{if } u\neq v\\ u & \text{if } u=v\end{cases},\ u,v>0,\ p\in\mathbb{R}\backslash\{-1,0\}.$$

(7) The p-power mean

$$M_p(u,v)=\left(\frac{u^p+v^p}{2}\right)^{\frac{1}{p}},\ u,v>0,p\in\mathbb{R}\backslash\{0\}.$$

(8) The weighted p-power mean

$$M_p\begin{pmatrix}\alpha_1,\alpha_2,\cdots,\alpha_n\\ u_1,u_2,\cdots,u_n\end{pmatrix}=\left(\sum_{i=1}^{n}\alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0\leq\alpha_i\leq 1$, $u_i>0$ $(i=1,2,\cdots,n)$ with $\sum_{i=1}^{n}\alpha_i=1$.

Using the above results, we have the following propositions, corollaries and remarks about the above special means:

3.1. Proposition. *In Corollary 2.5 and Corollary 2.9, let* $s\in(-\infty,1]\cup\left[1+\frac{1}{q},\infty\right)\setminus\{-1,0\}$, $q\geq 1, 0<a<b$ *and let* $f(t)=t^s$ *on* $[a,b]$. *Then we have the following trapezoid-type and Ostrowski-type inequalities.*

$$\begin{aligned}(3.1)\qquad &\left|A\left(A_\alpha^s(b,a),A_\alpha^s(a,b)\right)-L_s^s(a,b)\right|\\ \leq\ &\left[\frac{1}{4}-\alpha(1-2\alpha)\right]|s|(b-a)M_q\begin{pmatrix}\frac{I_1}{2(I_1+I_2)},\frac{I_2}{I_1+I_2},\frac{I_1}{2(I_1+I_2)}\\ a^{s-1},\left(\frac{a+b}{2}\right)^{s-1},b^{s-1}\end{pmatrix}\\ \leq\ &\left[\frac{1}{4}-\alpha(1-2\alpha)\right]|s|(b-a)M_q\left(a^{s-1},b^{s-1}\right)\end{aligned}$$

as $\alpha\in[0,1]$, $x=(1-\alpha)a+\alpha b$, $c=\frac{a+b}{2}$ *and* $y=\alpha a+(1-\alpha)b$.

$$\begin{aligned}&\left|x^s-L_s^s(a,b)\right|\\ \leq\ &(P_1+P_2)|s|(b-a)M_q\begin{pmatrix}\frac{P_1}{P_1+P_2},\frac{P_2}{P_1+P_2}\\ a^{s-1},b^{s-1}\end{pmatrix}\end{aligned}$$

as $x=c=y$ *and* $a\leq x\leq b$.

$$\begin{aligned} & \left|x^{s}-L_{s}^{s}\left(a,b\right)\right| \\ \leq\quad & 3\left(I_{1}+I_{4}\right)\left|s\right|\left(b-a\right)M_{q}\left(\begin{array}{c} \frac{I_{1}}{3(I_{1}+I_{4})},\frac{2}{3},\frac{I_{4}}{3(I_{1}+I_{4})} \\ a^{s-1},x^{s-1},b^{s-1}\end{array}\right) \\ \leq\quad & \left(P_{1}+P_{2}\right)\left|s\right|\left(b-a\right)M_{q}\left(\begin{array}{c} \frac{P_{1}}{P_{1}+P_{2}},\frac{P_{2}}{P_{1}+P_{2}} \\ a^{s-1},b^{s-1}\end{array}\right)\end{aligned}$$

as $x=c=y$ *and* $a<x<b$.

3.2. Corollary. *Let* $\alpha=0$ *and in the inequality* (3.1). *Then, using the Hermite-Hadamard inequality* (1.1), *we have the following Hermite-Hadamard-type inequalities.*

$$\begin{aligned} 0 &\leq A\left(a^{s},b^{s}\right)-L_{s}^{s}\left(a,b\right) \\ &\leq \frac{\left|s\right|\left(b-a\right)}{4}M_{q}\left(\begin{array}{c} \frac{1}{3},\frac{1}{3},\frac{1}{3} \\ a^{s-1},\left(\frac{a+b}{2}\right)^{s-1},b^{s-1}\end{array}\right) \\ &\leq \frac{\left|s\right|\left(b-a\right)}{4}M_{q}\left(a^{s-1},b^{s-1}\right)\end{aligned}$$

as $s\in\left(-\infty,0\right)\cup\left[1+\frac{1}{q},\infty\right)\setminus\left\{-1\right\}$.

$$\begin{aligned} 0 &\leq L_{s}^{s}\left(a,b\right)-A\left(a^{s},b^{s}\right) \\ &\leq \frac{s\left(b-a\right)}{4}M_{q}\left(\begin{array}{c} \frac{1}{4},\frac{1}{2},\frac{1}{4} \\ a^{s-1},\left(\frac{a+b}{2}\right)^{s-1},b^{s-1}\end{array}\right) \\ &\leq \frac{s\left(b-a\right)}{4}M_{q}\left(a^{s-1},b^{s-1}\right)\end{aligned}$$

as $s\in\left(0,1\right]$.

3.3. Corollary. *Let* $\alpha=\frac{1}{2}$ *and in the inequality* (3.1). *Then, using the Hermite-Hadamard inequality* (1.1), *we have the following Hermite-Hadamard-type inequalities.*

$$\begin{aligned} 0 &\leq L_{s}^{s}\left(a,b\right)-A^{s}\left(a,b\right) \\ &\leq \frac{\left|s\right|\left(b-a\right)}{4}M_{q}\left(\begin{array}{c} \frac{1}{6},\frac{2}{3},\frac{1}{6} \\ a^{s-1},\left(\frac{a+b}{2}\right)^{s-1},b^{s-1}\end{array}\right) \\ &\leq \frac{\left|s\right|\left(b-a\right)}{4}M_{q}\left(a^{s-1},b^{s-1}\right)\end{aligned}$$

as $s\in\left(-\infty,0\right)\cup\left[1+\frac{1}{q},\infty\right)\setminus\left\{-1\right\}$.

$$\begin{aligned} 0 &\leq A^{s}\left(a,b\right)-L_{s}^{s}\left(a,b\right) \\ &\leq \frac{s\left(b-a\right)}{4}M_{q}\left(\begin{array}{c} \frac{1}{6},\frac{2}{3},\frac{1}{6} \\ a^{s-1},\left(\frac{a+b}{2}\right)^{s-1},b^{s-1}\end{array}\right) \\ &\leq \frac{s\left(b-a\right)}{4}M_{q}\left(a^{s-1},b^{s-1}\right)\end{aligned}$$

as $s\in\left(0,1\right]$.

3.4. Corollary. *Let $\alpha = \frac{1}{4}$ in the inequality* (3.1). *Then, using the inequality* (1.2), *we have the Hermite-Hadamard-type inequalities.*

$$\begin{aligned} 0 &\leq L_s^s(a,b) - A\left(A_{\frac{1}{4}}^s(b,a), A_{\frac{1}{4}}^s(a,b)\right) \\ &\leq \frac{|s|(b-a)}{8} M_q \left(\begin{array}{c} \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \\ a^{s-1}, \left(\frac{a+b}{2}\right)^{s-1}, b^{s-1} \end{array} \right) \\ &\leq \frac{|s|(b-a)}{8} M_q\left(a^{s-1}, b^{s-1}\right) \end{aligned}$$

as $s \in (-\infty, 0) \cup \left[1 + \frac{1}{q}, \infty\right) \setminus \{-1\}$.

$$\begin{aligned} 0 &\leq A\left(A_{\frac{1}{4}}^s(b,a), A_{\frac{1}{4}}^s(a,b)\right) - L_s^s(a,b) \\ &\leq \frac{s(b-a)}{8} M_q \left(\begin{array}{c} \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \\ a^{s-1}, \left(\frac{a+b}{2}\right)^{s-1}, b^{s-1} \end{array} \right) \\ &\leq \frac{s(b-a)}{8} M_q\left(a^{s-1}, b^{s-1}\right) \end{aligned}$$

as $s \in (0, 1]$.

3.5. Proposition. *In Corollary 2.12 and Corollary 2.16, let* $s \in \left[1, 1 + \frac{1}{q}\right]$, $q \geq 1$, $0 \leq a \leq x \leq c \leq y \leq b$ *and let* $f(t) = t^s$ *on* $[a, b]$. *Then we have the following trapezoid-type and Ostrowski-type inequalities.*

$$\begin{aligned} & \left|A\left(A_\alpha^s(b,a), A_\alpha^s(a,b)\right) - L_s^s(a,b)\right| \\ \leq & \left[\frac{1}{8} - \alpha\left(\frac{1}{2} - \alpha\right)\right] s(b-a) \left[A^{s-1}_{\frac{I_1}{I_1+I_2}}\left(a, \frac{a+b}{2}\right) + A^{s-1}_{\frac{I_1}{I_1+I_2}}\left(b, \frac{a+b}{2}\right)\right] \\ \leq & \left[\frac{1}{4} - \alpha(1-2\alpha)\right] s(b-a) A^{s-1}(a,b). \end{aligned} \tag{3.2}$$

As $x = c = y$,

$$\begin{aligned} & |x - L_s^s(a,b)| \\ \leq & (P_1 + P_2) s(b-a) A^{s-1}_{\frac{P_1}{P_1+P_2}}(a,b) \end{aligned}$$

As $x = c = y$ *and* $a < x < b$,

$$\begin{aligned} & |x - L_s^s(a,b)| \\ \leq & 3(I_1 + I_4) s(b-a) \left[A_{\frac{1}{3}}^{s-1}(a,x) + A_{\frac{1}{3}}^{s-1}(b,x)\right] \\ \leq & (P_1 + P_2) s(b-a) A^{s-1}_{\frac{P_1}{P_1+P_2}}(a,b). \end{aligned}$$

3.6. Corollary. *Let* $\alpha = 0$ *and in the inequality* (3.2). *Then, using the Hermite-Hadamard inequality* (1.1), *we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq L_s^s(a,b) - A(a^s, b^s) \leq \frac{s(b-a)}{8}\left[A_{\frac{1}{6}}^{s-1}(a,b) + A_{\frac{1}{6}}^{s-1}(b,a)\right] \\ &\leq \frac{s(b-a)}{4} A^{s-1}(a,b). \end{aligned}$$

3.7. Corollary. *Let $\alpha = \frac{1}{2}$ in the inequality* (3.2)*. Then, using the Hermite-Hadamard inequality* (1.1)*, we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq A^s(a,b) - L_s^s(a,b) \leq \frac{s(b-a)}{8}\left[A_{\frac{1}{3}}^{s-1}(a,b) + A_{\frac{1}{3}}^{s-1}(b,a)\right] \\ &\leq \frac{s(b-a)}{4}A^{s-1}(a,b). \end{aligned}$$

3.8. Corollary. *Let $\alpha = \frac{1}{4}$ in the inequality* (3.2)*. Then, using the inequality* (1.2)*, we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq A\left(A_{\frac{1}{4}}^s(b,a), A_{\frac{1}{4}}^s(a,b)\right) - L_s^s(a,b) \\ &\leq \frac{s(b-a)}{16}\left[A_{\frac{1}{4}}^{s-1}(a,b) + A_{\frac{1}{4}}^{s-1}(b,a)\right] \\ &\leq \frac{s(b-a)}{8}A^{s-1}(a,b). \end{aligned}$$

3.9. Proposition. *In Corollary 2.5 and Corollary 2.9, let $q \geq 1, 0 < a \leq x \leq c \leq y \leq b$ and let $f(t) = \frac{1}{t}$ on $[a,b]$. Then we have the following trapezoid-type and Ostrowski-type inequalities.*

$$\begin{aligned} &\left|H^{-1}(A_\alpha(b,a), A_\alpha(a,b)) - L^{-1}(a,b)\right| \\ &\leq \left[\frac{1}{4} - \alpha(1-2\alpha)\right](b-a)M_q\begin{pmatrix} \frac{I_1}{2(I_1+I_2)}, \frac{I_2}{I_1+I_2}, \frac{I_1}{2(I_1+I_2)} \\ a^{-2}, \left(\frac{a+b}{2}\right)^{-2}, b^{-2} \end{pmatrix} \\ &\leq \left[\frac{1}{4} - \alpha(1-2\alpha)\right](b-a)M_q\left(a^{-2}, b^{-2}\right). \end{aligned} \tag{3.3}$$

As $x = c = y$,

$$\begin{aligned} &\left|\frac{1}{x} - L^{-1}(a,b)\right| \\ &\leq (P_1+P_2)(b-a)M_q\begin{pmatrix} \frac{P_1}{P_1+P_2}, \frac{P_2}{P_1+P_2} \\ a^{-2}, b^{-2} \end{pmatrix}. \end{aligned}$$

As $x = c = y$ and $a < x < b$,

$$\begin{aligned} &\left|\frac{1}{x} - L^{-1}(a,b)\right| \\ &\leq 3(I_1+I_4)(b-a)M_q\begin{pmatrix} \frac{I_1}{3(I_1+I_4)}, \frac{2}{3}, \frac{I_4}{3(I_1+I_4)} \\ a^{-2}, x^{-2}, b^{-2} \end{pmatrix} \\ &\leq (P_1+P_2)(b-a)M_q\begin{pmatrix} \frac{P_1}{P_1+P_2}, \frac{P_2}{P_1+P_2} \\ a^{-2}, b^{-2} \end{pmatrix}. \end{aligned}$$

3.10. Corollary. *Let $\alpha = 0$ and in the inequality* (3.3)*. Then, using the Hermite-Hadamard inequality* (1.1)*, we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq H^{-1}(a,b) - L^{-1}(a,b) \leq \frac{b-a}{4}M_q\begin{pmatrix} \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ a^{-2}, \left(\frac{a+b}{2}\right)^{-2}, b^{-2} \end{pmatrix} \\ &\leq \frac{b-a}{4}M_q\left(a^{-2}, b^{-2}\right). \end{aligned}$$

3.11. Corollary. *Let* $\alpha = \frac{1}{2}$ *in the inequality* (3.3). *Then, using the Hermite-Hadamard inequality* (1.1), *we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\le L^{-1}(a,b) - A^{-1}(a,b) \le \frac{b-a}{4} M_q \begin{pmatrix} \frac{1}{6}, \frac{2}{3}, \frac{1}{6} \\ a^{-2}, \left(\frac{a+b}{2}\right)^{-2}, b^{-2} \end{pmatrix} \\ &\le \frac{b-a}{4} M_q\left(a^{-2}, b^{-2}\right). \end{aligned}$$

3.12. Corollary. *Let* $\alpha = \frac{1}{4}$ *in the inequality* (3.3). *Then, using the inequality* (1.2), *we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\le L^{-1}(a,b) - H^{-1}\left(A_{\frac{1}{4}}(b,a), A_{\frac{1}{4}}(a,b)\right) \\ &\le \frac{b-a}{8} M_q \begin{pmatrix} \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \\ a^{-2}, \left(\frac{a+b}{2}\right)^{-2}, b^{-2} \end{pmatrix} \\ &\le \frac{b-a}{8} M_q\left(a^{-2}, b^{-2}\right). \end{aligned}$$

3.13. Proposition. *In Corollary 2.5 and Corollary 2.9, let* $q \ge 1, 0 < a \le x \le c \le y \le b$ *and let* $f(t) = \ln t$ *on* $[a,b]$. *Then we have the following trapezoid-type and Ostrowski-type inequalities.*

$$\begin{aligned} &\left|A\left(\ln A_\alpha(a,b), \ln A_\alpha(b,a)\right) - \ln I(a,b)\right| \\ &\le \left[\frac{1}{4} - \alpha(1-2\alpha)\right](b-a) M_q \begin{pmatrix} \frac{I_1}{2(I_1+I_2)}, \frac{I_2}{I_1+I_2}, \frac{I_1}{2(I_1+I_2)} \\ a^{-1}, \left(\frac{a+b}{2}\right)^{-1}, b^{-1} \end{pmatrix} \\ &\le \left[\frac{1}{4} - \alpha(1-2\alpha)\right](b-a) M_q\left(a^{-1}, b^{-1}\right). \end{aligned} \tag{3.4}$$

As $x = c = y$,

$$\begin{aligned} &\left|\ln x - \ln I(a,b)\right| \\ &\le (P_1+P_2)(b-a) M_q \begin{pmatrix} \frac{P_1}{P_1+P_2}, \frac{P_2}{P_1+P_2} \\ a^{-1}, b^{-1} \end{pmatrix}. \end{aligned}$$

As $x = c = y$ *and* $a < x < b$,

$$\begin{aligned} &\left|\ln x - \ln I(a,b)\right| \\ &\le 3(I_1+I_4)(b-a) M_q \begin{pmatrix} \frac{I_1}{3(I_1+I_4)}, \frac{2}{3}, \frac{I_4}{3(I_1+I_4)} \\ a^{-1}, x^{-1}, b^{-1} \end{pmatrix} \\ &\le (P_1+P_2)(b-a) M_q \begin{pmatrix} \frac{P_1}{P_1+P_2}, \frac{P_2}{P_1+P_2} \\ a^{-1}, b^{-1} \end{pmatrix} \end{aligned}$$

3.14. Corollary. *Let* $\alpha = 0$ *and in the inequality* (3.4). *Then, using the Hermite-Hadamard inequality* (1.1), *we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\le \ln I(a,b) - A(\ln a, \ln b) \le \frac{b-a}{4} M_q \begin{pmatrix} \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ a^{-1}, \left(\frac{a+b}{2}\right)^{-1}, b^{-1} \end{pmatrix} \\ &\le \frac{b-a}{4} M_q\left(a^{-1}, b^{-1}\right). \end{aligned}$$

3.15. Corollary. *Let* $\alpha = \frac{1}{2}$ *in the inequality* (3.3). *Then, using the Hermite-Hadamard inequality* (1.1), *we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\le \ln A(a,b) - \ln I(a,b) \le \frac{b-a}{4} M_q \begin{pmatrix} \frac{1}{6}, \frac{2}{3}, \frac{1}{6} \\ a^{-1}, \left(\frac{a+b}{2}\right)^{-1}, b^{-1} \end{pmatrix} \\ &\le \frac{b-a}{4} M_q\left(a^{-1}, b^{-1}\right). \end{aligned}$$

3.16. Corollary. *Let $\alpha = \frac{1}{4}$ in the inequality* (3.3) *. Then, using the inequality* (1.2)*, we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq A\left(\ln A_{\frac{1}{4}}(a,b), \ln A_{\frac{1}{4}}(b,a)\right) - \ln I(a,b) \\ &\leq \frac{b-a}{8} M_q \left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \\ a^{-1}, \left(\frac{a+b}{2}\right)^{-1}, b^{-1} \end{matrix} \right) \\ &\leq \frac{b-a}{8} M_q\left(a^{-1}, b^{-1}\right). \end{aligned}$$

4. Applications for the extended Trapezoid Quadrature Formula

Throughout in this section, let $0 \leq \alpha \leq 1, L_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a partition of the interval $[a,b]$, $\xi_i \leq x_i \leq \zeta_i$ in $[t_i, t_{i+1}]$, $l_i = t_{i+1} - t_i$, $(i = 0, 1, \cdots, n-1)$ let $P_1(i), P_2, I_j(i)$ $(j = 1, \cdots, 4; i = 1, \cdots, n)$ be defined as follows.

$$\begin{aligned} P_1(i) &= \frac{1}{3l_i^3}\left[(x_i - t_i)^2\left(\frac{3t_{i+1} - t_i}{2} - \xi_i\right) + (x_i - \xi_i)^2\left(\frac{3t_{i+1} - x_i}{2} - \xi_i\right)\right. \\ &\quad \left. + (\zeta_i - x_i)^2\left(\frac{3t_{i+1} - x_i}{2} - \zeta_i\right) + (t_{i+1} - \zeta_i)^3\right] \end{aligned}$$

and

$$\begin{aligned} P_2(i) = \frac{1}{3l_i^3}&\left[(\xi_i - t_i)^3 + (x_i - \xi_i)^2\left(\frac{x_i - 3t_i}{2} + \xi_i\right)\right. \\ + (\zeta_i - x_i)^2&\left.\left(\frac{x_i - 3t_i}{2} + \zeta_i\right) + (t_{i+1} - \zeta_i)^2\left(\frac{t_{i+1} - 3t_i}{2} + \zeta_i\right)\right]. \end{aligned}$$

As $t_i < x_i < t_{i+1}$,

$$I_1(i) = \frac{1}{3l_i^2(x_i - t_i)}\left[(\xi_i - t_i)^2\left(\frac{3x_i - t_i}{2} - \xi_i\right) + (x_i - \xi_i)^3\right],$$

$$I_2(i) = \frac{1}{3l_i^2(x_i - t_i)}\left[(x_i - \xi_i)^2\left(\frac{x_i - 3t_i}{2} + \xi_i\right) + (\xi_i - t_i)^3\right],$$

$$I_3(i) = \frac{1}{3l_i^2(t_{i+1} - x_i)}\left[(\zeta_i - x_i)^2\left(\frac{3t_{i+1} - x_i}{2} - \zeta_i\right) + (t_{i+1} - \zeta_i)^3\right]$$

and

$$I_4(i) = \frac{1}{3l_i^2(t_{i+1} - x_i)}\left[(t_{i+1} - \zeta_i)^2\left(\frac{t_{i+1} - 3x_i}{2} + \zeta_i\right) + (\zeta_i - x_i)^3\right].$$

Define the extended Trapezoid quadrature formula

$$\int_a^b f(t)dt = T(f, L_n, \xi, \zeta) + R(f, L_n, \xi, \zeta) \tag{4.1}$$

where

$$T(f, L_n, \xi, \zeta) = \sum_{i=0}^{n-1} \frac{x_i - t_i}{t_{i+1} - t_i} f(\xi_i) + \frac{t_{i+1} - x_i}{t_{i+1} - t_i} f(\zeta_i) \tag{4.2}$$

and the remainder term $R(f, L_n, \xi_i, \zeta_i)$ denotes the associated approximation error of $\int_a^b f(t)dt$ by $T(f, L_n, \xi, \zeta)$.

Now, we have the following special formulae.

(1) The trapezoid formula

$$T(f,L_n,\xi,\zeta)=T_0(f,L_n,\xi,\zeta)=\sum_{i=0}^{n-1}\frac{f(t_i)+f(t_{i+1})}{2}l_i \tag{4.3}$$

where $\xi_i=t_i$ and $\zeta_i=t_{i+1}$ $(i=0,1,\cdots,n-1)$.

(2) The midpoint formula

$$T(f,L_n,\xi,\zeta)=M(f,L_n,\xi,\zeta)=\sum_{i=0}^{n-1}f\left(\frac{t_i+t_{i+1}}{2}\right)l_i \tag{4.4}$$

where $\xi_i=\zeta_i=\frac{t_i+t_{i+1}}{2}$ $(i=0,1,\cdots,n-1)$.

(3) The Ostrowski formula

$$T(f,L_n,\xi,\zeta)=O(f,L_n,\xi,\zeta)=\sum_{i=0}^{n-1}f(x_i)l_i \tag{4.5}$$

where $\xi_i=\zeta_i=x_i$ $(i=0,1,\cdots,n-1)$.

4.1. Theorem. *Let f be defined as in Theorem 2.4 and let $\int_a^b f(t)dt, T(f,L_n,\xi,\zeta)$ and $R(f,L_n,\xi,\zeta)$ be defined as in the identity* (4.1)*. Then, the remainder term $R(f,L_n,\xi,\zeta)$ satisfies the following estimates.*

(1) We have the inequality

$$\begin{aligned}(4.6)\qquad &|R(f,L_n,\xi,\zeta)|\\ \leq\ &\sum_{i=0}^{n-1}(P_1(i)+P_2(i))l_i^2\left(\frac{P_1(i)|f'(\xi_i)|^q+P_2(i)|f'(\zeta_i)|^q}{P_1(i)+P_2(i)}\right)^{\frac{1}{q}}\\ \leq\ &\max\left\{|f'(a)|,|f'(b)|\right\}\sum_{i=0}^{n-1}(P_1(i)+P_2(i))l_i^2.\end{aligned}$$

(2) Let $t_i<x_i<t_{i+1}$ $(i=0,1,\cdots,n-1)$. Then we have the inequality

$$\begin{aligned}(4.7)\qquad &|R(f,L_n,\xi,\zeta)|\\ \leq\ &\sum_{i=0}^{n-1}\left\{\left(\sum_{j=1}^{4}I_j(i)\right)l_i^2\right.\\ &\left.\times\left(\frac{I_1(i)|f'(t_i)|^q+(I_2(i)+I_3(i))|f'(x_i)|^q+I_4(i)|f'(t_{i+1})|^q}{\sum_{j=1}^{4}I_j(i)}\right)^{\frac{1}{q}}\right\}\\ \leq\ &\sum_{i=0}^{n-1}(P_1(i)+P_2(i))l_i^2\left(\frac{P_1(i)|f'(t_i)|^q+P_2(i)|f'(t_{i+1})|^q}{P_1(i)+P_2(i)}\right)^{\frac{1}{q}}\\ \leq\ &\max\left\{|f'(a)|,|f'(b)|\right\}\sum_{i=0}^{n-1}(P_1(i)+P_2(i))l_i^2.\end{aligned}$$

Proof. Apply Theorem 2.4 on the intervals $[t_i,t_{i+1}]$ $(i=0,1,\cdots,n-1)$ to get the following inequalities.

(1) For all $i = 0, 1, \cdots, n-1$, we have the inequality

$$\left|\frac{f(\xi_i) + f(\zeta_i)}{2} l_i - \int_{t_i}^{t_{i+1}} f(s)ds\right| \tag{4.8}$$

$$\leq \quad (P_1(i) + P_2(i)) l_i^2 \left(\frac{P_1(i)|f'(t_i)|^q + P_2(i)|f'(t_{i+1})|^q}{P_1(i) + P_2(i)}\right)^{\frac{1}{q}}.$$

(2) Let $t_i < x_i < t_{i+1}$ $(i = 0, 1, \cdots, n-1)$. Then we have the inequality

$$\left|\frac{f(\xi_i) + f(\zeta_i)}{2} l_i - \int_{t_i}^{t_{i+1}} f(s)ds\right| \tag{4.9}$$

$$\leq \quad \left(\sum_{j=1}^{4} I_j(i)\right) l_i^2$$

$$\times \left(\frac{I_1(i)|f'(t_i)|^q + (I_2(i) + I_3(i))|f'(x_i)|^q + I_4(i)|f'(t_{i+1})|^q}{\sum\limits_{j=1}^{4} I_j(i)}\right)^{\frac{1}{q}} \Bigg\}$$

$$\leq \quad (P_1(i) + P_2(i)) l_i^2 \left(\frac{P_1(i)|f'(t_i)|^q + P_2(i)|f'(t_{i+1})|^q}{P_1(i) + P_2(i)}\right)^{\frac{1}{q}}.$$

Using the convexity of $|f'|^q$, we have the inequality

$$\left(\frac{P_1(i)|f'(t_i)|^q + P_2(i)|f'(t_{i+1})|^q}{P_1(i) + P_2(i)}\right)^{\frac{1}{q}} \tag{4.10}$$

$$\leq \quad \left[\frac{P_1(i)}{P_1(i) + P_2(i)} \left(\frac{b - t_i}{b - a}|f'(a)|^q + \frac{t_i - a}{b - a}|f'(b)|^q\right)\right.$$

$$\left. + \frac{P_2(i)}{P_1(i) + P_2(i)} \left(\frac{b - t_{i+1}}{b - a}|f'(a)|^q + \frac{t_{i+1} - a}{b - a}|f'(b)|^q\right)\right]^{\frac{1}{q}}$$

$$\leq \quad \left(\max\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}} = \max\left\{|f'(a)|, |f'(b)|\right\}$$

for all $i = 0, 1, \cdots, n-1$.

The inequalities (4.6) and (4.7) follow from the inequalities (4.10) − (4.10) and the generalized triangle inequality.

This completes the proof. ∎

4.2. Corollary. *In Theorem 4.1, let* $\xi_i = t_i, \zeta_i = t_{i+1}$ *and* $x_i = \frac{t_i + t_2}{2}$ $(i = 0, 1, \cdots, n-1)$. *Then* $P_1(i) = P_2(i) = \frac{1}{8}, I_1(i) = I_4(i) = \frac{1}{12}$, $I_2(i) = I_3(i) = \frac{1}{24}$ $(i = 0, 1, \cdots, n-1)$ *and the trapzoid-type error satisfies*

$$|R(f, L_n, \xi, \zeta)|$$

$$\leq \quad \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left[\frac{|f'(t_i)|^q + \left|f'\left(\frac{t_i + t_{i+1}}{2}\right)\right|^q + |f'(t_{i+1})|^q}{3}\right]^{\frac{1}{q}}$$

$$\leq \quad \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left[\frac{|f'(t_i)|^q + |f'(t_{i+1})|^q}{2}\right]^{\frac{1}{q}}$$

$$\leq \quad \max\left\{|f'(a)|, |f'(b)|\right\} \sum_{i=0}^{n-1} \frac{l_i^2}{4}$$

which improves Proposition 3 *in* [11].

4.3. Corollary. *In Theorem 4.1, let* $\xi_i = \zeta_i = x_i = \frac{t_i+t_{i+1}}{2} = (i = 0, 1, \cdots, n-1)$. *Then* $P_1(i) = P_2(i) = \frac{1}{8}, I_1(i) = I_4(i) = \frac{1}{24},\ I_2(i) = I_3(i) = \frac{1}{12}\ (i = 0, 1, \cdots, n-1)$ *and the midpoint-type error satisfies*

$$\begin{aligned}
& |R(f, L_n, \xi, \zeta)| \\
\leq\ & \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left[\frac{|f'(t_i)|^q + 4\left|f'\left(\frac{t_i+t_{i+1}}{2}\right)\right|^q + |f'(t_{i+1})|^q}{6} \right]^{\frac{1}{q}} \\
\leq\ & \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left[\frac{|f'(t_i)|^q + |f'(t_{i+1})|^q}{2} \right]^{\frac{1}{q}} \\
\leq\ & \max\left\{\left|f'(a)\right|, \left|f'(b)\right|\right\} \sum_{i=0}^{n-1} \frac{l_i^2}{4}.
\end{aligned}$$

4.4. Corollary. *In Theorem 4.1, let* $\xi_i = \zeta_i = x_i \in (t_i, t_{i+1})\ (i = 0, 1, \cdots, n-1)$. *Then*

$$\begin{aligned}
P_1(i) &= \frac{(x_i - t_i)^2 (3t_{i+1} - t_i - 2x)}{6(t_{i+1} - t_i)^3} + \frac{1}{3}\left(\frac{t_{i+1} - x}{t_{i+1} - t_i}\right)^3, \\
P_2(i) &= \frac{(t_{i+1} - x)^2 (t_{i+1} - 3t_i + 2x)}{6(t_{i+1} - t_i)^3} + \frac{1}{3}\left(\frac{x_i - t_i}{t_{i+1} - t_i}\right)^3, \\
I_1(i) &= \frac{1}{2} I_2(i) = \frac{1}{6}\left(\frac{x_i - t_i}{t_{i+1} - t_i}\right)^2, \\
I_4(i) &= \frac{1}{2} I_3(i) = \frac{1}{6}\left(\frac{t_{i+1} - x_i}{t_{i+1} - t_i}\right)^2,
\end{aligned}$$

$$\begin{aligned}
\sum_{j=0}^{4} I_j(i) &= P_1(i) + P_2(i) \\
&= 3(I_1(i) + I_4(i)) = \frac{1}{2} - \frac{(x_i - t_i)(t_{i+1} - x_i)}{(t_{i+1} - t_i)^2}
\end{aligned}$$

and the Ostrowski-type error satisfies

$$\begin{aligned}
& |R(f, L_n, \xi, \zeta)| \\
\leq\ & \sum_{i=0}^{n-1} \left\{ 3(I_1(i) + I_4(i)) l_i^2 \right. \\
& \left. \times \left[\frac{I_1(i)|f'(t_i)|^q + 2(I_1(i) + I_4(i))\left|f'\left(\frac{t_i+t_{i+1}}{2}\right)\right|^q + I_4(i)|f'(t_{i+1})|^q}{3(I_1(i) + I_4(i))} \right]^{\frac{1}{q}} \right\} \\
\leq\ & \sum_{i=0}^{n-1} (P_1(i) + P_2(i)) l_i^2 \left[\frac{P_1(i)|f'(t_i)|^q + P_2(i)|f'(t_{i+1})|^q}{P_1(i) + P_2(i)} \right]^{\frac{1}{q}} \\
\leq\ & \max\left\{\left|f'(a)\right|, \left|f'(b)\right|\right\} \sum_{i=0}^{n-1} (P_1(i) + P_2(i)) l_i^2.
\end{aligned}$$

Similarly, using Theorem 2.11 we can prove the following theorem.

4.5. Theorem. *Let* f *be defined as in Theorem 2.11 and let* $\int_a^b f(t)dt, T(f, L_n, \xi, \zeta)$ *and* $R(f, L_n, \xi, \zeta)$ *be defined as in the identity* (4.1). *Then, the remainder term* $R(f, L_n, \xi, \zeta)$ *satisfies the following estimates.*

(1) We have the inequality

$$|R(f, I_n, \xi, \zeta)| \leq \sum_{i=0}^{n-1} (P_1(i) + P_2(i)) l_i^2 \left| f'\left(\frac{P_1(i) t_i + P_2(i) t_{i+1}}{P_1(i) + P_2(i)} \right) \right|$$

for all $i = 0, 1, ..., n-1$.

(2) Let $t_i < x_i < t_{i+1}$ $(i = 0, 1, \cdots, n-1)$. Then we have the inequality

$$\begin{aligned} & |R(f, L_n, \xi, \zeta)| \\ \leq & \sum_{i=0}^{n-1} l_i^2 \left[(I_1(i) + I_2(i)) \left| f'\left(\frac{I_1(i) t_i + I_2(i) x_i}{I_1(i) + I_2(i)} \right) \right| \right. \\ & \left. + (I_3(i) + I_4(i)) \left| f'\left(\frac{I_3(i) x_i + I_4(i) t_{i+1}}{I_3(i) + I_4(i)} \right) \right| \right] \\ \leq & \sum_{i=0}^{n-1} (P_1(i) + P_2(i)) l_i^2 \left| f'\left(\frac{P_1(i) t_i + P_2(i) t_{i+1}}{P_1(i) + P_2(i)} \right) \right|. \end{aligned}$$

4.6. Corollary. *In Theorem 4.5, let* $\xi_i = t_i, \zeta_i = t_{i+1}$ *and* $x_i = \frac{t_i + t_2}{2}$ $(i = 0, 1, \cdots, n-1)$. *Then* $P_1(i) = P_2(i) = \frac{1}{8}, I_1(i) = I_4(i) = \frac{1}{12},\ I_2(i) = I_3(i) = \frac{1}{24}$ $(i = 0, 1, \cdots, n-1)$ *and the trapzoid-type error satisfies*

$$\begin{aligned} & |R(f, L_n, \xi, \zeta)| \\ \leq & \sum_{i=0}^{n-1} \frac{l_i^2}{8} \left(\left| f'\left(\frac{5t_i + t_{i+1}}{6} \right) \right| + \left| f'\left(\frac{t_i + 5t_{i+1}}{6} \right) \right| \right) \\ \leq & \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left| f'\left(\frac{t_i + t_{i+1}}{2} \right) \right| \end{aligned}$$

which improves Proposition 4 *in* [11].

4.7. Corollary. *In Theorem 4.5, let* $\xi_i = \zeta_i = x_i = \frac{t_i + t_{i+1}}{2} = (i = 0, 1, \cdots, n-1)$. *Then* $P_1(i) = P_2(i) = \frac{1}{8}, I_1(i) = I_4(i) = \frac{1}{24},\ I_2(i) = I_3(i) = \frac{1}{12}$ $(i = 0, 1, \cdots, n-1)$ *and the midpoint-type error satisfies*

$$\begin{aligned} & |R(f, L_n, \xi, \zeta)| \\ \leq & \sum_{i=0}^{n-1} \frac{l_i^2}{8} \left(\left| f'\left(\frac{2t_i + t_{i+1}}{3} \right) \right| + \left| f'\left(\frac{t_i + 2t_{i+1}}{3} \right) \right| \right) \\ \leq & \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left| f'\left(\frac{t_i + t_{i+1}}{2} \right) \right| \end{aligned}$$

4.8. Corollary. *In Theorem 4.5, let* $\xi_i = \zeta_i = x_i \in (t_i, t_{i+1})$ $(i = 0, 1, \cdots, n-1)$. *Then the Ostrowski-type error satisfies*

$$\begin{aligned} & |R(f, L_n, \xi, \zeta)| \\ \leq & \sum_{i=0}^{n-1} \frac{l_i^2}{2} \left[\left(\frac{x_i - t_i}{t_{i+1} - t_i} \right)^2 \left| f'\left(\frac{t_i + 2x_i}{3} \right) \right| + \left(\frac{t_{i+1} - x_i}{t_{i+1} - t_i} \right)^2 \left| f'\left(\frac{2x_i + t_{i+1}}{3} \right) \right| \right] \\ \leq & \sum_{i=0}^{n-1} (P_1(i) + P_2(i)) l_i^2 \left| f'\left(\frac{P_1(i) t_i + P_2(i) t_{i+1}}{P_1(i) + P_2(i)} \right) \right|, \end{aligned}$$

where $P_1(i), P_2(i)$ $(i = 0, 1, \cdots, n-1)$ *is defined as in Corollary 4.7 and* $t_i < x_i < t_{i+1}$ $(i = 0, 1, \cdots, n-1)$.

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